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On the Number of Limit Cycles in Generalized Abel Equations*

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Abstract. Given $p, q \in \mathbb{Z}_{\geq 2}$ with $p \neq q$, we study generalized Abel differential equations $\frac{dx}{d\theta} = A(\theta)x^p + B(\theta)x^q$, where A and B are trigonometric polynomials of degrees $n, m \geq 1$, respectively, and we are interested in the number of limit cycles (i.e., isolated periodic orbits) that they can have. More concretely, in this context, an open problem is to prove the existence of an integer, depending only on p, q, m, and n and that we denote by $\mathcal{H}_{p,q}(n,m)$, such that the above differential equation has at most $\mathcal{H}_{p,q}(n,m)$ limit cycles. In the present paper, by means of a second order analysis using Melnikov functions, we provide lower bounds of $\mathcal{H}_{p,q}(n,m)$ that, to the best of our knowledge, are larger than the previous ones appearing in the literature. In particular, for classical Abel differential equations (i.e., $p = 3$ and $q = 2$), we prove that $\mathcal{H}_{3,2}(n,m) \geq 2(n+m) - 1$.

Key words. generalized Abel equations, Melnikov theory, second order perturbation, limit cycles

AMS subject classifications. 34C07, 34C05, 37C10

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1. Introduction and statements of main results.

The study of the existence of periodic orbits in ordinary differential equations has been an interesting problem for years in many areas of mathematics, particularly in qualitative theory of differential equations. In this area of interest, when we focus on planar polynomial vector fields, one of the most renowned classical problems arises: to know the number and location of isolated periodic orbits, the so-called limit cycles, in terms of its degree n. The study of this problem began at the end of the 19th century with the seminal works by Poincaré, but takes its name after Hilbert because of his famous list of unsolved problems published in 1900. From the original list of 23 problems, the 16th is still open, in particular, its second part. More precisely (see [26, 36] for details), the "existential" Hilbert's 16th problem is to prove that for any $n \geq 2$ there exists a finite number $\mathcal{H}(n)$ such that any polynomial vector field of degree $\leq n$ has less than $\mathcal{H}(n)$ limit cycles.

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Motivated by Hilbert's 16th problem, a very related line of research is to investigate the periodic solutions of scalar differential equations

$$\dot{x} := \frac{dx}{d\theta} = \sum_{i=0}^{k} a_i(\theta)x^i,$$

where a_i are periodic analytic functions. In this context an isolated periodic solution is called a limit cycle and it occurs that its number increases with k. (The reader is referred to [15] for an enlightening explanation of this fact.) Linear differential equations have at most 1 limit cycle, whereas the quadratic ones have at most 2. The latter are known as Riccati equations and the upper bound follows from the fact that the return map is a Möbius function. Nevertheless the situation is more intricate for degree three. The well-known trigonometric Abel differential equation is written as

$$\dot{x} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x \tag{1}$$

with A, B, and C being trigonometric polynomials. Pliss [33] proves using the Schwarzian derivative that if A does not change sign then the maximum number of limit cycles is 3. However it was Lins Neto [31] who was the first to show that, in general, there is no upper bound for the number of limit cycles. Indeed, he proves that for every positive integer ℓ there exists an Abel differential equation with periodic coefficients having ℓ limit cycles. He does it by studying the perturbation $\dot{x} = \varepsilon A(\theta)x^3 + B(\theta)x^2$, where both coeficients are trigonometric polynomials of degree ℓ.

In this paper we are interested in the number of limit cycles of *generalized Abel equations* in terms of the trigonometric polynomial degree of their coefficients. More concretely, we study the differential equation

$$\dot{x} = A(\theta)x^p + B(\theta)x^q, \tag{2}$$

where A and B are trigonometric polynomials and $p, q \in \mathbb{Z}_{\geq 2}$ with $p \neq q$. We say that a solution $x = x(\theta)$ of this differential equation is a periodic orbit if it satisfies $x(-\pi) = x(\pi)$. As before, a periodic orbit is called a *limit cycle* if it is isolated in the set of periodic orbits. For fixed exponents p and q, we define the Hilbert number $\mathcal{H} = \mathcal{H}_{p,q}(n, m)$ as the maximum number of limit cycles that the differential equation (2) can have for any trigonometric polynomials A and B of degrees n and m, respectively. For the classical Abel equation, i.e., $(p, q) = (3, 2)$, it is known as the Smale–Pugh problem; see [37]. So far it is even unknown whether $\mathcal{H}$ exists. Our main contribution in the present paper is to provide a lower bound of $\mathcal{H}$ that, to the best of our knowledge, improves the previous ones appearing in the literature. We shall restrict ourselves to the case $n, m \geq 1$ because from the results in [16, 18, 31] the problem is completely solved when $n = 0$ or $m = 0$.

It is proved in [18] that the upper bound for the number of limit cycles of the differential equation (1) is three provided that A or B does not change sign. The authors use this result in order to bound the number of hyperbolic limit cycles in some planar polynomial differential systems. (This idea is also used in many other papers; see [1, 2, 12, 20, 25], for example.) The natural extension of this result to the equation $\dot{x} = A(\theta)x^p + B(\theta)x^q + C(\theta)x$ is considered in

[16] where, under the same hypothesis, it is proved that the upper bound is 5 (respectively, 4) when $\max(p,q)$ is odd (respectively, even). This result, particularized to $C=0$, gives $\mathcal{H}_{p,q}(n,m)$ in the case that $nm=0$. Therefore, as we explained before, for the the problem that we tackle in the present paper it is natural to assume that $n,m \geq 1$. Certainly the problem is much more difficult when the coefficients A and B do change sign. This is the case studied in [4] where, under some other hypotheses on the coefficients, it is proved that only one limit cycle exists. Other upper bounds for the number of periodic solutions are given in [29] under some conditions on the number of zeros of $B(\theta)$. More generally, also refer to the result in [3], where it is proved that the differential equation $\dot{x}=\sum_{i=0}^{m} a_i(\theta)x^{n_i}$ with $1 \leq n_i \leq n$ and a_i periodic analytic functions, can have at most $3n-1$ limit cycles provided some transversal conditions are verified. The extension of the aforesaid Lins Neto result to generalized Abel differential equations is also done in [16].

There are also some problems coming from planar polynomial differential systems that can be brought to a differential equation as in (1) or (2). Among others, the homogeneous nonlinear perturbations of the harmonic oscillator or the so-called rigid systems ($\dot{\theta}=1$); see [10] and [19], respectively. More recently, it is shown in [5] that Abel differential equations (2) have also limit cycles of alien type. They are not of small amplitude, like in a Hopf bifurcation, neither arising by the perturbation of an annulus that is foliated by periodic orbits. Among the long list of references to Abel differential equations (there are more than three hundred in the literature) there are some that reduce real problems to this type of differential equation. In [13] the authors computed an approximation of an unstable limit cycle that appears in an Abel equation arising in a tracking control problem that affects an elementary, nonminimum phase, second order bilinear power converter. The authors in [23] study a second order differential equation that describes the relativistic evolution of a causal dissipative cosmological fluid in a conformally flat space-time. They reduce this evolution equation to an Abel differential equation. The same authors, in a more recent work [24], consider quasi-stationary (traveling wave type) solutions of a nonlinear reaction-diffusion equation, which describes the evolution of glioblastomas. These aggressive primary brain tumors are characterized by extensive infiltration into the brain and are highly resistant to treatment. The second order nonlinear equation describing the glioblastoma growth through traveling waves is reduced to a differential equation of Abel type. The relationship between the Einstein–Friedmann and Abel equations is studied in [38]. In that work the authors demonstrate how the latter might be applied to the inflationary analysis in a spatially flat Friedmann universe filled with a real-valued scalar field. They use an Abel equation to provide the necessary and sufficient conditions for both slow-rolling and inflation to be estimated with respect to the initial value of the field.

Coming back to the original Hilbert's 16th problem, due to the difficulty in finding uniform upper bounds for even subclasses of polynomial differential systems, some weak versions have appeared during the past decades. One of them was proposed by Arnol'd [8] and it focuses on the study of limit cycles bifurcating from the period annulus of Hamiltonian systems. Closely related to this, our approach, in order to improve the lower bounds for $\mathcal{H}_{p,q}(n,m)$ when $n,m \geq 1$, is to consider a second order perturbation of the generalized Abel differential equation

$$\dot{x}=(\sin\theta+\varepsilon P_1(\theta)+\varepsilon^2 P_2(\theta))x^p+(\varepsilon Q_1(\theta)+\varepsilon^2 Q_2(\theta))x^q, \tag{3}$$

where the coefficients of the perturbation are trigonometric polynomials of degrees n and m, that is

$$P_i(\theta) = b_{i0} + \sum_{k=1}^{n} \big(a_{ik}\sin(k\theta) + b_{ik}\cos(k\theta)\big) \text{ and } Q_i(\theta) = d_{i0} + \sum_{k=1}^{m} \big(c_{ik}\sin(k\theta) + d_{ik}\cos(k\theta)\big)$$

for $i = 1, 2$. We note that the parameter space associated with (2) is $\mathbb{R}^{2(n+m+1)}$ and that by taking (3) we study the perturbation of a specific point, say $\xi_0 \in \mathbb{R}^{2(n+m+1)}$, corresponding to $\dot{x} = \sin\theta\, x^p$. The coefficients of P_i and Q_i, once determined, give a curve in $\mathbb{R}^{2(n+m+1)}$ passing through ξ_0 at $\varepsilon = 0$. In what follows, for the sake of convenience, we will treat these coefficients as parameters too, setting $\mu = (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$, where

$$\begin{aligned} &\mathbf{a} = (a_{1k}, a_{2k}; k = 1, 2, \dots, n), \quad \mathbf{c} = (c_{1k}, c_{2k}; k = 1, 2, \dots, m), \\ &\mathbf{b} = (b_{1k}, b_{2k}; k = 0, 1, \dots, n), \quad \mathbf{d} = (d_{1k}, d_{2k}; k = 0, 1, \dots, m). \end{aligned} \tag{4}$$

Thus $\mu \in \mathbb{R}^{4(n+m+1)}$, although as we already mentioned the "ambient" parameter space is $\mathbb{R}^{2(n+m+1)}$. That being said we denote by $x(\theta, \rho; \mu, \varepsilon)$ the solution of (3) with initial condition $x(-\pi, \rho; \mu, \varepsilon) = \rho$. One can readily prove (see Corollary 2.3) that the unperturbed system verifies $x(\pi, \rho; \mu, 0) = \rho$ for all $\rho \in I := (\kappa_p, +\infty)$, where

$$\kappa_p := \begin{cases} -\infty & \text{if } p \text{ is odd}, \\ -(2(p-1))^{-\frac{1}{p-1}} & \text{if } p \text{ is even}. \end{cases} \tag{5}$$

Then, thanks to the analytic dependence of solutions with respect to initial conditions and parameters, we can write the Taylor series of the Poincaré transition map as

$$x(\pi, \rho; \mu, \varepsilon) = \rho + \sum_{i=1}^{\infty} \varepsilon^i M_i(\rho; \mu),$$

where M_i is an analytic function on $I \times \mathbb{R}^{4(n+m+1)}$. Setting $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0)$, for our first main result we study the case $M_1 = 0$ and $M_2 \neq 0$ assuming that $n, m \geq 1$.

