A general continuous auction model with insiders *

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Abstract

We analyse in a unified way how the presence of a trader with privilege information makes the market to be efficient when the release time is known. We establish a general relation between the problem of finding an equilibrium and the problem of enlargement of filtrations. We also consider the case where the time of announcement is random. In such a case the market is not fully efficient and there exists equilibrium if the sensitivity of prices with respect to the global demand is time decreasing according with the distribution of the random time.

Key words: Market microstructure, insider trading, stochastic control, semimartingales.
JEL-Classification C61· D43· D44· D53· G11· G12· G14

1 Introduction

Models of financial markets with the presence of an insider or informational asymmetries have a great literature, see e.g Karatzas and Pikovsky (1996), Amendiger *et. al.* (1998), Imkeller *et. al.* (2001), Corcuera *et. al.* (2004), Biagini and Oksendal (2005), Kohatsu-Higa (2007), Di Nunno *et. al.* (2008) and the references there in. In most of these models prices are fixed exogenously, the insider does not affect the stock price dynamics and the privilege information is a functional of the stock price process: the maximum, the final value, etc. As pointed by Danilova (2010) in an equilibrium situation market prices are determined by the demand of market participants, so in such a situation the

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privilege information cannot be a functional of the stock price process because this implies the knowledge of future demand and it is unrealistic. Then the privilege information has to be something fixed exogenously like the announcement of the fundamental price, latter evolving independently of the demand and known by the insider. Another point is about the efficiency of the market in the sense that market prices converge to the fundamental one. In this paper we will show how the presence of an insider can be beneficial to the market in the sense that it makes the market be efficient. This problem has been addressed in different papers from the seminal papers of Kyle (1985) and Back (1992). See for instance Back and Pedersen (1998), Cho (2003), Lasserre (2004a, 2004b), Aase *et. al.* (2007), Campi and Cetin (2007), Danilova (2010) and Caldentey and Stacchetti (2010).

Here we analyze in a unified form how the presence of an insider makes the market be efficient when the insider knows the release time of the fundamental value of the asset. We also establish a general relationship between the problem of finding *rational* prices and the enlargement of filtrations problem. Moreover we consider the case when the time of the announcement is just a stopping time for all traders. In this latter case the market is not fully efficient, nevertheless there is an equilibrium where the sensitivity of prices is decreasing in time according with the probability that the announcement time is greater than the current time. In others words, prices are becoming more and more stable when the announcement is coming.

The paper is organize as follows. In the next section we describe the model that gives rise the stock prices. In the third section we discuss the optimal strategy . In section fourth and fith we discuss what happens when the release time is known or not respectively. In section six we review the results about the enlargement of filtration problem and provide new one. Finally we apply these results to find an equilibrium strategy.

2 The model

We consider a market with two assets, a risky asset S and a bank account with interest rate r equal to zero for the sake of simplicity. The period in which the participants trade is $[0, \infty)$. There is to be a convergence of *fundamental* and market price at (a possibly random) time τ. The fundamental price process is denoted by V, it will be the price of the asset after time τ, after that time we can consider that all traders have the same information so we cannot talk about insiders in the market.

In the paper of Caldentey and Stacchetti [7], authors consider that after time τ market price matches the fundamental price and that at the same time market makers use their pricing rule to set the market price, this situation leads the insider to a wild strategy. However we think that this situation is a bit artificial since if fundamental price is known from market makers then they do not need a pricing rule.

We shall denote the market price of the stock at time t by P_t. So, $P_t \neq V_t$

if $t < \tau$ and $P_t = V_t$ if $\tau \geq t$. The market is continuous in time and order driven. There are three kinds of traders. A (large) number of liquidity traders, who trade for liquidity or hedging reasons, an informed trader or insider, who has privilege information, and market makers, who set the price and clear the market.

We write $\mathbb{F}^P = \left(\mathcal{F}_t^P\right)_{t\geq 0}$ where $\mathcal{F}_t^P = \sigma(P_s, 0 \leq s \leq t)$. We denote by $\mathcal{S}_t = \sigma\{\tau \wedge s, 0 \leq s \leq t\}$ and $\mathbb{S} = (\mathcal{S}_t)_{t\geq 0}$. Let X be the demand process of the informed trader. At time t, her information is given by $\mathcal{H}_t$, where $\mathcal{H}_t = \sigma(P_s, \eta_s, \tau \wedge s, 0 \leq s \leq t)$, where η is a *signal process* or *firm value* in such a way that

$$V_t = \mathbb{E}(f(\eta_1)|\mathcal{H}_t),$$

where f is an increasing function. Then, V is an $\mathbb{H}$-martingale, where $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$. The informed trader tries to maximize her final wealth, that is, she is risk-neutral.

Let Z be the aggregate demand process of the noise traders, we assume that Z is an $\mathbb{H}$-martingale independent of η. We take for granted that all these processes are defined in the same, *complete*, probability space and the filtrations are *complete and right-continuous*.

We assume that market makers "clear" the market by fixing prices through a pricing rule, in terms of formulas

$$P_t = H(t, \xi_t), t \geq 0$$

with

$$\xi_t := \int_0^t \lambda(s)\mathrm{d}Y_s,$$

where λ is a positive function, $H \in C^{1,2}$ and $H(t, \cdot)$ is strictly increasing for every $t \geq 0$ and where $Y = X + Z$ is the total demand that market makers observe. We also assume that, due to the competition among market makers, the previous prices are *rational* or *competitive* in the sense that

$$P_t = \mathbb{E}(V_t|Y_s, \tau \wedge s, 0 \leq s \leq t), t \geq 0.$$

Note that $\mathcal{H}_t = \sigma(P_s, \eta_s, \tau \wedge s, 0 \leq s \leq t) = \sigma(Y_s, \eta_s, \tau \wedge s, 0 \leq s \leq t)$ and that (P_t) is then an $\mathbb{F}^Y \vee \mathbb{S}$ -martingale, where $\mathbb{F}^Y = \left(\mathcal{F}_t^Y\right)_{0\leq t\leq 1}$ and $\mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t)$.

3 The optimal strategy

Consider first a discrete model where trades are made at times $i = 1, 2, \ldots N$, and where N is random. If at time $i-1$, there is an order of buying $X_i - X_{i-1}$ shares, its cost will be $P_i(X_i - X_{i-1})$, so, there is a change in the bank account given by

$$-P_i(X_i - X_{i-1}).$$

Then the total change is

$$-\sum_{i=1}^{N} P_i(X_i - X_{i-1}),$$

and due to the convergence of market and fundamental prices, just after time N, there is the extra income: $X_N V_N$. So, the total wealth is

$$\begin{aligned} W_{N^+} &= -\sum_{i=1}^{N} P_i(X_i - X_{i-1}) + X_N V_N \\ &= -\sum_{i=1}^{N} P_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{N}(P_i - P_{i-1})(X_i - X_{i-1}) + X_N V_N \end{aligned}$$

Analogously, in the continuous model,

$$\begin{aligned} W_{\tau+} &= X_\tau V_\tau - \int_0^\tau P_{t-}\mathrm{d}X_t - [P,X]_\tau \\ &= \int_0^\tau X_{t-}\mathrm{d}V_t + \int_0^\tau V_{t-}\mathrm{d}X_t + [V,X]_\tau - \int_0^\tau P_{t-}\mathrm{d}X_t - [P,X]_\tau \\ &= \int_0^\tau (V_{t-} - P_{t-})\,\mathrm{d}X_t + \int_0^\tau X_{t-}\mathrm{d}V_t + [V,X]_\tau - [P,X]_\tau \end{aligned}$$

where (and throughout the whole article) P_{t-} denotes the left limit $\lim_{s\uparrow t} P_s$. We require that X is an $\mathbb{F}^{V,P}$-semimartingale, so that the integrals can be seen as Itô integrals, and to ensure the quadratic covariation $[P,X]$ is finite we also assume that P is an $\mathbb{F}^{V,P}$-semimartingale.