Theorem 1.1. *The function $M_1(\,\cdot\,; \mu)$ vanishes identically if and only if*

$$\mu \in \mathscr{L} = \{\mu \in \mathbb{R}^{4(n+m+1)} : b_{10} = d_{10} = \cdots = d_{1m} = 0\}.$$

For a fixed $\mu \in \mathscr{L}$, let $K^{\pm}$ be the number of zeros of $M_2(\,\cdot\,; \mu)$ on $I \cap \mathbb{R}^{\pm}$ taking multiplicities into account. Then the following properties hold:

(a) *If p and q are odd, then $K^{\pm} \leq n + m$. Moreover, there exists $\mu_0 \in \mathscr{L}$ such that $M_2(\,\cdot\,; \mu_0)$ has $2(n + m)$ simple zeros in $I \setminus \{0\}$.*

(b) *If p is odd and q is even, then $K^{\pm} \leq n + m$ and both equalities do not hold simultaneously. Moreover,*
   (i) *when $p < q$, there exists $\mu_0 \in \mathscr{L}$ such that $M_2(\,\cdot\,; \mu_0)$ has $2(n + m) - 1$ simple zeros in $I \setminus \{0\}$;*
   (ii) *when $p > q$, there exists $\mu_0 \in \mathscr{L}$ such that $M_2(\,\cdot\,; \mu_0)$ has $2(n + m - 1)$ simple zeros in $I \setminus \{0\}$;*

(c) *if p is even, q is odd, and $p < q$, then $K^+ + K^- \leq n + m$. Moreover, there exists $\mu_0 \in \mathscr{L}$ such that $M_2(\,\cdot\,;\mu_0)$ has $n + m$ simple zeros in $I \setminus \{0\}$;*

(d) *if p is even and either q is even or $p > q$, then $K^+ + K^- \leq n + m + 1$. Moreover, there exists $\mu_0 \in \mathscr{L}$ such that $M_2(\,\cdot\,;\mu_0)$ has $n + m + 1$ simple zeros in $I \setminus \{0\}$.*

Since $x(\pi, 0; \mu, \varepsilon) = 0$ for all ε, note that $M_i(0; \mu) = 0$ for all $i \geq 1$ but we stress that the zero $\rho = 0$ is not counted in the previous result. That being said, in order to be consistent with the above-mentioned papers on Abel differential equations, we do count this zero limit cycle in our next main result. Before giving its statement let us mention that in what follows we shall call M_i the *Melnikov function of ith order* for the perturbed differential equation (3). These functions are clearly analytic on I and, by applying the Weierstrass preparation theorem, the number of zeros (multiplicities taken into account) of the first nonidentically zero Melnikov function gives an upper bound for the number of roots of $x(\pi, \rho; \mu, \varepsilon) = \rho$ for $\varepsilon \approx 0$. In other words, it provides an upper bound for the number of limit cycles that bifurcate from the continuum of periodic orbits of the unperturbed differential equation. In its turn a lower bound is given by the number of simple zeros thanks to the implicit function theorem. In short this is how the general lower bound of the Hilbert number that we give in our next result follows from Theorem 1.1.

Theorem 1.2. *The Hilbert number for the Abel differential equation (2) with $p = 3$ and $q = 2$ verifies*

$$\mathcal{H}_{3,2}(n, m) \geq 2(n + m) - 1,$$

where $n, m \geq 1$ are, respectively, the degrees of the trigonometric polynomials A and B. Moreover, $\mathcal{H}_{3,2}(1, 3) \geq 8$ and $\mathcal{H}_{3,2}(4, 1) \geq 10$.

As we explained above, the general lower bound in the first assertion follows by the Melnikov theory. (The Melnikov theory for planar autonomous differential equations is equivalent to the so-called averaging theory; see [9].) By contrast the second assertion, which improves the bound by one limit cycle in two particular cases, follows by using Lyapunov constants. In order to make this clear and to facilitate the reading of the paper, for the reader's convenience we prove the second assertion separately in an appendix, where we also introduce the basic notions on Lyapunov constants.

There are two previous papers with results about the Hilbert number of Abel differential equations that should be referred. Recall that the general lower bound in Theorem 1.2 is obtained by a second order perturbation in ε. The authors in [6] give lower bounds for $\mathcal{H}_{3,2}(1, m)$ and $\mathcal{H}_{3,2}(n, 1)$ by a first order perturbation. On the other hand, the authors in [17] give a lower bound for $\mathcal{H}_{p,q}(n, 1)$ by a first order perturbation as well.

The paper is organized in the following way. In section 2 we study the perturbed differential equation $\dot{x} = h(x)f(\theta) + H(\theta, x; \varepsilon)$ and we give the expression of its first nonidentically zero Melnikov function (Theorem 2.1). This is a rather general result that, we believe, could be very useful in the development of further research on the issue. Next, in section 3, we recall the notion of Chebyshev system and explain the related basic results. We also state a key result from [17] that turns out to be very important for our purposes (Theorem 3.4). Section 4 is devoted to showing that the Melnikov function M_2 for the perturbed differential equation (3) belongs to an appropriate Chebyshev system (Proposition 4.4). The proofs of our two main results are given in section 5. Finally in the appendix we prove by using Lyapunov constants that $\mathcal{H}_{3,2}(1,3) \geq 8$ and $\mathcal{H}_{3,2}(4,1) \geq 10$ (Propositions A.2 and A.3, respectively), which improve in these particular cases the general lower bound that we obtain by using the Melnikov theory of second order. These two lower bounds also improve some previous ones obtained using Lyapunov constants as well. Here we follow a new approach using first and second order developments of the Lyapunov constants at some specific parameters having a center. Finally we explain some numerical evidence in order to increase the lower bound given in Theorem 1.2 for $\mathcal{H}_{3,2}(1,4)$ and $\mathcal{H}_{3,2}(2,3)$ by using Lyapunov constants.

2. Melnikov functions.

In this section we consider the perturbed differential equation

$$\frac{dx}{d\theta} = h(x)f(\theta) + H(\theta, x; \varepsilon), \tag{6}$$

where

- h is analytic on $\mathbb{R}$,
- f is a 2π-periodic analytic function with $\int_{-\pi}^{\pi} f(s)ds = 0$, and
- H is an analytic function on $\mathbb{R}\times\mathbb{R}\times(-\varepsilon_0, \varepsilon_0)$, for some $\varepsilon_0 > 0$, such that $\theta \mapsto H(\theta, x; \varepsilon)$ is 2π-periodic and $H(\theta, x; 0) \equiv 0$.

Given $\rho \in \mathbb{R}$, let $x(\theta, \rho; \varepsilon)$ denote the solution of (6) such that $x(-\pi, \rho; \varepsilon) = \rho$. (We shall sometimes use the more compact notation $x_\varepsilon(\theta, \rho)$ for the sake of brevity.) In this section we assume that the unperturbed differential equation, i.e., (6) with $\varepsilon = 0$, has a stripe of periodic orbits. More precisely, that there exists an open interval I of initial conditions such that $x(\pi, \rho; 0) = \rho$ for all $\rho \in I$. Note that, due to $H(\theta, x; 0) \equiv 0$, a necessary condition for this is that $\int_{-\pi}^{\pi} f(s)ds = 0$. Under this assumption, a sufficient condition for the existence of such an interval is that h vanishes at some point $x_0 \in \mathbb{R}$. This is precisely the setting that we have for the perturbed differential equation (3), for which $x_0 = 0$ and, as we will see, $I = (\kappa_p, +\infty)$, where $\kappa_p < 0$ is given in (5).

By the analytic dependence of solutions with respect to initial conditions and parameters, the solution $x(\theta, \rho; \varepsilon)$ is well-defined and analytic for all $(\theta, \rho, \varepsilon) \in [-\pi, \pi] \times U$, where U is an open neighborhood of $I \times \{0\}$ in $\mathbb{R}^2$. We can thus consider the Taylor series of $x(\pi, \rho; \varepsilon)$ at $\varepsilon = 0$,

$$x(\pi, \rho; \varepsilon) = \rho + \sum_{i=1}^{\infty} M_i(\rho)\varepsilon^i,$$

where each M_i is an analytic function on I. We aim to study the fixed points of $\rho \mapsto x(\pi, \rho; \varepsilon)$ that persist for small $\varepsilon \neq 0$ and to this end an explicit expression of the first $M_i \neq 0$ is needed. Our first result is addressed to this and in order to state it we introduce some more notation. We write the Taylor series of the perturbation at $\varepsilon = 0$ as

$$H(\theta, x; \varepsilon) = \sum_{i=1}^{\infty} \ell_i(\theta, x)\varepsilon^i.$$

We also use the differential operator

$$\Theta_x := h(x)\partial_x$$

and denote $\Theta_x^k = \Theta_x \circ \overset{(k)}{\cdots} \circ \Theta_x$. Furthermore we consider the *incomplete exponential Bell* *polynomials* $\mathcal{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$, which can be defined recursively by means of

$$\mathcal{B}_{n,k} = \sum_{i=1}^{n-k+1} \binom{n-1}{i-1} x_i \mathcal{B}_{n-i,k-1},$$

setting $\mathcal{B}_{0,0} = 1$, $\mathcal{B}_{n,0} = 0$ for $n \geq 1$, and $\mathcal{B}_{0,k} = 0$ for $k \geq 1$ (see, for instance, [14, 27]).

Theorem 2.1. *If $M_1 = M_2 = \cdots = M_{n-1} = 0$ then $M_n(\rho) = \frac{1}{n!}h(\rho)L_n(\pi,\rho)$ for all $\rho \in I$ such that $h(\rho) \neq 0$, where the sequence $\{L_n\}_{n\in\mathbb{Z}_{\geq 1}}$ is defined recursively by means of*

$$L_n(\theta,\rho) = \sum_{i=0}^{n-1} \frac{n!}{i!} \sum_{k=0}^{i} \int_{-\pi}^{\theta} \Theta_x^k \left(\frac{\ell_{n-i}(s,x)}{h(x)} \right) \Bigg|_{x=x_0(s,\rho)} \mathcal{B}_{i,k}\Big(L_1(s,\rho), L_2(s,\rho), \ldots, L_{i-k+1}(s,\rho)\Big) ds.$$

Proof. For each $(\theta,\rho) \in [-\pi,\pi] \times I$, let the Taylor series of $x(\theta,\rho;\varepsilon)$ at $\varepsilon = 0$ be written as

$$x_\varepsilon(\theta,\rho) = x_0(\theta,\rho) + \sum_{i=1}^{\infty} S_i(\theta,\rho)\varepsilon^i.$$

Notice in particular that, by definition, $M_i(\rho) = S_i(\pi,\rho)$.

Fix any $\rho_0 \in I$ such that $h(\rho_0) \neq 0$. We claim that there exists $\delta > 0$ small enough such that $M_n(\rho) = \frac{1}{n!}h(\rho)L_n(\pi,\rho)$ for all $\rho \in I$ with $|\rho - \rho_0| < \delta$. Clearly, due to the arbitrariness of ρ_0, the result will follow once we prove this claim. With this aim in view note first that if for a given $\rho \in I$ there exists $\theta^* \in [-\pi,\pi]$ such that $h(x_0(\theta^*,\rho)) = 0$ then $x_0(\theta,\rho) = \rho$ for all $\theta \in [-\pi,\pi]$. Thus, if we denote by $\hat{I}$ the connect component of $I \setminus \{x \in \mathbb{R} : h(x) = 0\}$ that contains ρ_0, then $h(x_0(\theta,\rho)) \neq 0$ for all $(\theta,\rho) \in [-\pi,\pi] \times \hat{I}$. In other words, setting $\tilde{I} := \{x_0(\theta,\rho) : \theta \in [-\pi,\pi], \rho \in \hat{I}\}$, we have that $0 \notin h(\tilde{I})$. Accordingly, if

$$G(\rho) := \int_{\rho_0}^{\rho} \frac{du}{h(u)}$$

then $G : \tilde{I} \to G(\tilde{I})$ is a well-defined diffeomorphism. One can readily verify that the coordinate change $y = G(x)$ brings the differential equation (6) to

$$\frac{dy}{d\theta} = f(\theta) + \widehat{H}(\theta,y;\varepsilon) \text{ with } \widehat{H}(\theta,y;\varepsilon) := \frac{H(\theta,x;\varepsilon)}{h(x)}\Bigg|_{x=G^{-1}(y)}. \tag{7}$$

For each $\rho \in G(\hat{I})$ we denote by $y_\varepsilon(\theta,\rho)$ the solution of (7) with initial condition $y_\varepsilon(-\pi,\rho) = \rho$. Due to $x_0(\theta, G^{-1}(\rho)) \in \tilde{I}$, for each fixed ρ there exists $\varepsilon > 0$ small enough such that

$$y_\varepsilon(\theta,\rho) = G\left(x_\varepsilon(\theta, G^{-1}(\rho))\right) \text{ for all } \theta \in [-\pi,\pi].$$

Consequently, by continuity, there exists $\delta > 0$ small enough such that if $|\rho - \rho_0| + |\varepsilon| < \delta$ then

$$x_\varepsilon(\theta,\rho) = G^{-1}\big(y_\varepsilon(\theta, G(\rho))\big) \text{ for all } \theta \in [-\pi,\pi].$$

Clearly $\widehat{H}$ is an analytic function in a neighborhood of any $(\theta, y, \varepsilon) \in [-\pi, \pi] \times G(\tilde{I}) \times \{0\}$ and so we can consider its Taylor series at $\varepsilon = 0$:

$$\widehat{H}(\theta, y; \varepsilon) = \sum_{i=1}^{\infty} \hat{\ell}_i(\theta, y)\varepsilon^i, \text{ where } \hat{\ell}_i(\theta, y) = \left.\frac{\ell_i(\theta, x)}{h(x)}\right|_{x=G^{-1}(y)}. \tag{8}$$

Let us consider at this point the Taylor series of $y_\varepsilon(\theta, \rho)$ at $\varepsilon = 0$, say

$$y_\varepsilon(\theta, \rho) = y_0(\theta, \rho) + \sum_{i=1}^{\infty} \widehat{S}_i(\theta, \rho)\varepsilon^i, \tag{9}$$

and set $\widehat{M}_i(\rho) := \widehat{S}_i(\pi, \rho)$, which is well-defined for any $\rho \in G(\hat{I})$ such that $|G^{-1}(\rho) - \rho_0| < \delta$. Then, taking the derivative with respect to θ on both sides of the above equality, from (7) and (8) we get that

$$\begin{aligned}
\partial_\theta \widehat{S}_n(\theta, \rho) &= \frac{1}{n!}\, \partial_\varepsilon^n \left(\sum_{i=1}^{\infty} \varepsilon^i \hat{\ell}_i(\theta, y_\varepsilon(\theta, \rho)) \right)\Bigg|_{\varepsilon=0} \\
&= \frac{1}{n!} \sum_{i=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{d^k \varepsilon^i}{d\varepsilon^k} \partial_\varepsilon^{n-k} \left(\hat{\ell}_i(\theta, y_\varepsilon(\theta, \rho)) \right)\Big|_{\varepsilon=0} \\
&= \frac{1}{n!} \sum_{i=1}^{n} \binom{n}{i} i!\; \partial_\varepsilon^{n-i} \left(\hat{\ell}_i(\theta, y_\varepsilon(\theta, \rho)) \right)\Big|_{\varepsilon=0}.
\end{aligned}$$

Accordingly

$$\partial_\theta \widehat{S}_n(\theta, \rho) = \sum_{i=1}^{n} \frac{1}{(n-i)!}\, \partial_\varepsilon^{n-i} \left(\hat{\ell}_i(\theta, y_\varepsilon(\theta, \rho)) \right)\Big|_{\varepsilon=0} = \sum_{i=0}^{n-1} \frac{1}{i!} \partial_\varepsilon^i \left(\hat{\ell}_{n-i}(\theta, y_\varepsilon(\theta, \rho)) \right)\Big|_{\varepsilon=0}. \tag{10}$$

By applying Faà di Bruno's formula for the chain rule (see [14, 27]) we can assert that

$$\partial_\varepsilon^i \Big(\hat{\ell}_j(\theta, y_\varepsilon(\theta, \rho)) \Big) = \sum_{k=0}^{i} \partial_y^k \Big(\hat{\ell}_j(\theta, y_\varepsilon(\theta, \rho)) \Big) \mathcal{B}_{i,k} \Big(\partial_\varepsilon y_\varepsilon(\theta, \rho), \partial_\varepsilon^2 y_\varepsilon(\theta, \rho), \ldots, \partial_\varepsilon^{i-k+1} y_\varepsilon(\theta, \rho) \Big).$$

Thus, on account of $\partial_\varepsilon^k y_\varepsilon(\theta, \rho)\big|_{\varepsilon=0} = k!\widehat{S}_k(\theta, \rho)$, we get

$$\begin{aligned}
&\partial_\varepsilon^i \left(\hat{\ell}_j(\theta, y_\varepsilon(\theta, \rho)) \right)\Big|_{\varepsilon=0} \\
&\qquad = \sum_{k=0}^{i} \partial_y^k \left(\hat{\ell}_j(\theta, y_0(\theta, \rho)) \right) \mathcal{B}_{i,k} \Big(\widehat{S}_1(\theta, \rho), 2\hat{S}_2(\theta, \rho), \ldots, (i-k+1)!\widehat{S}_{i-k+1}(\theta, \rho) \Big).
\end{aligned}$$

Therefore, since $\widehat{S}_n(-\pi, \rho) = 0$, from (10) it follows that

(11)

$$\begin{aligned}
&\widehat{S}_n(\theta, \rho) \\
&\quad = \sum_{i=0}^{n-1} \frac{1}{i!} \sum_{k=0}^{i} \int_{-\pi}^{\theta} \partial_y^k \Big(\hat{\ell}_{n-i}(s, y_0(s, \rho)) \Big) \mathcal{B}_{i,k}(\widehat{S}_1(s, \rho), 2\widehat{S}_2(s, \rho), \ldots, (i-k+1)!\widehat{S}_{i-k+1}(s, \rho))\, ds.
\end{aligned}$$

Recall now that $x_\varepsilon(\theta,\rho) = G^{-1}\big(y_\varepsilon(\theta, G(\rho))\big) = x_0(\theta,\rho) + \sum_{i=1}^{\infty} S_i(\theta,\rho)\varepsilon^i$ for all $\theta \in [-\pi,\pi]$ provided that $|\rho-\rho_0| + |\varepsilon| < \delta$. Hence, since $(G^{-1})'(y) = h(G^{-1}(y))$ by definition,

$$\begin{aligned}
M_n(\rho) = S_n(\theta,\rho)|_{\theta=\pi} &= \frac{1}{n!}\partial_\varepsilon^n\Big(G^{-1}\big(y_\varepsilon(\theta, G(\rho))\big)\Big)\Big|_{\varepsilon=0,\theta=\pi} \\
&= \frac{1}{n!}\partial_\varepsilon^{n-1}\Big((G^{-1})'\big(y_\varepsilon(\theta,G(\rho))\big)\partial_\varepsilon y_\varepsilon(\theta,G(\rho))\Big)\Big|_{\varepsilon=0,\theta=\pi} \\
&= \frac{1}{n!}\partial_\varepsilon^{n-1}\Big(h\big(x_\varepsilon(\theta,\rho)\big)\partial_\varepsilon y_\varepsilon(\theta,G(\rho))\Big)\Big|_{\varepsilon=0,\theta=\pi} \\
&= \frac{1}{n!}\sum_{k=0}^{n-1}\binom{n-1}{k}\,\partial_\varepsilon^{n-1-k}\Big(h\big(x_\varepsilon(\theta,\rho)\big)\Big)\partial_\varepsilon^{k+1}\Big(y_\varepsilon(\theta,G(\rho))\Big)\Big|_{\varepsilon=0,\theta=\pi} \\
&= \frac{1}{n!}\sum_{k=0}^{n-1}\binom{n-1}{k}\,\partial_\varepsilon^{n-1-k}\Big(h\big(x_\varepsilon(\theta,\rho)\big)\Big)\Big|_{\varepsilon=0,\theta=\pi}(k+1)!\widehat{M}_{k+1}(G(\rho)),
\end{aligned}$$

where in the last equality we use (11) and $\widehat{M}_i(\rho) = \widehat{S}_i(\pi,\rho)$. If $\widehat{M}_1 = \widehat{M}_2 = \cdots = \widehat{M}_{n-1} = 0$ then only the term for $k = n-1$ remains and

$$M_n(\rho) = h(\rho)\widehat{M}_n(G(\rho)) = h(\rho)\widehat{S}_n(\pi, G(\rho))$$

for all $\rho \in (\rho_0-\delta, \rho_0+\delta)$. Note that by arguing recursively we get the same equality assuming $M_1 = M_2 = \cdots = M_{n-1} = 0$.

It only remains to express $\widehat{S}_n(\theta,\rho)$ in terms of the solution $x_0(\theta,\rho)$ of (6). To this end notice that, on account of $(G^{-1})'(y) = h(G^{-1}(y))$ once again, from (8) we get

$$\partial_y^k\hat{\ell}_i(\theta,y) = \Theta_x^k\left(\frac{\ell_i(\theta,x)}{h(x)}\right)\Big|_{x=G^{-1}(y)}.$$

Accordingly, due to $y_0(\theta,\rho) = G\left(x_0(\theta, G^{-1}(\rho))\right)$, we can assert that

$$\partial_y^k\Big(\hat{\ell}_i(s, y_0(s,\rho))\Big) = \Theta_x^k\left(\frac{\ell_i(s,x)}{h(x)}\right)\Big|_{x=x_0(s,G^{-1}(\rho))}.$$

Taking this into account and setting $L_i(\theta,\rho) := i!\widehat{S}_i(\theta, G(\rho))$ for all $i \in \mathbb{N}$, from (9) we obtain

$$\begin{aligned}
&L_n(\theta,\rho) = n!\widehat{S}_n(\theta,G(\rho)) \\
&\quad = \sum_{i=0}^{n-1}\frac{n!}{i!}\sum_{k=0}^{i}\int_{-\pi}^{\theta}\Theta_x^k\left(\frac{\ell_i(s,x)}{h(x)}\right)\Big|_{x=x_0(s,\rho)}\mathcal{B}_{i,k}\Big(L_1(s,\rho), L_2(s,\rho), \ldots, L_{i-k+1}(s,\rho)\Big)ds
\end{aligned}$$

and, hence, $M_n(\rho) = h(\rho)\widehat{S}_n(\pi, G(\rho)) = \frac{1}{n!}h(\rho)L_n(\pi,\rho)$ for all $\rho \in (\rho_0-\delta, \rho_0+\delta)$. This proves the claim and concludes the proof of the result. ∎

Note that if we take $\hat{\rho} \in I$ such that $h(\hat{\rho}) = 0$ then $x_0(\theta,\hat{\rho}) = \hat{\rho}$ for all θ. It happens then that the function $L_n(\theta,\rho)$ is not well-defined at $\rho = \hat{\rho}$ due to the denominator $h(x_0(s,\rho))$

in its integrand. However, since M_n is continuous (in fact analytic) at $\rho = \hat{\rho}$, the limit of $h(\rho)L_n(\pi,\rho)$ as ρ tends to $\hat{\rho}$ exists and is equal to $n!M_n(\hat{\rho})$. Thus the singularity of $h(\rho)L_n(\pi,\rho)$ at $\rho = \hat{\rho}$ is removable. Our next result shows this for the perturbation associated with the differential equation (3), for which we have $\hat{\rho} = 0$. In its statement recall that κ_p is given in (5).

Remark 2.2. *For the reader's convenience we give the first terms in the recurrence of Theorem* 2.1. *Since* $\mathcal{B}_{0,0} = 1$, $\mathcal{B}_{1,0} = \mathcal{B}_{2,0} = 0$, $\mathcal{B}_{1,1}(x_1) = x_1$, $\mathcal{B}_{2,1}(x_1,x_2) = x_2$, *and* $\mathcal{B}_{2,2}(x_1) = x_1^2$, *we get*

$$L_1(\theta,\rho) = \int_{-\pi}^{\theta} \frac{\ell_1(s,x)}{h(x)}\bigg|_{x=x_0(s,\rho)} ds,$$

$$L_2(\theta,\rho) = 2\int_{-\pi}^{\theta} \left(\frac{\ell_2(s,x)}{h(x)} + \Theta_x\left(\frac{\ell_1(s,x)}{h(x)}\right)L_1(s,\rho)\right)\bigg|_{x=x_0(s,\rho)} ds,$$

and

$$\begin{aligned} L_3(\theta,\rho) = 3\int_{-\pi}^{\theta} \Bigg(&2\frac{\ell_3(s,x)}{h(x)} + 2\Theta_x\left(\frac{\ell_2(s,x)}{h(x)}\right)L_1(s,\rho) \\ &+ \Theta_x\left(\frac{\ell_1(s,x)}{h(x)}\right)L_2(s,\rho) + \Theta_x^2\left(\frac{\ell_1(s,x)}{h(x)}\right)(L_1(s,\rho))^2\Bigg)\bigg|_{x=x_0(s,\rho)} ds. \end{aligned}$$

Corollary 2.3. *Following the previous notation, for the differential equation* (3) *we have*

$$x(\theta,\rho;0) = \frac{\rho}{\left(1+\rho^{p-1}(p-1)(1+\cos\theta)\right)^{1/(p-1)}},$$

so that $x(\pi,\rho;0) = \rho$ *for all* $\rho \in I = (\kappa_p,+\infty)$. *Moreover*

$$M_1(\rho) = \rho^p \int_{-\pi}^{\pi} \left(P_1(\theta) + Q_1(\theta)x_0(\theta,\rho)^{q-p}\right) d\theta$$

and if $M_1(\rho) \equiv 0$ *then*

$$M_2(\rho) = \rho^p \int_{-\pi}^{\pi} \left(P_2(\theta) + Q_2(\theta)x_0(\theta,\rho)^{q-p} + (q-p)Q_1(\theta)S(\theta,\rho)x_0(\theta,\rho)^{q-1}\right) d\theta,$$

where

$$S(\theta,\rho) = \int_{-\pi}^{\theta} \left(P_1(s) + Q_1(s)x_0(s,\rho)^{q-p}\right) ds.$$

Proof. The statement concerning the unperturbed equation follows from an easy computation. The second part follows by Theorem 2.1 particularizing the expressions given in Remark 2.2 for the case $f(\theta) = \sin\theta$, $h(x) = x^p$, and $\ell_i(\theta,x) = P_i(\theta)x^p + Q_i(\theta)x^q$ for $i = 1,2$. ■

3. Extended complete Chebyshev (ECT)-systems. We begin this section by recalling some properties about Chebyshev systems, then we extend some results in [17]. All of them will be necessary in order to prove our main results.