First we consider strategies of the form

$$\mathrm{d}X_t = \theta_t \mathrm{d}t,$$

where θ is an $\mathbb{H}$-adapted process, with $\int_0^\tau |\theta_t|\mathrm{d}t < \infty$, a.s..

Assumption 1.

$$\mathbb{E}\left(\int_0^\tau |\partial_2 H(t,\xi_t)\theta_s|\,\mathrm{d}s\right) < \infty,$$

Assumption 2.

$$\mathbb{E}\left(\int_0^\tau |X_s|^2\,\sigma_V^2(s)\mathrm{d}s\right) < \infty,$$

where $\sigma_V^2(s) := \frac{\mathrm{d}[V,V]_s}{\mathrm{d}s}$.

The wealth at time τ_+ is given by

$$W_{\tau+} := \int_0^\tau (V_t - P_t)\theta_t \mathrm{d}t + \int_0^\tau X_t \mathrm{d}V_t$$

where $P_t = H(t,\xi_t)$, with $\xi_t := \int_0^t \lambda(s)(\theta(s)\mathrm{d}s + \mathrm{d}Z_s)$, and where Z is an $\mathbb{H}$-martingale. In the following we will consider two kinds of stopping times: τ

bounded, or τ is conditionally independent of $(V_s, P_s, Z_s)_{t\leq s}$ given $\mathcal{H}_t$. In both cases we have that $E(\int_0^\tau X_t \mathrm{d}V_t) = 0$.

Denote

$$J(\theta) := E\left(W_{\tau+}\right) = E\left(\int_0^\tau (V_t - H(t,\xi_t))\theta_t \mathrm{d}t\right),$$

then, if θ is optimal, for all β, such that $\theta + \varepsilon\beta$ is admissible with $\varepsilon > 0$ small enough, we will have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(\theta + \varepsilon\beta) \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E\left(\int_0^\tau (V_t - H(t, \int_0^t \lambda(s)((\theta(s) + \varepsilon\beta_s)\mathrm{d}s + \mathrm{d}Z_s)))\left(\theta(t) + \varepsilon\beta_t\right)\mathrm{d}t\right) \\
&= E\left(\int_0^\tau (V_t - H(t,\xi_t))\beta_t \mathrm{d}t\right) + E\left(\int_0^\tau -\partial_2 H(t,\xi_t)\theta_t \left(\int_0^t \lambda(s)\beta(s)\mathrm{d}s\right)\mathrm{d}t\right) \\
&= E\left(\left(\left(\int_0^\tau (V_t - H(t,\xi_t)) - \lambda(t)\int_t^\tau \partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\right)\beta_t \mathrm{d}t\right),
\end{aligned}$$

then, since we can take $\beta_t = \mathbf{1}_{[u,u+h]}(t)\ \alpha_u$, with α_u $\mathcal{H}_u$-measurable and bounded. Then

$$E\left(\int_u^{u+h}\left(E(\mathbf{1}_{[0,\tau]}(t)\left(V_t - H(t,\xi_t)\right)\middle|\,\mathcal{H}_t) - \lambda(t)E\left(\int_t^\infty \mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\middle|\,\mathcal{H}_t\right)\right)\mathrm{d}t\middle|\,\mathcal{H}_u\right) = 0 \tag{1}$$

and this means that

$$M_t := \int_0^t \left(E(\mathbf{1}_{[0,\tau]}(u)V_u|\mathcal{H}_u) - E(\mathbf{1}_{[0,\tau]}(u)H(u,\xi_u)|\mathcal{H}_u) - \lambda(u)\int_u^\infty E\left(\mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s|\mathcal{H}_u\right)\mathrm{d}s\right)\mathrm{d}u$$

is an $\mathbb{H}$-martingale and this implies that, for all $t \geq 0$,

$$E(\mathbf{1}_{[0,\tau]}(t)V_t|\mathcal{H}_t) - E(\mathbf{1}_{[0,\tau]}(t)H(t,\xi_t)|\mathcal{H}_t) - \lambda(t)\int_t^\infty E\left(\mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s|\mathcal{H}_t\right)\mathrm{d}s = 0, a.e.$$

Since $\mathbb{S} \subseteq \mathbb{H}$, or equivalently τ is an $\mathbb{H}$-stopping time, and $\mathbb{F}^{P,V} \subseteq \mathbb{H}$, then we can write in the set $\tau > t$,

$$V_t - H(t,\xi_t) - \lambda(t)E\left(\int_t^\tau \partial_2 H(s,\xi_s)\theta_s \mathrm{d}s\middle|\,\mathcal{H}_t\right) = 0, \text{a.e..}$$

4 Case when τ is bounded and known by the insider

4.1 Market efficiency

If $\sigma(\tau) \in \mathcal{H}_0$, then

$$V_t - H(t,\xi_t) - \lambda(t)\int_t^\tau E\left(\partial_2 H(s,\xi_s)\theta_s|\mathcal{H}_t\right)\mathrm{d}s = 0, a.e. \quad 0 \leq t \leq \tau.$$

$E\left(\int_t^\tau |\partial_2 H(s,\xi_s)\theta_s|\,\mathrm{d}s|\mathcal{H}_t\right)$ is a supermartingale:

$$\begin{aligned} E\left(E\left(\int_t^\tau |\partial_2 H(s,\xi_s)\theta_s|\,\mathrm{d}s\middle|\mathcal{H}_t\right)\middle|\mathcal{H}_s\right) &= E\left(\int_t^\tau |\partial_2 H(s,\xi_s)\theta_s|\,\mathrm{d}s\middle|\mathcal{H}_s\right) \\ &\leq E\left(\int_s^\tau |\partial_2 H(s,\xi_s)\theta_s|\,\mathrm{d}s\middle|\mathcal{H}_s\right). \end{aligned}$$

and by Assumption 1

$$E\left(E\left(\int_t^\tau |\partial_2 H(s,\xi_s)\theta_s|\,\mathrm{d}s\middle|\mathcal{H}_s\right)\middle|\tau\right) < \infty, \text{ a.s..}$$

So, for fixed τ, $E\left(\int_t^\tau |\partial_2 H(s,\xi_s)\theta_s|\,\mathrm{d}s\middle|\mathcal{H}_s\right)$ converges in L^1 to zero, when $t \to \tau$, so it converges a.s to zero. Then we have that

$$V_\tau - H(\tau-,\xi_{\tau-}) = 0, a.s.$$

So, optimal strategies lead the market price to the fundamental one making the market be efficient.

Remark 1 *This is the case in Campi and Çetin (2007), where they take* $V_t = \mathbf{1}_{\{\bar{\tau}>1\}}$, $\bar{\tau}$ *is an* $\mathbb{H}$*-stopping time and* $\tau = \bar{\tau} \wedge 1$ *and* τ *is known by the insider, that is* $\tau \in \mathcal{H}_0$ *and it is bounded. Then they obtain*

$$\mathbf{1}_{\{\bar{\tau}>1\}} - H(\bar{\tau} \wedge 1, \xi_{\bar{\tau}\wedge 1}) = 0, a.s..$$

They also assume that $\bar{\tau}$ *is the first passage time of a standard Brownian motion that is independent of* Z.

Remark 2 *If we take* $V_t \equiv V$ *and* $\tau \equiv 1$ *then we are in Back's framework (1992). Market prices converge to* V *when* $t \to 1$.

4.2 Price pressure

For the sake of simplicity we are going to assume that Z is a *continuous* $\mathbb{H}$-martingale, even though similar calculations can be done in case of a jump part in the process Z. Nevertheless the presence of jumps and the independence of Z and V have been shown to be incompatible with the existence of rational prices in important cases (see Corcuera *et. al* (2010)). By using Itô's formula, we have

$$\begin{aligned} &E\left(\int_t^\tau \frac{1}{\lambda(s)}\partial_2 H(s,\xi_s)\mathrm{d}\xi_s\,|\mathcal{H}_t\right) \\ =\ &E\left(\frac{V_\tau}{\lambda(\tau)}\middle|\mathcal{H}_t\right) - \frac{H(t,\xi_t)}{\lambda(t)} \\ &-E\left(\int_t^\tau \left(-\frac{\lambda'(s)}{\lambda^2(s)}H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22}H(s,\xi_s)\lambda(s)\sigma_s^2\right)\mathrm{d}s\middle|\mathcal{H}_t\right), \end{aligned}$$

where $\sigma_s^2 := \frac{\mathrm{d}[Z,Z]}{\mathrm{d}s}$. So,

$$
\begin{aligned}
0 &= V_t - \lambda(t) E\left(\frac{V_\tau}{\lambda(\tau)} \middle| \mathcal{H}_t \right) \\
&\quad + \lambda(t) E\left(\int_t^\tau \left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22} H(s,\xi_s)\lambda(s)\sigma_s^2 \right) \mathrm{d}s \middle| \mathcal{H}_t \right)
\end{aligned}
$$

Now if $\tau \in \mathcal{H}_0$, then

$$
\begin{aligned}
& \frac{V_t}{\lambda(t)} - \frac{V_\tau}{\lambda(\tau)} + \int_t^\tau E\left(-\frac{\lambda'(s)}{\lambda^2(s)} H(s,\xi_s) + \frac{\partial_1 H(s,\xi_s)}{\lambda(s)} + \frac{1}{2}\partial_{22} H(s,\xi_s)\lambda(s)\sigma_s^2 \middle| \mathcal{H}_t \right) \mathrm{d}s \\
= \; & 0.
\end{aligned}
$$

By differentiating and identifying the predictive and martingale parts we have that

$$
\frac{\lambda'(t)}{\lambda^2(t)} V_t - \frac{\lambda'(t)}{\lambda^2(t)} H(t,\xi_t) + \frac{\partial_1 H(t,\xi_t)}{\lambda(t)} + \frac{1}{2}\partial_{22} H(t,\xi_t)\lambda(t)\sigma_t^2 = 0
$$

Then writing $\bar{H}(t,y) := \frac{H(t,y)}{\lambda(t)}$, $V_t(\omega) = v_t$ and $\xi_t(\omega) = y_t$ we have the following equation for $\bar{H}$:

$$
\frac{\lambda'(t)}{\lambda^2(t)} v_t + \partial_1 \bar{H}(t,y_t) + \frac{1}{2}\partial_{22}\bar{H}(t,y_t)\lambda^2(t)\sigma_t^2 = 0.
$$

Proposition 3 *Assume that the law of ξ and M, where $M_\cdot := \int_0^\cdot \lambda(u)\mathrm{d}Z_u$, are equivalent. Then the price pressure is constant and*

$$
\partial_1 \bar{H}(t,y_t) + \frac{1}{2}\partial_{22}\bar{H}(t,y_t)\lambda^2(t)\sigma_t^2 = 0.
$$

Proof. Since Z and V are independent, this implies that

$$
\frac{\lambda'(t)}{\lambda^2(t)} v_t + \partial_1 \bar{H}(t,y_t) + \frac{1}{2}\partial_{22}\bar{H}(t,y_t)\lambda^2(t)\sigma_t^2 = 0.
$$

is satisfied for certain trajectories of (V,M), that only depend on the value of V, but corresponding to different values of ω. Moreover the pricing rule has to be independent of the values of V, so the previous equation should be satisfied for any value of (V,M). Then, by Itô's formula, we have

$$
\begin{aligned}
\bar{H}(\tau, M_\tau) &= \bar{H}(t, M_t) + \int_t^\tau \partial_2 \bar{H}(s,M_s)\lambda(s)\mathrm{d}Z_s + \int_t^\tau \partial_1 \bar{H}(s,M_s)\mathrm{d}s \\
&\quad + \int_t^\tau \frac{1}{2}\partial_{22}\bar{H}(s,M_s)\lambda^2(s)\sigma_s^2 \mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left(\bar{H}(\tau, M_\tau)|M_t = y_t, V_t = v_t\right) \\
= \; & \bar{H}(t, y_t) + E\left(\int_t^\tau \partial_1 \bar{H}(s, M_s)\lambda(s)\mathrm{d}s \,\middle|\, M_t = y_t, V_t = v_t\right) \\
& + E\left(\int_t^\tau \frac{1}{2}\partial_{22}\bar{H}(s, M_s)\lambda^2(s)\sigma_s^2\mathrm{d}s \,\middle|\, M_t = y_t, V_t = v_t\right) \\
= \; & \bar{H}(t, y_t) - E\left(\int_t^\tau \frac{\lambda'(s)}{\lambda^2(s)}V_s\mathrm{d}s \,\middle|\, M_t = y_t, V_t = v_t\right) = \bar{H}(t, y_t) - v_t\left(\frac{1}{\lambda(t)} - \frac{1}{\lambda(\tau)}\right),
\end{aligned}
$$

so

$$
H(t, y_t) = \lambda(t)E\left(\bar{H}(\tau, M_\tau)|M_t = y_t\right) - v_t\left(1 - \frac{\lambda(t)}{\lambda(\tau)}\right),
$$

but the price function cannot depend on the values of V, so $\lambda(t) = \lambda_0$ and the price pressure is constant. ■

Remark 4 *Note that we finally have that*

$$
H(t, y) = E\left(H(\tau, \lambda_0 Z_\tau)|\lambda_0 Z_t = y\right).
$$

4.3 More general strategies and a verification theorem

Define

$$
J(f(v), t, y) := \int_y^{H^{-1}(\tau, \lambda_0 \cdot)(v)} \frac{f(v) - H(t, \lambda_0 x)}{\lambda_0}\mathrm{d}x,
$$

then, by Itô's formula and assuming again, for the sake of simplicity that Z is continuous.

$$
\begin{aligned}
J(V_\tau, \tau, \xi_\tau) \; = \; & J(0,0,0) + \int_0^\tau \partial_1 J\mathrm{d}s + \int_0^\tau \partial_2 J\mathrm{d}\xi_s + \int_0^\tau \partial_0 J\mathrm{d}V_s \\
& + \frac{1}{2}\int_0^\tau \partial_{22}J\lambda_0^2\sigma_s^2\mathrm{d}s + \frac{1}{2}\int_0^\tau \partial_{00}J\sigma_V^2\mathrm{d}s.
\end{aligned}
$$