Definition 3.1. *Let $f_0, f_1, \ldots, f_n$ be analytic functions on an open interval $I \subset \mathbb{R}$. The ordered set $(f_0, f_1, \ldots, f_n)$ is an ECT-system on I if for each $k = 0, 1, 2, \ldots, n$ every nontrivial linear combination*

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_k f_k(x)$$

has at most k isolated zeros on I counted with multiplicities.

Definition 3.2. *Let $f_0, f_1, \ldots, f_k$ be analytic functions on an open interval $I \subset \mathbb{R}$. Then*

$$W[f_0, f_1, \ldots, f_k](x) = \det\left(f_j^{(i)}(x)\right)_{0 \le i,j \le k} = \begin{vmatrix} f_0(x) & \cdots & f_k(x) \\ f_0'(x) & \cdots & f_k'(x) \\ \vdots & \ddots & \vdots \\ f_0^{(k)}(x) & \cdots & f_k^{(k)}(x) \end{vmatrix}$$

is the Wronskian *of $(f_0, f_1, \ldots, f_k)$ at $x \in I$.*

The following is a well-known result (see, for instance, [28, 32]) that enables us to characterize Chebyshev systems in terms of Wronskians.

Lemma 3.3. *$(f_0, f_1, \ldots, f_n)$ is an ECT-system on an open interval $I \subset \mathbb{R}$ if and only if, for each $k = 0, 1, 2, \ldots, n$,*

$$W[f_0, f_1, \ldots, f_k](x) \neq 0 \text{ for all } x \in I.$$

To study the zeros of the Melnikov functions in Corollary 2.3 we will apply [17, Theorem A], which we state next for the reader's convenience. In its statement g is an analytic function, I_g is the connected component of $\{y \in \mathbb{R} : 1 - yg(\theta) > 0 \text{ for all } \theta \in [-\pi, \pi]\}$ containing the origin, and, for each $k \in \mathbb{Z}_{\ge 0}$ and $\alpha \in \mathbb{R}$,

$$\mathcal{T}_{k,\alpha}(y) := \int_{-\pi}^{\pi} \frac{g^k(\theta)}{\big(1 - yg(\theta)\big)^{\alpha}} d\theta \text{ for all } y \in I_g. \tag{12}$$

Theorem 3.4. *Consider $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}_{\ge 0}$. The following hold:*

(a) *If $\alpha \notin \mathbb{Z}_{\le 0}$ then $(\mathcal{T}_{0,\alpha}, \mathcal{T}_{1,\alpha}, \ldots, \mathcal{T}_{n,\alpha})$ is an ECT-system on I_g.*

(b) *If $\alpha \in \mathbb{Z}_{\le 0}$ then $(\mathcal{T}_{0,\alpha}, \mathcal{T}_{1,\alpha}, \ldots, \mathcal{T}_{n,\alpha})$ is an ECT-system on I_g if and only if $n \le -\alpha$.*

The following technical lemmas extend some of the results in [17].

Lemma 3.5. *If $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}_{\ge 0}$ then $(y^{\alpha}\mathcal{T}_{k,\alpha}(y))' = \alpha y^{\alpha-1}\mathcal{T}_{k,\alpha+1}(y)$ for all $y \in I_g \cap (0, +\infty)$.*

Proof. This is an easy consequence of the following computation:

$$(y^{\alpha}\mathcal{T}_{k,\alpha}(y))' = \alpha y^{\alpha-1} \int_{-\pi}^{\pi} \frac{g(\theta)^k}{\big(1 - yg(\theta)\big)^{\alpha}} d\theta + \alpha y^{\alpha} \int_{-\pi}^{\pi} \frac{g(\theta)^{k+1}}{\big(1 - yg(\theta)\big)^{\alpha+1}} d\theta = \alpha y^{\alpha-1}\mathcal{T}_{k,\alpha+1}(y).$$

∎

Lemma 3.6. *If $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ and $n \in \mathbb{Z}_{\geq 0}$ then*
(a) $\big(1, y^\alpha \mathcal{T}_{0,\alpha}(y), \dots, y^\alpha \mathcal{T}_{n,\alpha}(y)\big)$ *is an ECT-system on* $I_g \cap (0, +\infty)$, *and*
(b) $\big(1, (-y)^\alpha \mathcal{T}_{0,\alpha}(y), \dots, (-y)^\alpha \mathcal{T}_{n,\alpha}(y)\big)$ *is an ECT-system on* $I_g \cap (-\infty, 0)$.

Proof. For the sake of shortness let us prove the first assertion only (the other one follows in exactly the same way). We claim that, for each $k = 0, 1, \dots, n$,

$$W\left[1, y^\alpha \mathcal{T}_{0,\alpha}(y), \dots, y^\alpha \mathcal{T}_{k,\alpha}(y)\right] = (\alpha y^{\alpha-1})^{k+1} W\left[\mathcal{T}_{0,\alpha+1}, \mathcal{T}_{1,\alpha+1}, \dots, \mathcal{T}_{k,\alpha+1}\right](y)$$

for all $y \in I_g \cap (0, +\infty)$. Notice that, by applying Lemma 3.3 and Theorem 3.4, the result will follow once we show the claim. To this end a computation shows that

$$\begin{aligned} W\left[1, y^\alpha \mathcal{T}_{0,\alpha}(y), \dots, y^\alpha \mathcal{T}_{k,\alpha}(y)\right] &= W\left[(y^\alpha \mathcal{T}_{0,\alpha}(y))', \dots, (y^\alpha \mathcal{T}_{k,\alpha}(y))'\right] \\ &= W\left[\alpha y^{\alpha-1} \mathcal{T}_{0,\alpha+1}(y), \dots, \alpha y^{\alpha-1} \mathcal{T}_{k,\alpha+1}(y)\right] = (\alpha y^{\alpha-1})^{k+1} W\left[\mathcal{T}_{0,\alpha+1}, \dots, \mathcal{T}_{k,\alpha+1}\right](y), \end{aligned}$$

where the second equality follows by applying Lemma 3.5 and the third one by the so-called Hesse–Christoffel's identity (see [30, 34] for instance). This proves the validity of the claim and hence the result follows. ∎

4. Melnikov functions for the differential equation (3).

Recall that Corollary 2.3 provides an expression of M_2 assuming that $M_1 = 0$. Our goal in this section is to write it as a linear combination of functions belonging to an ECT-system. This will be done in Proposition 4.4. With this aim in view we first particularize the integrals $\mathcal{T}_{k,\alpha}$ defined in (12) with a specific choice of function g and parameter α that is very related to the solution of the unperturbed system. In order to stress this, and for the reader's convenience, we introduce the following additional notation

$$\mathcal{I}_k(y) := \int_{-\pi}^{\pi} \frac{g^k(\theta)}{\big(1 - y g(\theta)\big)^\alpha} d\theta \text{ with } g(\theta) = -(p-1)(1 + \cos\theta) \text{ and } \alpha = \frac{q-p}{p-1}. \tag{13}$$

Related to this we also define

$$\mathcal{C}_k(y) := \int_{-\pi}^{\pi} \frac{\cos(k\theta)}{\big(1 + y(p-1)(1 + \cos\theta)\big)^{\frac{q-p}{p-1}}} d\theta. \tag{14}$$

As will be clear in a moment, these integrals constitute the building blocks for the Melnikov functions of the perturbed differential equation (3). In what follows recall that κ_p is given in (5).

Proposition 4.1. $\big(1, \rho^{q-p} \mathcal{I}_0(\rho^{p-1}), \dots, \rho^{q-p} \mathcal{I}_n(\rho^{p-1})\big)$ *is an ECT-system on the open intervals* $(\kappa_p, 0)$ *and* $(0, +\infty)$ *for every* $n \in \mathbb{Z}_{\geq 0}$. *In addition the following equality between linear spans holds:*

$$\langle 1, \rho^{q-p} \mathcal{I}_0(\rho^{p-1}), \dots, \rho^{q-p} \mathcal{I}_n(\rho^{p-1}) \rangle = \langle 1, \rho^{q-p} \mathcal{C}_0(\rho^{p-1}), \dots, \rho^{q-p} \mathcal{C}_n(\rho^{p-1}) \rangle.$$

Proof. Note (see (12) and (13)), that $\mathcal{I}_k(y)$ is $\mathcal{T}_{k,\alpha}(y)$ particularized with $g(\theta) = -(p-1)(1+\cos\theta)$ and $\alpha = \frac{q-p}{p-1}$. One can readily see that in this case the connected component of

$$\big\{ y \in \mathbb{R} : 1 - y g(\theta) > 0 \text{ for all } \theta \in [-\pi, \pi] \big\}$$

containing the origin turns out to be $I_g = (\frac{-1}{2(p-1)}, +\infty)$. Moreover, since $\frac{q-p}{p-1} \in (-1,+\infty)\setminus\{0\}$ due to $p, q \in \mathbb{Z}_{\geq 2}$ with $p \neq q$, we have that $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq -1}$. Consequently, by applying Lemma 3.6,

(i) $\big(1, y^\alpha \mathcal{I}_0(y), \dots, y^\alpha \mathcal{I}_n(y)\big)$ is an ECT-system on $(0,+\infty)$, and
(ii) $\big(1, (-y)^\alpha \mathcal{I}_0(y), \dots, (-y)^\alpha \mathcal{I}_n(y)\big)$ is an ECT-system on $(\frac{-1}{2(p-1)}, 0)$.

Next we make the substitution $y = \rho^{p-1}$. This formally corresponds to composing each function with $\rho \mapsto \rho^{p-1}$, which restricted to $(-\infty, 0)$ and $(0,+\infty)$ is a diffeomorphism. In the first case we get that

$$\big(1, \rho^{q-p}\mathcal{I}_0(\rho^{p-1}), \dots, \rho^{q-p}\mathcal{I}_n(\rho^{p-1})\big) \tag{15}$$

is an ECT-system on $(0,+\infty)$ for every p and that it is an ECT-system on $(-\infty, 0)$ in the case that p is odd. In the second case we obtain that

$$\big(1, (-\rho)^{q-p}\mathcal{I}_0(\rho^{p-1}), \dots, (-\rho)^{q-p}\mathcal{I}_n(\rho^{p-1})\big)$$

is an ECT-system on $(-(2(p-1))^{-\frac{1}{p-1}}, 0)$ if p is even. On account of the fact that $(-1)^{q-p}$ is constant this implies that (15) is an ECT-system on $(-(2(p-1))^{-\frac{1}{p-1}}, 0)$ if p is even. Accordingly, on account of the definition of κ_p given in (5), we have so far proved the validity of the first assertion in the statement.

Finally, the assertion concerning the linear spans follows by noting that if we define $E_k := \langle 1, \cos\theta, \dots, \cos(k\theta)\rangle$ then $\dim(E_k) = k+1$ and

$$E_k = \langle 1, \cos\theta, \dots, \cos^k\theta\rangle = \langle 1, g(\theta), \dots, g^k(\theta)\rangle.$$

To get the first equality one can use that $\cos(k\theta) = T_k(\cos\theta)$, where T_k is the Chebyshev polynomial of the first kind and degree k. This proves the result. ∎

The next result provides a more explicit expression of the Melnikov functions M_1 and M_2 for (3). In its statement we point out that a_{ik}, b_{ik}, c_{ik}, and d_{ik} are the coefficients of the trigonometric polynomials in the perturbation and $\mathcal{C}_k$ is the function defined in (14).