Where ∂_i, $i = 0, 1, 2$ indicates the partial derivative with respect to the first second or third argument of J respectively. Then if the strategy is optimal $J(V_\tau, \tau, \xi_\tau) = 0$, since any optimal strategy satisfies $H^{-1}(\tau, \lambda_0 \cdot)(f^{-1}(V_\tau)) = \xi_\tau$, so

$$
\begin{aligned}
E(J(0,0,0)) \; = \; & E\left(\int_0^\tau \frac{V_s - H(s, \xi_s)}{\lambda_0}\theta_s\mathrm{d}s\right) - E\left(\int_0^\tau \partial_1 J\mathrm{d}s\right) \\
& - E\left(\frac{1}{2}\int_0^\tau \partial_{22}J\lambda_0^2\sigma_s^2\mathrm{d}s - \frac{1}{2}\int_0^\tau \partial_{00}J\sigma_V^2(s)\mathrm{d}s\right),
\end{aligned}
$$

where $\sigma_V^2(s) := \frac{\mathrm{d}[V,V]_s}{\mathrm{d}s}$. Then if

$$
E\left(\int_0^\tau \partial_1 J\mathrm{d}s + \frac{1}{2}\int_0^\tau \partial_{22}J\lambda_0^2\sigma_s^2\mathrm{d}s\right) = 0,
$$

we will have that $E\left(J(0,0,0)+\frac{1}{2}\int_0^\tau \partial_{00}J\sigma_V^2 \mathrm{d}s\right) = E\left(\int_0^\tau \frac{V_s - H(s,\lambda_0\xi_s)}{\lambda_0}\theta_s \mathrm{d}s\right)$.

But

$$\partial_2 J = -\frac{f(v) - H(t,\lambda_0 y)}{\lambda_0} \tag{2}$$

so we deduce from the equation for $\bar{H}$ that

$$\partial_1 J + \frac{1}{2}\partial_{22}J\lambda_0^2\sigma_t^2 = C(t) \tag{3}$$

but since $J(V_\tau, t, \xi_\tau) = 0$ for all t, we obtain that $C(t) \equiv 0$. Then we have the following theorem.

Theorem 5 $E\left(J(0,0,0)+\frac{1}{2}\int_0^\tau \partial_{00}J\sigma_V^2(s)\mathrm{d}s\right)$ *is the maximum expected profit and it can be reached by a strategy X if and only if it satisfies the following properties:*

> (i) *X has continuous paths,*
>
> (ii) *the Doob's decomposition of X does not have martingale part,*
>
> (iii) *the strategy drives the price to V_τ.that is $\lim_{t\to\tau} P_t = V_\tau$.*

Proof. By using Itô's formula, we have

$$\begin{aligned} J(V_\tau,\tau,\xi_\tau) &= J(0,0,0) + \int_0^\tau \partial_1 J(V_t,t,\xi_{t-})\mathrm{d}t + \int_0^\tau \partial_2 J(V_t,t,\xi_{t-})\mathrm{d}\xi_t + \\ &+ \frac{1}{2}\int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\mathrm{d}[\xi^c,\xi^c]_t + \int_0^\tau \partial_{02}J(V_t,t,\xi_{t-})\mathrm{d}[\xi^c,V]_t \\ &+ \frac{1}{2}\int_0^\tau \partial_{00}J(V_t,t,\xi_{t-})\sigma_V^2\mathrm{d}t \\ &+ \sum_{0\le t\le 1}\left(\Delta J(V_t,t,\xi_t) - \frac{\partial J}{\partial y}(V_t,t,\xi_{t-})\Delta\xi_t\right). \end{aligned}$$

By construction, $\xi_0 = 0$, and we have $\mathrm{d}\xi_t = \lambda_0 \mathrm{d}Y_t$

$$\mathrm{d}[\xi^c,\xi^c]_t = \lambda_0^2\mathrm{d}[X^c,X^c]_t + 2\lambda_0^2\mathrm{d}[X^c,Z]_t + \lambda_0^2\sigma_t^2\mathrm{d}t,$$

and

$$\partial_{02}J(V_t,t,\xi_{t-})\mathrm{d}[\xi^c,V]_t = -\mathrm{d}[X,V]_t$$

so using (2) and (3), we get

$$\begin{aligned} J(V_\tau,\tau,\xi_\tau) &= J(0,0,0) + \int_0^\tau \partial_0 J(V_t,t,\xi_{t-})\mathrm{d}V_t + \int_0^\tau (P_{t-} - V_t)(\mathrm{d}X_t + \mathrm{d}Z_t) \\ &+ \frac{1}{2}\int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\lambda_0^2\mathrm{d}[X^c,X^c]_t - [X,V]_\tau + \frac{1}{2}\int_0^\tau \partial_{00}J(V_t,t,\xi_{t-})\sigma_V^2\mathrm{d}t \\ &+ \int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\lambda_t^2\mathrm{d}[X^c,Z^c] + \sum_{0\le t\le 1}\left(\Delta J(t,\xi_t) - \frac{\partial J}{\partial y}(t,\xi_{t-})\Delta\xi_t\right) \end{aligned}$$

Subtracting $[P,X]_\tau$ from both sides and substituting, we obtain

$$\begin{aligned}
&\int_0^\tau (V_t - P_{t-})\mathrm{d}X_t - [P,X]_\tau + [X,V]_\tau - \left(J(0,0,0) + \frac{1}{2}\int_0^\tau \partial_{00}J(V_t,t,\xi_{t-})\sigma_V^2 \mathrm{d}t\right)\\
= \quad & -J(V_\tau,\tau,\xi_\tau) + \int_0^\tau \partial_0 J(V_t,t,\xi_{t-})\mathrm{d}V_t + \int_0^\tau (P_{t-} - V_t)\mathrm{d}Z_t\\
&+\frac{1}{2}\int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\lambda_0^2 \mathrm{d}[X^c,X^c]_t + \int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\lambda_t^2 \mathrm{d}[X^c,Z^c]\\
&+\sum_{0\le t\le 1}\left(\Delta J(t,\xi_t) - \frac{\partial J}{\partial y}\Delta\xi_t\right) - [P,X]_\tau.
\end{aligned}$$

We will show that the expectation of the left hand side is non-positive by evaluating the right hand side. Note that

$$[P,X]_\tau \equiv [P^c,X^c]_\tau + \sum_{0\le t\le \tau}\Delta P_t \Delta X_t.$$

Itô's formula for H shows that the continuous local martingale part of P is $\int \frac{\partial H}{\partial y}(t,\xi_{t-})\mathrm{d}\xi_t^c$, so by using 2, we obtain

$$\begin{aligned}
[P^c,X^c]_\tau &= \left[\int \frac{\partial H}{\partial y}(t,\xi_{t-})\mathrm{d}\xi_t^c, X^c\right]_\tau = \int_0^\tau \frac{\partial H}{\partial y}(t,\xi_{t-})\mathrm{d}\left[\xi^c,X^c\right]_t\\
&= \int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\lambda_0^2 \mathrm{d}\left[X^c,X^c\right]_t + \int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\lambda_0^2 \mathrm{d}\left[X^c,Z\right]_t,
\end{aligned}$$

and also

$$\begin{aligned}
\lambda_0\partial_2 J(V_t,t,\xi_{t-})\Delta X_t + \Delta P_t\Delta X_t &= (P_{t-} - V_t)\Delta X_t + \Delta P_t \Delta X_t\\
&= (P_t - V_t)\Delta X_t = \lambda_0\partial_2 J(V_t,t,\xi_t)\Delta X_t.
\end{aligned}$$

Substituting them for $[P,X]_t$ in the right hand side of equation, it simplifies to

$$\begin{aligned}
&-J(V_\tau,\tau,\xi_\tau) + \int_0^\tau \partial_0 J(V_t,t,\xi_{t-})\mathrm{d}V_t + \int_0^\tau (P_{t-} - V_t)\mathrm{d}Z_t - \frac{1}{2}\int_0^\tau \partial_{22}J(V_t,t,\xi_{t-})\lambda_0^2 \mathrm{d}[X^c,X^c]_t\\
&+\sum_{0\le t\le 1}\big(J(V_t,t,\xi_t) - J(V_t,t,\xi_{t-}) - \lambda_0\partial_2 J(V_t,t,\xi_t)\Delta X_t\big)
\end{aligned}$$

1. We know that $\lambda_0\partial_{22}J(V_\tau,\tau,\xi_\tau) = \partial_2 H(\tau,\xi_\tau) > 0$ and that $\lambda_0\partial_2 J(V_\tau,\tau,\xi_\tau) = -V_\tau + H(\tau,\lambda_0\xi_\tau)$ so we have a maximum value of $-J(V_\tau,\tau,\xi_\tau)$ if and only if $-V_\tau + H(\tau,\lambda_0\xi_\tau) = 0$ and in that case $J(V_\tau,\tau,\xi_\tau) = 0$.