Proposition 4.2. *The following holds for the perturbed differential equation* (3).

(a) *The first Melnikov function is given by* $M_1(\rho) = 2\pi b_{10}\rho^p + \sum_{k=0}^m d_{1k}\rho^q \mathcal{C}_k\big(\rho^{p-1}\big)$. *Moreover,* $M_1 = 0$ *if and only if* $b_{10} = d_{10} = \cdots = d_{1m} = 0$.
(b) *If* $M_1 = 0$ *then the second Melnikov function is given by* $M_2(\rho) = M_{21}(\rho) + (q-p)\rho^p M_{22}(\rho)$, *where*

$$M_{21}(\rho) := 2\pi b_{20}\rho^p + \sum_{k=0}^m d_{2k}\rho^q \mathcal{C}_k(\rho^{p-1})$$

and

$$M_{22}(\rho) := \sum_{\substack{1\leq k\leq n\\ 1\leq l\leq m}} \frac{b_{1k}c_{1l}}{k}\int_{-\pi}^{\pi} \sin(k\theta)\sin(l\theta)x_0(\theta,\rho)^{q-1}d\theta.$$

Proof. By applying Corollary 2.3, we get that the first Melnikov function is written as

$$M_1(\rho) = 2\pi b_{10}\rho^p + \rho^p \sum_{k=0}^{m} d_{1k} \int_{-\pi}^{\pi} \cos(k\theta) x_0(\theta,\rho)^{q-p}\, d\theta = 2\pi b_{10}\rho^p + \sum_{k=0}^{m} d_{1k}\rho^q \mathcal{C}_k\big(\rho^{p-1}\big),$$

where in the first equality we use that the solution $x_0(\theta,\rho)$ of the unperturbed equation is even in θ, whereas in the second one we take (14) into account. Then the second assertion in (a) follows by Proposition 4.1, which shows that the ordered set $\big(\rho^p, \rho^q\mathcal{C}_0(\rho^{p-1}), \dots, \rho^q\mathcal{C}_n(\rho^{p-1})\big)$ is an ECT-system on $(\kappa_p, 0)$ and $(0, +\infty)$.

Let us turn next to the proof of (b). If $M_1 = 0$ then, by Corollary 2.3 again, the second Melnikov function is written as

$$M_2(\rho) = \rho^p \int_{-\pi}^{\pi} \big(P_2(\theta) + Q_2(\theta)x_0(\theta,\rho)^{q-p} + (q-p)Q_1(\theta)S(\theta,\rho)x_0(\theta,\rho)^{q-1}\big)\, d\theta,$$

where

$$S(\theta,\rho) = \int_{-\pi}^{\theta} \big(P_1(s) + Q_1(s)x_0(s,\rho)^{q-p}\big)\, ds.$$

That the first summand in M_2 is written as $\rho^p \int_{-\pi}^{\pi} (P_2(\theta) + Q_2(\theta)x_0(\theta,\rho)^{q-p})\, d\theta = M_{21}(\rho)$ can be shown exactly as we did in (a). With regard to the second summand note that, on account of (a),

$$P_1(\theta) = \sum_{k=1}^{n} (a_{1k}\sin(k\theta) + b_{1k}\cos(k\theta)) \text{ and } Q_1(\theta) = \sum_{l=1}^{m} c_{1l}\sin(l\theta).$$

Thus, $Q_1(\theta)x_0(\theta,\rho)^{q-1}$ and $\int_{-\pi}^{\theta} Q_1(s)x_0(s,\rho)^{q-p}ds$ are odd and even functions in θ, respectively. Hence the second summand in M_2 is written as

$$\begin{aligned}
\int_{-\pi}^{\pi} Q_1(\theta)S(\theta,\rho)x_0(\theta,\rho)^{q-1}d\theta &= \int_{-\pi}^{\pi} Q_1(\theta)x_0(\theta,\rho)^{q-1}\left(\int_{-\pi}^{\theta} P_1(s)ds\right) d\theta \\
&= \int_{-\pi}^{\pi} Q_1(\theta)x_0(\theta,\rho)^{q-1}\left(\sum_{k=1}^{n} \frac{b_{1k}}{k}\sin(k\theta)\right) d\theta \\
&= \sum_{\substack{1\le k\le n\\ 1\le l\le m}} \frac{b_{1k}c_{1l}}{k} \int_{-\pi}^{\pi} \sin(k\theta)\sin(l\theta)x_0(\theta,\rho)^{q-1}d\theta,
\end{aligned}$$

which is equal to $M_{22}(\rho)$. This completes the proof. ∎

As is clear from the previous result, the function $\mathcal{S}_{kl}$ that we introduce in the next statement is a building block of the second Melnikov function.

Proposition 4.3. *For each $r \in \mathbb{Z}_{\ge 0}$, define $\mathscr{B}_r := \langle \rho^{q-p}\mathcal{C}_0\big(\rho^{p-1}\big), \dots, \rho^{q-p}\mathcal{C}_r\big(\rho^{p-1}\big)\rangle$. Then*

$$\mathcal{S}_{kl}(\rho) := \int_{-\pi}^{\pi} \sin(k\theta)\sin(l\theta)x_0(\theta,\rho)^{q-1}d\theta$$

belongs to $\mathscr{B}_{k+l-1}$ for all $k, l \in \mathbb{Z}_{\ge 1}$. Furthermore,

$$\mathscr{B}_{k+l-1} = \langle \rho^{q-p}\mathcal{C}_0\big(\rho^{p-1}\big), \dots, \rho^{q-p}\mathcal{C}_k\big(\rho^{p-1}\big), \mathcal{S}_{k2}(\rho), \dots, \mathcal{S}_{kl}(\rho)\rangle$$

for all $k \ge 0$ and $l \ge 2$.

Proof. In order to prove the first assertion we can suppose without loss of generality that $k \geq l$. It is well known that $\cos(i\theta) = T_i(\cos\theta)$ and $\sin((i+1)\theta) = \sin\theta U_i(\cos\theta)$, where T_i and U_i are the ith degree Chebyshev polynomials of the first and second kind, respectively. (The reader is referred to [35] for the formulas relating the Chebyshev polynomials that we shall use hereafter.) Thus

$$\begin{aligned} \mathcal{S}_{kl}(\rho) &= \int_{-\pi}^{\pi} U_{k-1}(\cos\theta)U_{l-1}(\cos\theta)\sin^2\theta\, x_0(\theta,\rho)^{q-1}d\theta \\ &= \frac{1}{p-q}\int_{-\pi}^{\pi}\big(U_{k-1}(\cos\theta)U_{l-1}(\cos\theta)\sin\theta\big)' x_0(\theta,\rho)^{q-p}d\theta, \end{aligned} \tag{16}$$

where the second equality follows using that $x_0(\theta,\rho)$ is the solution of the unperturbed differential equation (3) to perform an integration by parts. Since

$$U_{k-1}(x)U_{l-1}(x) = \sum_{r=0}^{l-1} U_{k-l+2r}(x)$$

and, thanks to $(x^2-1)U_r'(x) = (r+1)T_{r+1}(x) - xU_r(x)$,

$$\big(U_r(\cos\theta)\sin\theta\big)' = (r+1)T_{r+1}(\cos\theta) = (r+1)\cos((r+1)\theta),$$

we get that

$$\big(U_{k-1}(\cos\theta)U_{l-1}(\cos\theta)\sin\theta\big)' = \sum_{\substack{r=k-l+1\\ \text{step } 2}}^{k+l-1} r\cos(r\theta).$$

On account of (14), the substitution of this identity in (16) yields

$$\mathcal{S}_{kl}(\rho) = \frac{\rho^{q-p}}{p-q}\sum_{\substack{r=k-l+1\\ \text{step } 2}}^{k+l-1} r\,\mathcal{C}_r(\rho^{p-1}). \tag{17}$$

Accordingly the first assertion is true. With regard to the second one, it suffices to show that

$$\rho^{q-p}\mathcal{C}_{k+r}\big(\rho^{p-1}\big) \in \big\langle \rho^{q-p}\mathcal{C}_0\big(\rho^{p-1}\big),\dots,\rho^{q-p}\mathcal{C}_k\big(\rho^{p-1}\big),\mathcal{S}_{k2}(\rho),\dots,\mathcal{S}_{kl}(\rho)\big\rangle$$

for all $r = 1,2,\dots,l-1$, which can be proved by induction on r taking (17) into account again. This completes the proof of the result. ∎

In what follows we will need to further emphasize the dependence of the Melnikov functions on the perturbative parameters. For this reason we use the notation $\mu = (\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ as introduced in (4), so that $\mu \in \mathbb{R}^{4(n+m+1)}$, and we denote the ith order Melnikov function associated with the perturbed differential equation given in (3) by $M_i(\rho;\mu)$.

Proposition 4.4. *Setting $\mathscr{L} = \{\mu \in \mathbb{R}^{4(n+m+1)} : M_1(\rho;\mu) = 0$ for all $\rho \in I\}$, there exists a surjective map $\beta : \mathscr{L} \longrightarrow \mathbb{R}^{n+m+1}$ such that*

$$M_2(\rho;\mu) = \beta_1(\mu)\rho^p + \rho^q\sum_{k=0}^{n+m-1}\beta_{k+2}(\mu)\mathcal{I}_k(\rho^{p-1}).$$

Proof. We claim that there exists a surjective map $\hat{\beta}:\mathscr{L}\longrightarrow\mathbb{R}^{n+m+1}$ such that

$$M_2(\rho;\mu)=\hat{\beta}_1(\mu)\rho^p+\rho^q\sum_{k=0}^{n+m-1}\hat{\beta}_{k+2}(\mu)\mathcal{C}_k(\rho^{p-1}).$$

The result will follow once we prove this because, by Proposition 4.1, we know that the linear spans $\langle 1,\rho^{q-p}\mathcal{I}_0(\rho^{p-1}),\dots,\rho^{q-p}\mathcal{I}_k(\rho^{p-1})\rangle$ and $\langle 1,\rho^{q-p}\mathcal{C}_0(\rho^{p-1}),\dots,\rho^{q-p}\mathcal{C}_k(\rho^{p-1})\rangle$ are equal and have dimension $k+2$ for all $k\in\mathbb{Z}_{\geq 0}$. In order to prove the claim let us fix any $\mu=(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})\in\mathscr{L}$. Then, by applying Proposition 4.2,

$$M_2(\rho;\mu)=2\pi b_{20}\rho^p+\sum_{k=0}^{m}d_{2k}\rho^q\mathcal{C}_k(\rho^{p-1})+(q-p)\rho^p\sum_{\substack{1\leq k\leq n\\1\leq l\leq m}}\frac{b_{1k}c_{1l}}{k}\mathcal{S}_{kl}(\rho). \tag{18}$$

By the first assertion in Proposition 4.3, for all $k=1,2,\dots,n$ and $l=1,2,\dots,m$,

$$\mathcal{S}_{kl}(\rho)\in\langle\rho^{q-p}\mathcal{C}_0\big(\rho^{p-1}\big),\dots,\rho^{q-p}\mathcal{C}_{n+m-1}\big(\rho^{p-1}\big)\rangle.$$

On account of (18), this shows that $M_2(\rho;\mu)\in\langle\rho^p,\rho^q\mathcal{C}_0\big(\rho^{p-1}\big),\dots,\rho^q\mathcal{C}_{n+m-1}\big(\rho^{p-1}\big)\rangle$, which has dimension $n+m+1$ by Proposition 4.1. Hence there exists a unique $\hat{\beta}=\hat{\beta}(\mu)\in\mathbb{R}^{n+m+1}$ such that

$$M_2(\rho;\mu)=\hat{\beta}_1\rho^p+\rho^q\sum_{k=0}^{n+m-1}\hat{\beta}_{k+2}\mathcal{C}_k(\rho^{p-1}).$$

It only remains to be proved that $\hat{\beta}:\mathscr{L}\longrightarrow\mathbb{R}^{n+m+1}$ is a surjective map. To this end it suffices to verify that $\mathbb{R}^{n+m+1}=\hat{\beta}(\mathscr{L}\cap\mathscr{P})$, where

$$\mathscr{P}:=\{\mu\in\mathbb{R}^{4(n+m+1)}:b_{11}=0,c_{1l}=0,\ \text{for } l=1,\dots,m-1,\ \text{and } c_{1m}=1\}.$$

Indeed, if $\mu\in\mathscr{L}\cap\mathscr{P}$ then from (18) we get that

$$M_2(\rho;\mu)=2\pi b_{20}\rho^p+\sum_{k=0}^{m}d_{2k}\rho^q\mathcal{C}_k(\rho^{p-1})+(q-p)\rho^p\sum_{k=2}^{n}\frac{b_{1k}}{k}\mathcal{S}_{km}(\rho).$$

Hence the inclusion $\mathbb{R}^{n+m+1}\subset\hat{\beta}(\mathscr{L}\cap\mathscr{P})$ follows from the second assertion in Proposition 4.3. Since the other inclusion is clear, this proves the validity of the claim and so the result follows. ∎

5. Proof of the main results.

This section is devoted to proving Theorems 1.1 and 1.2. Let us advance that in order to complete the proof of the latter we will need some additional results that for the sake of simplicity in the exposition we gather in the appendix.

Proof of Theorem 1.1. The fact that $M_1(\,\cdot\,;\mu)\equiv 0$ if and only if $b_{10}=d_{10}=\cdots=d_{1m}=0$ follows from (a) in Proposition 4.2. By applying Proposition 4.4, there exists a surjective map β such that if $\mu\in\mathscr{L}$ then

$$M_2(\rho;\mu)=\rho^p\widetilde{M}_2(\rho;\mu)\ \text{with}\ \widetilde{M}_2(\rho;\mu):=\beta_1(\mu)+\rho^{q-p}\sum_{k=0}^{n+m-1}\beta_{k+2}(\mu)\mathcal{I}_k(\rho^{p-1}). \tag{19}$$

It is clear that M_2 and the rescaled $\widetilde{M}_2$ have the same number of positive and negative zeros counted with multiplicities. For the sake of convenience we also define

$$N(\rho;\mu) := \rho^{q-p} \sum_{k=0}^{n+m-1} \beta_{k+2}(\mu)\mathcal{I}_k(\rho^{p-1}).$$

Recall (see (12) and (13)), that $\mathcal{I}_k(y)$ is equal to $\mathcal{T}_{k,\alpha}(y)$ particularized with $g(\theta) = -(p-1)(1+\cos\theta)$ and $\alpha = \frac{q-p}{p-1}$. By abusing notation, in what follows we will write $\mathcal{I}_k(y) = \mathcal{T}_{k,\alpha}(y)$ for simplicity. Taking this into account, by applying Lemma 3.5, we get that the derivative of N is

$$N'(\rho;\mu) = (q-p)\rho^{q-p-1} \sum_{k=0}^{n+m-1} \beta_{k+2}(\mu)\mathcal{T}_{k,\alpha+1}(\rho^{p-1}). \tag{20}$$

In addition, the application of Theorem 3.4 to these $\mathcal{T}_{k,\alpha}$ and $\mathcal{T}_{k,\alpha+1}$ easily shows the following:

Claim 1: N has at most $n+m-1$ zeros in $(\kappa_p, 0)$ counted with multiplicities. The same is true in the interval $(0,+\infty)$.

Claim 2: N' has at most $n+m-1$ zeros in $(\kappa_p, 0)$ counted with multiplicities. The same is true in the interval $(0,+\infty)$.

Claim 3: In the case that p is even, N has at most $n+m-1$ zeros in $(\kappa_p,+\infty)\setminus\{0\}$ counted with multiplicities. The same is true for N'.

Above we have omitted the dependence on μ for the sake of shortness. We will also do it in what follows when there is no risk of confusion. Let us prove next each one of the assertions in the statement of the result:

(a) From (19), by applying Proposition 4.1, we get that $K^{\pm} \le n+m$. In addition, thanks to the surjectivity of β, we can assert that there exists $\mu_0 \in \mathbb{R}^{4(n+m+1)}$ such that $\widetilde{M}_2(\rho;\mu_0)$ has exactly $n+m$ positive simple zeros. Since $\widetilde{M}_2(\rho;\mu_0)$ is an even function in ρ in the case that p and q are odd, we can conclude that it has the same number of negative simple zeros.

(b) Exactly as before, $K^{\pm} \le n+m$. We prove by contradiction that the equalities cannot hold simultaneously. So suppose that M_2 has $n+m$ positive zeros, say $0 < \rho_1^+ \le \cdots \le \rho_{n+m}^+$, and $n+m$ negative zeros, say $0 > -\rho_1^- \ge \cdots \ge -\rho_{n+m}^-$, counted with multiplicities. Note that $N(\rho)$ is an odd function because $q-p$ is odd and $p-1$ is even. Therefore

$$N(\rho_i^+) = \widetilde{M}_2(\rho_i^+) - \beta_1 = -\beta_1 \text{ and } N(\rho_j^-) = -\widetilde{M}_2(-\rho_j^-) + \beta_1 = \beta_1 \tag{21}$$

for all $i,j \in \{1,2,\ldots,n+m\}$. This shows in particular that $\beta_1 \ne 0$ (otherwise we get a contradiction with Claim 1), which in turn implies that $\rho_i^+ \ne \rho_j^-$ for all i and j.

On the other hand, $N' = \widetilde{M}_2'$ has $n+m-1$ positive zeros, say $0 < \varrho_1^+ \le \cdots \le \varrho_{n+m-1}^+$, and $n+m-1$ negative zeros, say $0 > -\varrho_1^- \ge \cdots \ge -\varrho_{n+m-1}^-$, counted with multiplicities, satisfying $\varrho_i^{\pm} \in [\rho_i^{\pm}, \rho_{i+1}^{\pm}]$ for all i. Since N' is even, taking Claim 2 into account we can assert that $\varrho_i^+ = \varrho_i^-$ for all i. In particular

$$[\rho_i^+, \rho_{i+1}^+] \cap [\rho_i^-, \rho_{i+1}^-] \ne \emptyset \text{ for all } i. \tag{22}$$

For each $i = 1, 2, \ldots, n+m$, let J_i be the open interval with endpoints ρ_i^+ and ρ_i^-. Then, on account of (22), it is easy to show that $J_1, J_2, \ldots, J_{n+m}$ are pairwise nonintersecting. Since, due to (21), each J_i contains at least one zero of N, this contradicts Claim 1. Accordingly, either $K^+ < n+m$ or $K^- < n+m$.

(i) By Theorem 3.4, we have that

$$\big(\rho^{q-p}\mathcal{I}_0(\rho^{p-1}), \rho^{q-p}\mathcal{I}_1(\rho^{p-1}), \ldots, \rho^{q-p}\mathcal{I}_{n+m-1}(\rho^{p-1})\big)$$

is an ECT-system on $(0, +\infty)$. Thus there exist $\beta_2^*, \ldots, \beta_{n+m+1}^* \in \mathbb{R}$ such that the function

$$\rho \longmapsto \rho^{q-p} \sum_{k=0}^{n+m-1} \beta_{k+2}^* \mathcal{I}_k(\rho^{p-1})$$

has $n+m-1$ positive simple zeros and $n+m-1$ negative simple zeros. Here we use that the above function is odd due to the parity assumption on p and q. For the same reason, $\rho = 0$ is a zero with odd multiplicity. Hence, taking $\beta_1^* \approx 0$, the function

$$\rho \longmapsto \beta_1^* + \rho^{q-p} \sum_{k=0}^{n+m-1} \beta_{k+2}^* \mathcal{I}_k(\rho^{p-1})$$

has $2(n+m)-1$ simple zeros on $(\kappa_p, +\infty) \setminus \{0\}$. We use at this point the surjectivity of $\mu \mapsto \beta(\mu)$ to choose some $\mu_0 \in \mathscr{L}$ such that $\beta(\mu_0) = (\beta_1^*, \beta_2^*, \ldots, \beta_{n+m+1}^*)$. Then by construction (see (19)), $M_2(\rho; \mu_0)$ has $2(n+m)-1$ simple zeros on $(\kappa_p, +\infty) \setminus \{0\}$.

(ii) Exactly the same argument as before, but choosing $\beta_1^* = 0$ guarantees the existence of $\mu_0 \in \mathscr{L}$ such that $M_2(\rho; \mu_0)$ has $2(n+m-1)$ simple zeros on $(\kappa_p, +\infty) \setminus \{0\}$.

(c) Let $\rho_1^+ \leq \cdots \leq \rho_{K^+}^+$ and $\rho_1^- \leq \cdots \leq \rho_{K^-}^-$ be the zeros of $\widetilde{M}_2(\rho; \mu)$ on $(0, +\infty)$ and $(\kappa_p, 0)$, respectively. Then $\widetilde{M}_2' = N'$ has at least $K^+ - 1$ zeros on $[\rho_1^+, +\infty)$ and at least $K^- - 1$ zeros on $(\kappa_p, \rho_1^-]$, counted with multiplicities. Since p is even, it is clear (cf. (20)), that

$$\mathcal{E}(y; \mu) := \sum_{k=0}^{n+m-1} \beta_{k+2}(\mu) \mathcal{T}_{k,\alpha+1}(y)$$

has at least $K^+ - 1$ zeros on $\big[(\rho_1^+)^{p-1}, +\infty\big)$ and at least $K^- - 1$ zeros on the interval $(\frac{-1}{2(p-1)}, (\rho_1^-)^{p-1}]$, counted with multiplicities. (Here we use that $\kappa_p^{p-1} = \frac{-1}{2(p-1)}$.) By Theorem 3.4,

$$(\mathcal{T}_{0,\alpha+1}, \mathcal{T}_{1,\alpha+1}, \ldots, \mathcal{T}_{n+m-1,\alpha+1})$$

is an ECT-system on $(\frac{-1}{2(p-1)}, +\infty)$. Hence $\mathcal{E}$ can have at most $n+m-1$ zeros counted with multiplicities in this interval. If $\mathcal{E}(0) = 0$ then $\mathcal{E}$ has at least $K^+ + K^- - 1$ zeros counted with multiplicites in this interval. Consequently $K^+ + K^- \leq n+m$. If $\mathcal{E}(0) \neq 0$ then the multiplicity of $\widetilde{M}_2' = N'$ at $\rho = 0$ is exactly $q-p-1$, which is an even number by assumption. On account of this, and the fact that $M_2(\rho_1^\pm) = 0$

with $\rho_1^- < 0 < \rho_1^+$, one can easily conclude the existence of a zero of $\widetilde{M}_2' = N'$ on $(\rho_1^-, \rho_1^+) \setminus \{0\}$. Thus N' has at least $n+m-1$ zeros on $(\kappa_p, +\infty) \setminus \{0\}$ counted with multiplicities. By Claim 3 we get the upper bound $K^+ + K^- \leq n+m$ also in this case.

Exactly as we did in the previous cases, Proposition 4.1 and the surjectivity of $\mu \longmapsto \beta(\mu)$ ensure the existence of some $\mu_0 \in \mathscr{L}$ such that $M_2(\,\cdot\,;\mu_0)$ has $n+m$ simple zeros in, for instance, the interval $(0, +\infty)$.

(d) The number of zeros of $N' = \widetilde{M}_2'$ on $(\kappa_p, +\infty) \setminus \{0\}$ counted with multiplicities is at least $K^+ + K^- - 2$. Then, by Claim 3 again, we get $K^+ + K^- \leq n+m+1$. In order to prove that this upper bound is sharp we will use that, by Theorem 3.4, $\big(\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_{n+m-1}\big)$ is an ECT-system on $(\frac{-1}{2(p-1)}, +\infty)$ and we consider two cases:

Case 1: q even and $p < q$. We take $\beta_2^*, \ldots, \beta_{n+m+1}^* \in \mathbb{R}$ such that

$$\mathcal{F}(\rho) := \sum_{k=0}^{n+m-1} \beta_{k+2}^* \mathcal{I}_k(\rho^{p-1}) \tag{23}$$

vanishes at $0 < \rho_1 < \rho_2 < \cdots < \rho_{n+m-1}$ with multiplicity one and satisfies $\mathcal{F}(0) \neq 0$. Thus $\rho^{q-p}\mathcal{F}(\rho)$ has a zero at $\rho = 0$ of multiplicity $q-p$, which is an even number, and vanishes with multiplicity one at ρ_i, $i = 1, 2, \ldots, n+m-1$. Consequently we can choose β_1^* small enough such that $\beta_1^* + \rho^{q-p}\mathcal{F}(\rho)$ has $n+m-1$ simple zeros near $\rho_1 < \rho_2 < \cdots < \rho_{n+m-1}$, together with one positive and one negative zero near $\rho = 0$, both being simple as well. Now, as we did before, we use the surjectivity of $\mu \longmapsto \beta(\mu)$ to choose some $\mu_0 \in \mathscr{L}$ such that $\beta(\mu_0) = (\beta_1^*, \beta_2^*, \ldots, \beta_{n+m+1}^*)$. Then by construction (see (19)), $M_2(\rho;\mu_0)$ has $n+m+1$ simple zeros on $(\kappa_p, +\infty) \setminus \{0\}$ as desired.