2. The processes $\int_0^\cdot \partial_0 J(V_t,t,\xi_{t-})\mathrm{d}V_t$ and $\int_0^\cdot (P_{t-} - V_t)\mathrm{d}Z_t$ are $\mathbb{F}^{P,V}$-martingale, so they vanish when we take expectations.

3. By (2) and H being increasing monotone, we have that $\partial_{22}J > 0$, and the measure $\mathrm{d}[X^c,X^c] \ge 0$,

4. $\partial_{22} J > 0$ (convexity) implies that

$$J(v,t,x+h) - J(v,t,x) - \frac{\partial J}{\partial y}(v,t,x+h)h \leq 0.$$

So,

$$\sum_{0\leq t\leq 1} \left(J(V_t, t, \xi_{t-} + \lambda_t \Delta Y_t) - J(t, \xi_{t-}) - \frac{\partial J}{\partial y}(V_t, t, \xi_t)\lambda_t \Delta X_t \right) \leq 0,$$

and has its maximum if and only if $\Delta Y_t = 0$, that is if and only if X is continuous.

■

5 Case when τ is unknown

In the general case

$$V_t - H(t,\xi_t) - \lambda(t) \int_t^\infty E\left(\mathbf{1}_{[0,\tau]}(s)\partial_2 H(s,\xi_s)\theta_s | \mathcal{H}_t\right) \mathrm{d}s = 0, a.e. \quad t \geq 0.$$

Then if we assume that $\{\tau \wedge s, s \geq t\}$ is conditionally independent of $(V_t, P_t, Z_t)_{0\leq t\leq s}$ given $\mathcal{H}_t$, we will have (provided that $P(\tau > t) > 0$).

$$\begin{aligned} & V_t - H(t,\xi_t) - \lambda(t) \int_t^\infty P\left(\tau > s | \mathcal{H}_t\right) E(\partial_2 H(s,\xi_s)\theta_s | \mathcal{H}_t) \,\mathrm{d}s \\ = \quad & V_t - H(t,\xi_t) - \frac{\lambda(t)}{P(\tau > t)} \int_t^\infty P\left(\tau > s\right) E(\partial_2 H(s,\xi_s)\theta_s | \mathcal{H}_t) \,\mathrm{d}s = 0, a.e. \quad t \geq 0. \end{aligned}$$

Then

$$\frac{(V_t - H(t,\xi_t))\, P(\tau > t)}{\lambda(t)} - \int_t^\infty P\left(\tau > s\right) E(\partial_2 H(s,\xi_s)\theta_s | \mathcal{H}_t) \,\mathrm{d}s = 0.$$

By Assumption 1, $\int_t^\infty P\left(\tau > s\right) E(|\partial_2 H(s,\xi_s)\theta_s|\, | \mathcal{H}_t) \,\mathrm{d}s$ converges in L^1 to zero when t goes to infinity:

$$\lim_{t\to\infty} E\left(\int_t^\infty P(\tau > s) \left|\partial_2 H(s,\xi_s)\theta_s\right| \mathrm{d}s \right) = 0,$$

and since it is a supermartingale it converges a.s. to zero. Then we have that

$$\lim_{t\to\infty} \frac{(V_t - H(t,\xi_t))\, P(\tau > t)}{\lambda(t)} = 0. \tag{4}$$

Then proceeding in a similar way than before and assuming again that Z is continuous and that $\sigma_s^2 := \frac{\mathrm{d}[Z,Z]}{\mathrm{d}s}$,

$$\begin{aligned}
& E\left(\int_t^\infty P(\tau > s)\, \partial_2 H(s,\xi_s)\theta_s \mathrm{d}s \middle| \mathcal{H}_t\right) \\
= & \lim_{T\to\infty} E\left(\frac{H(T,\xi_T)P(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t\right) - \frac{H(t,\xi_t)P(\tau > t)}{\lambda(t)} \\
& - E\left(\int_t^\infty (\partial_s \left(\frac{P(\tau > s)}{\lambda(s)}\right) H(s,\xi_s) + \frac{P(\tau > s)}{\lambda(s)}\partial_1 H(s,\xi_s) \right. \\
& \left. + \frac{1}{2}\partial_{22} H(s,\xi_s) P(\tau > s)\, \lambda(s)\sigma_s^2)\mathrm{d}s \middle| \mathcal{H}_t\right).
\end{aligned}$$

then by (4)

$$\begin{aligned}
\lim_{T\to\infty} E\left(\frac{H(T,\xi_T)P(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t\right) &= \lim_{T\to\infty} E\left(\frac{V_T P(\tau > T)}{\lambda(T)} \middle| \mathcal{H}_t\right) \\
&= V_t \lim_{T\to\infty} \frac{P(\tau > T)}{\lambda(T)} := V_t C,
\end{aligned}$$

$$\begin{aligned}
0 \;=\; & V_t\left(1 - \frac{C\lambda(t)}{P(\tau > t)}\right) - \frac{\lambda(t)}{P(\tau > t)} E\left(\int_t^\infty \left(\partial_s \left(\frac{P(\tau > s)}{\lambda(s)}\right) H(s,\xi_s) \right.\right. \\
& \left.\left. + \frac{P(\tau > s)}{\lambda(s)}\partial_1 H(s,\xi_s) + \frac{1}{2}\partial_{22} H(s,\xi_s) P(\tau > s)\, \lambda(s)\sigma_s^2\right) \mathrm{d}s \middle| \mathcal{H}_t\right),
\end{aligned}$$

and we will have a solution if and only if $C = \frac{P(\tau > t)}{\lambda(t)}$, then the price pressure is not constant. Here the situation is analogous to that in Cho (2003) where he considers a risk-averse insider, he concludes that a risk-averse would do most of his trading early to avoid the risk that the prices gets closer to the asset value, unless the trading conditions become more favorable over time, this is exactly what happens when the insider does not know the release time. He would try to trade early to use his information before the anouncement unless the price pressure decreases over time making more favorable trading later and this is what happens in equilibrium.

Note also that we have that

$$\partial_1 H(s,\xi_s) + \frac{1}{2}\partial_{22} H(s,\xi_s)\lambda^2(s)\sigma_s^2 = 0.$$

By defining (conjecture, check)

$$J(v,t,y) := \lim_{\tau\to\infty} \int_y^{H^{-1}(\tau,\lambda_0\cdot)(v)} \frac{f(v) - H(t,\lambda_0 x)}{\lambda_0} \mathrm{d}x,$$

we would obtain a similar theorem to Theorem 5.