Case 2: $p > q$. In this case we choose $\beta_2^*, \ldots, \beta_{n+m+1}^* \in \mathbb{R}$ such that

$$\sum_{k=0}^{n+m-1} \beta_{k+2}^* \mathcal{I}_k(y)$$

has exactly $n+m-1$ simple zeros in $(\frac{-1}{2(p-1)}, +\infty)$, one of them being $y = 0$ and the other ones positive. Thus the function $\mathcal{F}(\rho)$ in (23) has exactly $n+m-2$ positive simple zeros, say $\rho_2 < \rho_3 < \cdots < \rho_{n+m-1}$, and there exists an analytic function η such that

$$\mathcal{F}(\rho) = \sum_{k=0}^{n+m-1} \beta_{k+2}^* \mathcal{I}_k(\rho^{p-1}) = \eta(\rho)\rho^{p-1} \text{ with } \eta(0) \neq 0.$$

Let us suppose, without loss of generality, that $\eta(0) > 0$. Then, since p is even, there exists $\hat{\rho} \in (0, \rho_2)$ such that $\rho\mathcal{F}(\rho) > 0$ for all $\rho \in (-\hat{\rho}, \hat{\rho}) \setminus \{0\}$. Let us define at this point

$$\widehat{M}_2(\rho; \beta_1, \sigma) := \eta(\rho)\rho^{p-1} + \beta_1\rho^{p-q} + \sigma\mathcal{I}_0(\rho^{p-1})$$

and split the proof into two subcases depending on the parity of q.

Subcase 2a: q odd. In this case $\widehat{M}_2(\rho; \beta_1, 0) = \rho^{p-q}\big(\eta(\rho)\rho^{q-1} + \beta_1\big)$ with $p-q$ odd and $q-1 > 0$ even. Hence, by continuity, there exist small enough $\beta_1^* < 0$ and

$\tilde{\rho} \in (0, \hat{\rho})$ such that $\widehat{M}_2(\rho; \beta_1^*, 0)$ has $n+m-2$ positive simple zeros near $\rho_2, \ldots, \rho_{n+m-1}$ and, moreover,

$$\widehat{M}_2(-\hat{\rho}; \beta_1^*, 0) < 0,\ \widehat{M}_2(-\tilde{\rho}; \beta_1^*, 0) > 0,\ \widehat{M}_2(\tilde{\rho}; \beta_1^*, 0) < 0,\ \widehat{M}_2(\hat{\rho}; \beta_1^*, 0) > 0.$$

Note also (see (13)), that $\mathcal{I}_0(\rho^{p-1}) > 0$ for all ρ. Thus, by continuity, we can take $\sigma^* > 0$ small enough, such that $\widehat{M}_2(\rho; \beta_1^*, \sigma^*)$ still has $n+m-2$ positive simple zeros near $\rho_2, \ldots, \rho_{n+m-1}$ and also verifies $\widehat{M}_2(0; \beta_1^*, \sigma^*) > 0$, together with

$$\widehat{M}_2(-\hat{\rho}; \beta_1^*, \sigma^*) < 0,\ \widehat{M}_2(-\tilde{\rho}; \beta_1^*, \sigma^*) > 0,\ \widehat{M}_2(\tilde{\rho}; \beta_1^*, \sigma^*) < 0,\ \widehat{M}_2(\hat{\rho}; \beta_1^*, \sigma^*) > 0.$$

Therefore $\widehat{M}_2(\rho; \beta_1^*, \sigma^*)$ vanishes at least once in each interval $(-\hat{\rho}, -\tilde{\rho})$, $(0, \tilde{\rho})$, and $(\tilde{\rho}, \hat{\rho})$. So the total number of zeros of $\widehat{M}_2(\rho; \beta_1^*, \sigma^*)$ on $(\kappa_p, +\infty) \setminus \{0\}$ is at least $n+m+1$. On account of the surjectivity of $\beta : \mathscr{L} \longrightarrow \mathbb{R}^{n+m+1}$, we can take $\mu_0 \in \mathscr{L}$ such that

$$\beta(\mu_0) = (\beta_1^*, \beta_2^* + \sigma^*, \beta_3^*, \ldots, \beta_{n+m-1}^*).$$

Hence, by construction, $M_2(\rho; \mu_0) = \rho^q \widehat{M}_2(\rho; \beta_1^*, \sigma^*)$ has at least $n+m+1$ zeros on $(\kappa_p, +\infty) \setminus \{0\}$. Finally, since we have already proved that $K^+ + K^- \leq n+m+1$, these zeros must be simple, as desired.

Subcase 2b: q even. In this case $\widehat{M}_2(\rho; \beta_1, 0) = \rho^{p-q}\big(\eta(\rho)\rho^{q-1} + \beta_1\big)$ with $p-q$ even and $q-1 > 0$ odd. Then, by continuity, we can take small enough $\beta_1^* < 0$ and $\tilde{\rho} \in (0, \hat{\rho})$ such that $\widehat{M}_2(\rho; \beta_1^*, 0)$ has $n+m-2$ positive simple zeros near $\rho_2, \ldots, \rho_{n+m-1}$ and

$$\widehat{M}_2(\hat{\rho}; \beta_1^*, 0) > 0,\ \ \widehat{M}_2(-\hat{\rho}; \beta_1^*, 0) < 0,\ \ \widehat{M}_2(\tilde{\rho}; \beta_1^*, 0) < 0.$$

Due to $\mathcal{I}_0(\rho^{p-1}) > 0$ for all ρ, taking $\sigma^* > 0$ small enough, $\widehat{M}_2(\rho; \beta_1^*, \sigma^*)$ still has $n+m-2$ positive simple zeros near $\rho_2, \ldots, \rho_{n+m-1}$ and, additionally,

$$\widehat{M}_2(\hat{\rho}; \beta_1^*, \sigma^*) > 0,\ \ \widehat{M}_2(-\hat{\rho}; \beta_1^*, \sigma^*) < 0,\ \ \widehat{M}_2(\tilde{\rho}; \beta_1^*, \sigma^*) < 0,\ \ \widehat{M}_2(0; \beta_1^*, \sigma^*) > 0.$$

Thus $\widehat{M}_2(\rho; \beta_1^*, \sigma^*)$ vanishes at least once in each interval $(-\hat{\rho}, 0)$, $(0, \tilde{\rho})$, and $(\tilde{\rho}, \hat{\rho})$. Exactly as we did in the previous subcase, there exists $\mu_0 \in \mathscr{L}$ such that $M_2(\rho; \mu_0) = \rho^q \widehat{M}_2(\rho; \beta_1^*, \sigma^*)$ has exactly $n+m+1$ simple zeros on $(\kappa_p, +\infty) \setminus \{0\}$. ■

Remark 5.1. *If we only consider K^+ in Theorem 1.1, then by (19) and Proposition 4.1, $K^+ \leq n+m$ always holds and the upper bound can be achieved.*

Proof of Theorem 1.2. Taking $p = 3$ and $q = 2$, from point (ii) in assertion (b) of Theorem 1.1 we know that there exists $\mu_0 \in \mathbb{R}^{4(n+m+1)}$ with $M_1(\rho; \mu_0) = 0$ for all $\rho \in I$ and such that $M_2(\rho; \mu_0)$ has $2(n+m-1)$ simple zeros in $I \setminus \{0\}$. Hence, due to $x(\pi, \rho; \mu_0, \varepsilon) = \rho + \varepsilon^2 M_2(\rho; \mu_0) + o(\varepsilon^2)$, by applying the implicit function theorem we can assert that $\rho \mapsto x(\pi, \rho; \mu_0, \varepsilon)$ has at least $2(n+m-1)$ fixed points in $I \setminus \{0\}$ for all $\varepsilon \approx 0$. Since $x(\pi, 0; \mu_0, \varepsilon) = 0$ for all ε, the first assertion follows. With regard to the second assertion, the bound $\mathcal{H}_{3,2}(1,3) \geq 8$ is proved in Proposition A.2, whereas $\mathcal{H}_{3,2}(4,1) \geq 10$ is proved in Proposition A.3. ■

Appendix A. Improvements using Lyapunov constants. Our goal in this appendix is to use Lyapunov constants in order to improve the general lower bound $\mathcal{H}_{3,2}(n,m) \geq 2(n+m)-1$

in the case that $(n,m) \in \{(1,3),(4,1)\}$. At the end we shall discuss the difficulties we have found to tackle two other particular cases using the same approach.

In the last two decades, there have been several works about the Hilbert number $\mathcal{H}_{3,2}(n,m)$. Unfortunately the problem is far from being solved even for the case $n=m=1$. We gather in the following theorem the main results obtained in [6].

Theorem A.1. *For any nonnegative integers n and m, we have $\mathcal{H}_{3,2}(n,0)=\mathcal{H}_{3,2}(0,m)=2$, $\mathcal{H}_{3,2}(n,1)\geq n+2$, and $\mathcal{H}_{3,2}(1,m)\geq 2m+1$. Moreover, $\mathcal{H}_{3,2}(3,1)\geq 7$ and $\mathcal{H}_{3,2}(2,2)\geq 7$.*

As it occurs with our proof of Theorem 1.2, the general lower bounds in the above result follow by using Melnikov functions, whereas the improvements for the particular cases follow by means of Lyapunov constants, that enable us to study those limit cycles bifurcating from $\rho=0$. With this aim let us consider the differential equation $\dot{x}=A(\theta)x^3+B(\theta)x^2$ and write its trigonometric polynomials as

$$A(\theta)=b_0+\sum_{k=1}^{n}\big(a_k\sin(k\theta)+b_k\cos(k\theta)\big) \text{ and } B(\theta)=d_0+\sum_{k=1}^{m}\big(c_k\sin(k\theta)+d_k\cos(k\theta)\big). \tag{24}$$

If $x(\theta,\rho)$ denotes the solution with initial condition $x(0,\rho)=\rho$ then the hyperbolic limit cycles near $\rho=0$ can be viewed as simple zeros of the displacement map

$$\Delta(\rho):=\frac{x(2\pi,\rho)-\rho}{2\pi}=\sum_{j=2}^{\infty}X_j\rho^j. \tag{25}$$

The coefficients X_j of the Taylor development of Δ at $\rho=0$ are polynomial on a_k,b_k,c_k,d_k and the first X_j which is not identically zero is called the *jth order Lyapunov constant* of the corresponding Abel equation. Usually the Lyapunov constants appear in the context of planar polynomial vector fields when studying the stability of equilibrium points of monodromic type. They are polynomial in the coefficients of the vector field when we restrict the analysis to the trace zero class and typically they are computed writing the planar differential equation in polar coordinates; see [11]. In fact, (1) is the third degree truncation of this type of equation and the zero trace class here is reduced to $C=0$, which is precisely the equation that we are analyzing. Hence, the standard Lyapunov scheme applies, so that the Lyapunov constants that we obtain are polynomials in the coefficients of A and B. As in the standard scheme here we consider the transition map from $\theta=0$ to $\theta=2\pi$. Only a constant translation $\theta\mapsto\theta-\pi$ is needed to obtain the transition from $\theta=-\pi$ to $\theta=\pi$. Clearly, the number of limit cycles does not depend on this initial angle. Moreover, the Lyapunov constants are always defined modulus the vanishing of all the previous ones, i.e., $X_j:=X_j|_{X_1=\cdots=X_{j-1}=0}$. For simplicity, by abusing notation we keep the same symbol to denote them. As is usual, we say that $\rho=0$ is a *weak focus of order k* if $X_k\neq 0$ and $X_j=0$ for $j=2,\ldots,k-1$. Moreover, we say that $\rho=0$ is a *center* if all the solutions in a neighborhood of $\rho=0$ are periodic, i.e., $\Delta(\rho)\equiv 0$. This theory was initially developed for planar ordinary differential systems having an equilibrium point of *center-focus* type (see [7]) but, writing in polar coordinates, both are equivalent. We point out that here, contrary to what happens in the standard planar scheme, the coefficients with an even subindex do not vanish identically. The bifurcation phenomenon is known as the *degenerated Hopf bifurcation*. In our context, since we have that $X_1 = 0$, one limit cycle is always missing when we perturb a weak focus or a center. Note that if b_k and d_k in (24) are all zero then the equation $\dot{x} = A(\theta)x^3 + B(\theta)x^2$ is invariant by the change $\theta \mapsto -\theta$ and, consequently, it has a *reversible center* at the origin, so that all X_j vanish. Therefore, when $b = (b_0, \ldots, b_n)$ and $d = (d_0, \ldots, d_m)$ are nonzero parameters, we can write the truncated first order Taylor series of X_j at $(b, d) = (0, 0)$ as

$$X_j^1 := \sum_{k=0}^{n} f_{j,k}(a,c)b_k + \sum_{k=0}^{m} g_{j,k}(a,c)d_k \text{ for all } j \geq 2, \tag{26}$$

where $a = (a_1, \ldots, a_n)$ and $c = (c_1, \ldots, c_m)$. We remark that the above expression has no constant term because the return map is identically zero when $b = d = 0$ for every a and c. Here we will use the ideas developed in [21] to work only with these linear developments in order to increase the number of limit cycles of small amplitude for families of centers. In [22] it is proved that these linear developments can also be obtained by computing the Taylor series at $\rho = 0$ of the first Melnikov function M_1 as introduced in section 2.