Remark 6 *In Caldentey and Stacchetti (2010) authors assume that V is an arithmetic Brownian motion and τ follows an exponential distribution with scale parameter μ, independent of $(V_t, P_t, Z_t)_{0 \le t \le \tau}$. Then, on the set $t < \tau$*

$$V_t - H(t, \xi_t) - \lambda(t) \int_t^\infty e^{-\mu(s-t)} E\left(\partial_2 H(s, \xi_s)\theta_s | \mathcal{H}_t\right) \mathrm{d}s = 0, a.e,$$

then, proceeding in a similar way we have a solution if and only if $\lambda(t) = \lambda_0 e^{-\mu t}$.

6 Enlargement of filtrations

We have seen that the total demand of assets in equilibrium is given by

$$Y_t = Z_t + \int_0^t \theta(\eta_t; Y_u, 0 \le u \le s)\mathrm{d}s, \quad 0 \le t \le T \tag{5}$$

where Z is a martingale independent of , so Z is an $\mathbb{F}^{Z,\eta}$ martingale and, since $\mathbb{F}^{Y,\eta} \subseteq \mathbb{F}^{Z,\eta}$ and Z is adapted to $\mathbb{F}^{Y,\eta}$, it is also an $\mathbb{F}^{Y,\eta}$-martingale. On the other hand Y is supposed to be, in equilibrium, an $\mathbb{F}^{Y}$-martingale. Consequently (5) becomes the Doob-Meyer decomposition of the $\mathbb{F}^{Y}$-martingale Y when we enlarge the filtration $\mathbb{F}^{Y}$ with the process η. We are then into a problem of *enlargement of filtrations.*

6.1 Initial enlargement of filtrations

Consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a random variable L $\mathcal{F}$-measurable with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mathcal{G}_t := \cap_{s>t} (\mathcal{F}_t \vee \sigma(L))$ and $\mathbb{G} = (\mathcal{G}_t)$.

Condition A. For all t, there exists a σ-finite measure η_t in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Q_t(\omega, \cdot) \ll \eta_t$ where $Q_t(\omega, \mathrm{d}x)$ is a regular version of $L|\mathcal{F}_t$.

Proposition 7 ***Condition A*** *is equivalent to $Q_t(\omega, \mathrm{d}x) \ll \eta(\mathrm{d}x)$ where η is the law of L.*

Proof. By **Condition A** we have that $Q_t(\omega, \mathrm{d}x) = q_t^x(\omega)\eta_t(\mathrm{d}x)$, where $q_t^x(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$ measurable then we can write $Q_t(\omega, \mathrm{d}x) = \hat{q}_t^x(\omega)\eta(\mathrm{d}x)$ with $\hat{q}_t^x(\omega) = \frac{q_t^x(\omega)}{E(q_t^x(\omega))}$. ■

Proposition 8 *Under **Condition A** there exists $q_t^x(\omega)$ $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable such that $Q_t(\omega, \mathrm{d}x) = q_t^x(\omega)\eta(\mathrm{d}x)$ and, for fixed x, q_t^x is an $\mathbb{F}$-martingale.*

Proof. See Jacod (1985) Lemma 1.8. ■

Theorem 9 *Let M be a continuous local $\mathbb{F}$-martingale and $k_t^x(\omega)$ such that*

$$\langle q^x, M \rangle_t = \int_0^t k_s^x q_{s-}^x \mathrm{d}\langle M, M \rangle_s$$

then

$$M - \int_0^{\cdot} k_s^L \mathrm{d}\langle M, M\rangle_s,$$

is a $\mathbb{G}$-martingale.

Proposition 10 ***Proof.*** *Except for a localization procedure (see details in Jacod (1985) Theorem 2.1) the proof is the following: let $Z \in \mathcal{F}_s$ and g be Borelian and bounded, then*

$$\begin{aligned}
E(Zg(L)(M_t - M_s)) &= E(E(Zg(L)(M_t - M_s)|\mathcal{F}_t)) \\
&= E(Z(M_t - M_s)E(g(L)|\mathcal{F}_t)) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(M_t - M_s)q_t^x) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(M_t q_t^x - M_s q_s^x)) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(\langle M, q^x\rangle_t - \langle M, q^x\rangle_s)) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(\langle M, q^x\rangle_t - \langle M, q^x\rangle_s)) \\
&= \int_{\mathbb{R}} g(x)\eta(\mathrm{d}x)E(Z(\int_s^t k_u^x q_{u-}^x \mathrm{d}\langle M, M\rangle_u)) \\
&= E(Zg(L)(\int_s^t k_u^x q_{u-}^x \mathrm{d}\langle M, M\rangle_u))
\end{aligned}$$

■

Example 11 *Take $M_t = B_t$ where B is a standard Brownian motion, take $L = B_1$ then*

$$q_t^x(\omega) \sim \frac{1}{(1-t)^{1/2}} \exp\left\{-\frac{1}{2(1-t)}(B_t(\omega) - x)^2 + \frac{x^2}{2}\right\},$$

by Ito's formula

$$\mathrm{d}_t q_t^x = q_t^x \frac{x - B_t}{1-t} \mathrm{d}B_t,$$

then $k_s^x = \frac{x-B_t}{1-t}$ and

$$B - \int_0^{\cdot} \frac{B_1 - B_s}{1-s} \mathrm{d}s$$

is an $\mathbb{F}^B \vee \sigma(B_1)$ martingale. Note that, by the Lévy theorem, $B - \int_0^{\cdot} \frac{B_1 - B_s}{1-s}\mathrm{d}s$ is a (standard) $\mathbb{G} := \mathbb{F}^B \vee \sigma(B_1)$ -Brownian motion and since B_1 is $\mathcal{G}_0$-measurable, it is independent of W.

Example 12 *Note that if the filtration $\mathbb{F}$ is that generated by a Brownian motion, B, then for any $\mathbb{F}$-martingale*

$$\mathrm{d}M_t = \sigma_t \mathrm{d}B_t$$

and

$$\mathrm{d}\langle M, M\rangle_t = \sigma_t^2 \mathrm{d}t.$$

Also, assuming that

$$q_t^x(\omega) = h_t^x(B_t)$$

and $h \in C^{1,2}$ we will have that

$$\mathrm{d}_t q_t^x = \partial h_t^x(B_t)\mathrm{d}B_t,$$

and

$$k_t^x = \frac{\partial \log h_t^x(B_t)}{\sigma_t}.$$

Example 13 *In fact the previous example is a particular case of the following one: let Y be the Brownian semimartingale*

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\mathrm{d}B_s + \int_0^t b(Y_s)\mathrm{d}s,$$

and assume that

$$Y_1|\mathcal{F}_t \sim \pi(1-t, Y_t, x)\mathrm{d}x.$$

with π smooth. We know that $(\pi(1-t, Y_t, x))_t$ is an $\mathbb{F}$-martingale, then

$$\mathrm{d}\pi(1-t, Y_t, x) = \frac{\partial \pi}{\partial y}(1-t, Y_t, x)\sigma(Y_s)\mathrm{d}B_s$$

and by the Jacod theorem

$$\int_0^t \sigma(Y_s)\mathrm{d}B_s - \int_0^t \frac{\partial \log \pi}{\partial y}(1-s, Y_s, Y_1)\sigma^2(Y_s)\mathrm{d}s$$

is an $\mathbb{F} \vee \sigma(Y_1)$-martingale, and we can write

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\mathrm{d}\tilde{B}_s + \int_0^t b(Y_s)\mathrm{d}s + \int_0^t \frac{\partial \log \pi}{\partial y}(1-s, Y_s, Y_1)\sigma^2(Y_s)\mathrm{d}s,$$

where $\tilde{B}$ is an $\mathbb{F} \vee \sigma(Y_1)$-Brownian motion.

Example 14 *Let B a Brownian motion and $\tau = \inf\{t > 0, B_t = -1\}$ it is well known that*

$$P[\tau \leq s|\mathcal{F}_t] = 2\Phi(-\frac{1+B_t}{\sqrt{s-t}})\mathbf{1}_{\{\tau\wedge s > t\}} + \mathbf{1}_{\{s < \tau \wedge t\}},$$

where Φ is the c.d.f. of a standard normal distribution. Then in $t < s \wedge \tau$ we have, by Itô´s formula,

$$P[\tau \leq s|\mathcal{F}_t] = 2\Phi(-\frac{1}{\sqrt{s}}) + \sqrt{\frac{2}{\pi}}\int_0^t \frac{1}{\sqrt{s-u}} e^{-\frac{(1+B_u)^2}{2(s-u)}}\mathrm{d}B_u,$$

so

$$\mathrm{d}\langle P[\tau \leq s|\mathcal{F}_\cdot], B\rangle_t = -\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{s-t}}e^{-\frac{(1+B_t)^2}{2(s-t)}}\mathrm{d}t,$$

and

$$\begin{aligned}
&\alpha_t^s Q_t(\cdot, \mathrm{d}s) \\
=& \frac{\partial}{\partial s}\left(\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{s-t}}e^{-\frac{(1+B_t)^2}{2(s-t)}}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{(s-t)^3}} - \frac{(1+B_t)^2}{\sqrt{(s-t)^5}}\right)e^{-\frac{(1+B_t)^2}{2(s-t)}},
\end{aligned}$$

finally

$$Q_t(\cdot, \mathrm{d}s) = \frac{\partial}{\partial s}P[\tau > s|\mathcal{F}_t] = \frac{e^{-\frac{(1+B_t)^2}{2(s-t)}}}{\sqrt{2\pi}\sqrt{(s-t)^3}}(1+B_t),$$

and

$$\alpha_t^s = \frac{\frac{\partial}{\partial s}\left(\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{s-t}}e^{-\frac{(1+B_t)^2}{2(s-t)}}\right)}{\frac{\partial}{\partial s}P[\tau > s|\mathcal{F}_t]} = \frac{1}{1+B_t} - \frac{1+B_t}{s-t}.$$

Consequently

$$B_t - \int_0^{t\wedge\tau}\left(\frac{1}{1+B_s} - \frac{1+B_s}{\tau - s}\right)\mathrm{d}s, \quad t \geq 0,$$

is a $\mathbb{G}$-martingale.

6.2 Progressive enlargement of filtrations

In the progressive enlargement of filtrations $\mathbb{G} = (\mathcal{G}_t)$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathbb{H} = (\mathcal{H}_t)$ is another filtration. The case where $\mathcal{H}_t = \sigma(\mathbf{1}_{\{\tau \leq t\}})$ with τ a random time has been extensely studied, see for instance Jeulin (1980), Jeulin and Yor (1985) or Mansuy and Yor (2006), among others, however few studies has been developed in the general setting. One exception is when $\mathcal{H}_t = \sigma(J_t)$, for $J_t = \inf_{s \geq t} X_s$ and when X is a 3-dimensional Bessel process, see section 1.2.2 in Mansuy and Yor (2006), but this case can be reduced in fact to a case with random times taking into account that

$$\{J_t < a\} = \{t < \Lambda_a\},$$

where $\Lambda_a = \sup\{t, X_t = a\}$. Another one is the case when $\mathcal{H}_t = \sigma(L_t)$, for $L_t = G(X, Y_t)$, with X and $\mathcal{F}_T$-measurable random variable Y a process independent of $\mathcal{F}_T$ and G a Borelian function, see Corcuera *et al. (2004)*. A particular case is given by the proposition:

Proposition 15 *Assume that B is a Brownian motion and that $\mathbb{F} = \mathbb{F}^B$. Let W be another Brownian motion independent of B,consider the process $V_t := B_1 + \int_t^1 \sigma_s \mathrm{d}W_s$, with $\int_t^1 \sigma_s^2 \mathrm{d}s < \infty$, for all $0 \leq t \leq 1$.Then provided that*

$$\int_0^t \frac{1}{1 + \int_s^1 \sigma_u^2 \mathrm{d}u - s}\mathrm{d}s < \infty,$$

we have that the Doob-Meyer decomposition of B in $\mathbb{F}^{B,V}$ is given by

$$B_t = \tilde{W}_t + \int_0^t \frac{V_s - B_s}{1 + \int_s^1 \sigma_u^2 \mathrm{d}u - s} \mathrm{d}s, 0 \leq t < 1$$

where $\tilde{W}$ is a Brownian motion but correlated with V.

Proof. $\tilde{W}$ is a centered Gaussian processes and for $0 \leq s \leq t < 1$

$$\begin{aligned}
E(\tilde{W}_t \tilde{W}_s) &= E\left(\left(B_t - \int_0^t \frac{V_u - B_u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u\right)\left(B_s - \int_0^s \frac{V_u - B_u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u\right)\right) \\
&= s - E\left(B_t \int_0^s \frac{V_u - B_u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u\right) - E\left(B_s \int_0^t \frac{V_u - B_u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u\right) \\
&\quad + E\left(\int_0^t \frac{V_u - B_u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u \int_0^s \frac{V_u - B_u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u\right) \\
&= s - \int_0^s \frac{t - u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u - \int_0^s \frac{s - u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u \\
&\quad + 2E\left(\int_0^s \left(\int_0^r \frac{(V_r - B_r)(V_u - B_u)}{\left(1 + \int_r^1 \sigma_v^2 \mathrm{d}v - r\right)\left(1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u\right)} \mathrm{d}u\right) \mathrm{d}r\right) \\
&\quad + E\left(\int_s^t \left(\int_0^s \frac{(V_r - B_r)(V_u - B_u)}{\left(1 + \int_r^1 \sigma_v^2 \mathrm{d}v - r\right)\left(1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u\right)} \mathrm{d}u\right) \mathrm{d}r\right) \\
&= s - \int_0^s \frac{s + t - 2u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u + 2\int_0^s \frac{s - u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u + \int_0^s \frac{t - s}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u \\
&= s.
\end{aligned}$$

On the other hand, for $t \geq s$

$$E(\tilde{W}_t V_s) = s - \int_0^s \frac{1 + \int_s^1 \sigma_v^2 \mathrm{d}v - u}{1 + \int_u^1 \sigma_v^2 \mathrm{d}v - u} \mathrm{d}u > 0,$$

provided that σ_v is not identically null (a.e.). ■

Remark 16 *It is important to note that contrarily to the case of initial enlargement, the innovation process $\tilde{W}$ is not necesarily independent of the additional information. Then this fact makes the application of enlargement of filtrations in our framework more involved. In other words, in most of the models, we assume that the privilege information (V) is independent of the demand process of liquidity traders $(\tilde{W})$ so the previous Proposition cannot be used with our models. Instead we have to look for processes such that their Doob-Meyer decomposition is of the form*

$$X_t = \tilde{W}_t + \int_0^t \theta(V_t; X_u, 0 \leq u \leq s) \mathrm{d}s, \quad 0 \leq t \leq T,$$

where $\tilde{W}$ and V are independent.

Now consider the case when $\mathcal{H}_t = \sigma(V_t)$ for

$$V_t = V_0 + \int_0^t \sigma_s \mathrm{d}W_s^1,$$

where σ_s is a deterministic funtion, V_0 is a zero mean normal r.v., $\left(W^1, W^2\right)$ is a 2-dimensional Brownian motion independent of V_0. We have the following proposition:

Proposition 17 *Assume that $Var(V_1) = 1$ and that*

$$\int_0^t \frac{\mathrm{d}s}{Var(V_s) - s} < \infty \text{ for all } 0 \le t < 1,$$

then

$$B_t = W_t^2 + \int_0^t \frac{V_s - B_s}{Var(V_s) - s} \mathrm{d}s, 0 \le t \le 1$$

is a Brownian motion with $B_1 = V_1$.

Proof. Denote $v_r := Var(V_r)$

$$B_t = \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right) \mathrm{d}W_u^2 + \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right) \frac{V_u}{v_u - u}\mathrm{d}u,$$

so B is a centered Gaussian process, and for $s \le t < 1$,

$$\begin{aligned}
E\left(B_t B_s\right) &= \exp\left(-\int_s^t \frac{1}{v_r - r}\mathrm{d}r\right) \\
&\quad + E\left(\int_0^t \int_0^s \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right) \exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right) \frac{V_u V_v}{\left(v_u - u\right)\left(v_v - v\right)}\mathrm{d}u\mathrm{d}v\right) \\
&= \exp\left(-\int_s^t \frac{1}{v_r - r}\mathrm{d}r\right) \int_0^s \exp\left(-2\int_u^s \frac{1}{v_r - r}\mathrm{d}r\right) \mathrm{d}u \\
&\quad + \int_s^t \int_0^s \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right) \exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right) \frac{v_v}{\left(v_u - u\right)\left(v_v - v\right)}\mathrm{d}u\mathrm{d}v \\
&\quad + 2\int_0^s \int_0^u \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right) \exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right) \frac{v_v}{\left(v_u - u\right)\left(v_v - v\right)}\mathrm{d}u.
\end{aligned}$$

Then , since

$$\int_0^s \exp\left(-\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right) \frac{v_v}{v_v - v}\mathrm{d}v = s,$$

and

$$2\int_0^s \exp\left(-2\int_v^s \frac{1}{v_r - r}\mathrm{d}r\right) \frac{v_v}{v_v - v}\mathrm{d}v = 2s + \int_0^s \exp\left(-2\int_u^s \frac{1}{v_r - r}\mathrm{d}r\right) \mathrm{d}u$$

we obtain that $E\left(B_t B_s\right) = s$. So for $0 \leq t < 1$ we have that (B_t) is a standard Brownian motion. On the other hand

$$\begin{aligned}
E(B_t V_t) &= E\left(\int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\frac{V_u V_t}{v_u - u}\mathrm{d}u\right) \\
&= \int_0^t \exp\left(-\int_u^t \frac{1}{v_r - r}\mathrm{d}r\right)\frac{v_u}{v_u - u}\mathrm{d}u \\
&= t,
\end{aligned}$$

therefore

$$\begin{aligned}
E((B_t - V_t)^2) &= E(B_t^2) + E({V_t}^2) - 2E(B_t V_t) \\
&= t + v_t - 2t = v_t - t,
\end{aligned}$$

and, since by hypothesis $v_1 = 1$, this means that

$$\lim_{t\to 1} B_t \overset{L^2}{=} V_1,$$

then for all $0 \leq t < 1$

$$E\left(\int_0^t \frac{|V_s - B_s|}{v_s - s}\mathrm{d}s\right) < \int_0^t \frac{E\left(\left(V_s - B_s\right)^2\right)^{\frac{1}{2}}}{v_s - s}\mathrm{d}s = \int_0^t \sqrt{v_s - s}\mathrm{d}s < \sqrt{2},$$

and this implies, by the monotone convergence theorem, that

$$\lim_{t\to 1}\int_0^t \frac{|V_s - B_s|}{v_s - s}\mathrm{d}s = \int_0^1 \frac{|V_s - B_s|}{v_s - s}\mathrm{d}s < \infty$$

and that $B_1 = \lim_{t\to 1} B_t$ is well defined. Now, we have, by the uniqueness of the limit in probability, that $V_1 = B_1$ a.s. ■

6.3 Application to find the equilibrium strategy

In this section we shall apply the results of the previous section to find the equilibrium strategy of the insider. We will see trough different examples how this can be done. These different examples correspond to different models that are extensions of the Kyle-Back model.

Example 18 *(Back 92) Assume that Z is a Brownian motion with variance σ^2 and $V \equiv V_1 = f(\eta_1)$. If the strategy is optimal $V_1 = H(1, Y_1)$, and if η_1 has a continuous cumulative distribution function we can assume that $Y_1 \equiv N(0, \sigma^2)$ by choosing f conveniently. It is also assumed that η_1 (and consequently Y_1) is independent of Z. Then by the calculations in the Example 11 we have that*

$$Y_t = Z_t + \int_0^t \frac{Y_1 - Y_s}{1 - s}\mathrm{d}s,$$

is a Brownian motion with variance σ^2. So the equilibrium strategy is

$$X_t = \int_0^t \frac{Y_1 - Y_s}{1-s} \mathrm{d}s, 0 \leq t < 1.$$

Example 19 *(Aase et. al (2007))*

$$Z_t = \int_0^t \sigma_s dW_s$$

where σ is deterministic and $V \equiv Y_1$ is a $N(0, \int_0^1 \sigma_s^2 \mathrm{d}s)$ independent of Z. Then $V|\mathcal{F}_t^Y \sim N(Y_t, \int_t^1 \sigma_s^2 \mathrm{d}s)$ and by the results in the Example 12

$$Y_t = Z_t + \int_0^t \frac{Y_s - Y_1}{\int_t^1 \sigma_s^2 \mathrm{d}s} \sigma_s^2 \mathrm{d}s,$$

has the same law as Z. We have a similar result if σ is random.

Example 20 *(Campi, Cetin, Danilova (CCD)2009) (Which is the connection with 5.41 in CCD 2010?) If $dZ_t = \sigma(Y_t)dW_t$ and $V \equiv V_1 = f(\eta_1)$. Where $\eta_t = \int_0^t \sigma(\eta_s)dB_s$, and independent of Z, then by the results in the Example 13*

$$dY_t = \sigma(Y_t)dW_t + \sigma^2(Y_t)\frac{\partial_y G(1-t, Y_t, V_1)}{G(1-t, Y_t, V_1)}dt$$

where $G(t, y, z)$ is the transition density of V., is a martingale.

Example 21 *(Campi and Cetin (2007)) If we want the aggregate process Y to be a Brownian motion that reaches the value -1 for the first time at time τ, and Z is also a Brownian motion then, by the results in the Example 14:*

$$Y_t = Z_t + \int_0^t \left(\frac{1}{1+Y_s} - \frac{1+Y_s}{\tau - s} \right) \mathbf{1}_{[0,\tau]}(s) \mathrm{d}s,$$

so, in this case $\eta \equiv \tau$.

Example 22 *(Back and Pedersen (1998), Wu (1999), Danilova (2008)) The insider receives a continuous signal*

$$\eta_t = \eta_0 + \int_0^t \sigma_s dW_s,$$

where σ_s is deterministic, η_0 is a zero mean normal random variable, W is a Brownian motion, both independent of the Brownian motion Z. It is assumed that $var(\eta_1) = var(\eta_0) + \int_0^1 \sigma_s^2 ds = 1$, then, by Proposition 17,

$$Y_t = Z_t + \int_0^t \frac{\eta_t - Y_t}{var(\eta_t) - t} dt, \ 0 \leq t \leq 1.$$

is a Brownian motion.

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