In the next two propositions, since we treat the cases $n = 1$ and $m = 1$, we can follow similarly as with the perturbed problem (3), taking the coefficients of $\sin\theta$ and $\cos\theta$ to be 1 and 0, respectively, since we can rescale x and do a translation in θ if necessary. In the proofs we will see that we are using all the other free parameters. This also shows that the weak-focus order is maximal in the considered families.

Proposition A.2. *Let us consider the Abel differential equation $\dot{x} = A(\theta)x^3 + B(\theta)x^2$ taking the trigonometric polynomials A and B as introduced in* (24) *with $(n, m) = (1, 3)$ and $(a_1, b_1) = (1, 0)$. Then there exist parameters (b_0, c, d) such that the origin is a weak focus of order 9 unfolding 7 nonzero limit cycles of small amplitude. Consequently $\mathcal{H}_{3,2}(1, 3) \geq 8$.*

Proof. In this case it turns out that all the Lyapunov constants that we need in order to prove the result are linear with respect to $(b_0, d_0, d_1, d_2, d_3)$, so that $X_j = X_j^1$. The proof follows by finding a transversal intersection point on the zero level set of $X_2, \ldots, X_8$ in which X_9 is nonvanishing. Then 7 nonzero limit cycles of small amplitude bifurcate from the origin and $\rho = 0$ is still a hyperbolic solution that remains. The complete expressions are quite large and we only show the first ones:

$$\begin{aligned}
X_2 &= d_0,\\
X_3 &= b_0,\\
X_4 &= 2^{-1}d_1,\\
X_5 &= 12^{-1}(-3c_1 + c_3)d_2 - 12^{-1}c_2 d_3,\\
X_6 &= 16^{-1}(3c_1c_2 - 2)d_2 + 32^{-1}(4c_1^2 + c_2^2)d_3,\\
X_7 &= 432^{-1}(-1674c_1^3 - 1620c_1^2c_2 - 504c_1^2c_3 - 432c_1c_2^2 + 162c_1c_2c_3 + 150c_1c_3^2 + 225c_2^2c_3\\
&\quad + 234c_2c_3^2 + 64c_3^3 - 864c_1 - 162c_2 + 72c_3)d_2 + 864^{-1}(648c_1^3 - 828c_1^2c_2 + 216c_1^2c_3\\
&\quad - 1242c_1c_2^2 - 720c_1c_2c_3 - 369c_2^3 - 414c_2^2c_3 - 128c_2c_3^2 + 144c_1 - 360c_2)d_3.
\end{aligned} \tag{27}$$

Clearly X_2, X_3, X_4, X_5 have degree one with respect to b_0, d_0, d_1, d_2. Then, when $3c_1 - c_3 \neq 0$ we can write $X_j = u_j$ for $j = 2, \dots, 5$ and we get

$$X_j|_{u_2=\cdots=u_5=0} = C_j \frac{f_{j-5}(c_1, c_2, c_3)}{3c_1 - c_3} d_3 \text{ for } j = 6, 7, 8, 9$$

with some nonvanishing rational numbers C_j and

$$\begin{aligned}
f_1 &= 12c_1^3 - 4c_1^2c_3 - 3c_1c_2^2 - c_2^2c_3 + 4c_2,\\
f_2 &= 216c_1^4 + 96c_1^3c_2 - 54c_1^2c_2^2 - 36c_1^2c_2c_3 - 24c_1^2c_3^2 - 27c_1c_2^3 - 36c_1c_2^2c_3\\
&\quad + 4c_1c_2c_3^2 - 9c_2^3c_3 - 6c_2^2c_3^2 + 48c_1^2 + 72c_1c_2 - 16c_1c_3 + 36c_2^2 + 24c_2c_3,\\
f_3 &= 18792c_1^5 + 16128c_1^4c_2 + 5832c_1^4c_3 - 828c_1^3c_2^2 - 672c_1^3c_2c_3 - 1824c_1^3c_3^2 - 4536c_1^2c_2^3\\
&\quad - 6156c_1^2c_2^2c_3 - 1344c_1^2c_2c_3^2 - 736c_1^2c_3^3 - 1197c_1c_2^4 - 3024c_1c_2^3c_3 - 1224c_1c_2^2c_3^2 + 224c_1c_2c_3^3\\
&\quad - 399c_2^4c_3 - 504c_2^3c_3^2 - 184c_2^2c_3^3 + 13248c_1^3 + 10068c_1^2c_2 - 1728c_1^2c_3 + 4752c_1c_2^2\\
&\quad + 2688c_1c_2c_3 - 896c_1c_3^2 + 1608c_2^3 + 1584c_2^2c_3 + 780c_2c_3^2 + 432c_1 + 1728c_2 - 144c_3,\\
f_4 &= 80352c_1^6 + 99252c_1^5c_2 + 48384c_1^5c_3 + 26352c_1^4c_2^2 + 26388c_1^4c_2c_3 + 1440c_1^4c_3^2\\
&\quad - 20034c_1^3c_2^3 - 28800c_1^3c_2^2c_3 - 7944c_1^3c_2c_3^2 - 5376c_1^3c_3^3 - 14364c_1^2c_2^4 - 30618c_1^2c_2^3c_3\\
&\quad - 15120c_1^2c_2^2c_3^2 - 1728c_1^2c_2c_3^3 - 1152c_1^2c_3^4 - 2646c_1c_2^5 - 9576c_1c_2^4c_3 - 8235c_1c_2^3c_3^2\\
&\quad - 1728c_1c_2^2c_3^3 + 480c_1c_2c_3^4 - 882c_2^5c_3 - 1596c_2^4c_3^2 - 1104c_2^3c_3^3 - 288c_2^2c_3^4 + 110538c_1^4\\
&\quad + 101808c_1^3c_2 + 16470c_1^3c_3 + 33678c_1^2c_2^2 + 13344c_1^2c_2c_3 - 11934c_1^2c_3^2 + 11196c_1c_2^3\\
&\quad + 9369c_1c_2^2c_3 + 4992c_1c_2c_3^2 - 1946c_1c_3^3 + 3600c_2^4 + 3732c_2^3c_3 + 2925c_2^2c_3^2\\
&\quad + 1328c_2c_3^3 + 17280c_1^2 + 22698c_1c_2 - 4032c_1c_3 + 10800c_2^2 + 6156c_2c_3 - 576c_3^2.
\end{aligned}$$

For $d_3 \neq 0$, the solutions of the system of equations defined by $\{f_1 = f_2 = f_3 = 0\}$ are written as

$$c^* = (c_1^*, c_2^*, c_3^*) = \left(\beta, -\frac{\beta^3\, p_2(\beta)}{126976}, \frac{\beta\, p_3(\beta)}{31744}\right),$$

where β is a simple real zero of $p(x) = 50625x^{16} - 207900x^{12} + 112644x^8 - 26880x^4 + 4096$, that one can verify it has exactly 4 simple real zeros near ± 0.7796202641 and ± 1.369217569, and

$$\begin{aligned}
p_2(x) &= 1771875x^{12} - 7456500x^8 + 4643340x^4 - 842624,\\
p_3(x) &= 16875x^{12} - 177300x^8 + 398508x^4 - 18304.
\end{aligned}$$

The proof will follow once we check that f_4, the denominator $3c_1 - c_3$, and the determinant of the Jacobian matrix of (f_1, f_2, f_3) with respect to $c = (c_1, c_2, c_3)$ are all different from zero evaluated at $c = c^*$. Straightforward computations show that

$$\begin{aligned}
f_4(c^*) &= (151875\beta^{12} + 3649500\beta^8 - 2148180\beta^4 + 184448)/1984 =: p_4(\beta),\\
\det \mathrm{Jac}_{(f_1,f_2,f_3)}(c^*) &= \beta(172800\beta^{12} + 843264\beta^8 - 516096\beta^4 + 65536) =: p_5(\beta),\\
3c_1 - c_3|_{c^*} &= -\beta(16875\beta^{12} - 177300\beta^8 + 398508\beta^4 - 113536)/31744 =: p_6(\beta).
\end{aligned}$$

For $i = 4, 5, 6$, one can verify that the resultant between $p_i(x)$ and $p(x)$ is different from zero. Consequently this implies that $p_i(\beta) \neq 0$ for $i = 4, 5, 6$, as desired. This concludes the proof of the result. ∎

Proposition A.3. *Let us consider the Abel differential equation $\dot{x} = A(\theta)x^3 + B(\theta)x^2$ taking the trigonometric polynomials A and B as introduced in (24) with $(n, m) = (4, 1)$ and $(c_1, d_1) = (1, 0)$. Then there exist parameters (a, b, d_0) such that the origin is a weak focus of order 11 unfolding 9 nonzero limit cycles of small amplitude. Consequently $\mathcal{H}_{3,2}(4, 1) \geq 10$.*

Proof. The proof follows similarly as in the previous result, but here we must use the linear Taylor developments of some X_j as we have previously explained. This is so because in this case not all Lyapunov constants are of degree 1 in the parameters $(b_0, b_1, b_2, b_3, b_4, d_0)$. For simplicity we will write $a = (a_1, a_2, a_3, a_4)$. Since the expressions of the Lyapunov constants are very large, for the sake of shortness we only show the first ones:

$$
\begin{aligned}
X_2 &= d_0,\\
X_3 &= b_0,\\
X_4 &= -2^{-1}b_1,\\
X_5 &= 4^{-1}b_2,\\
X_6 &= 48^{-1}((2a_2 - a_4 - 6)b_3 + a_3b_4),\\
X_7 &= 24^{-1}(-4a_1 + 6a_2 - 3a_4 - 18)b_3 + 16^{-1}(-a_2 + 2a_3 + 1)b_4,\\
X_8 &= 960^{-1}((-60a_1^2 + 240a_1a_2 - 120a_1a_4 + 135a_2^2 + 80a_2a_3 - 15a_2a_4 - 40a_3a_4\\
&\qquad - 30a_4^2 - 1840a_1 + 575a_2 - 240a_3 - 643a_4 - 2610)b_3 + (-45a_1a_2 + 120a_1a_3\\
&\qquad + 75a_2a_3 + 40a_3^2 + 30a_3a_4 + 145a_1 - 420a_2 + 463a_3 + 420)b_4).
\end{aligned}
$$

The next necessary X_k, for $k = 9, 10, 11$, have degree 3 in a_3, a_4 and we linearize them with respect to a_3, a_4. As above, we start simplifying with the first that are linear, $X_2, \ldots, X_5$, and X_6, writing, when $a_3 \neq 0$, $X_k = u_k$ for $k = 2, \ldots, 6$. Then, naming $b_3 = u_7$ we can write

$$
X_j|_{u_2=\cdots=u_6=0} = X_j^1 + O_2(b_3) = \frac{C_j}{a_3} f_{j-6}(a)\, u_7 + O_2(u_7) \text{ for } j = 7, 8, 9, 10, 11 \tag{28}
$$

with some C_j nonvanishing rational numbers and

$$
\begin{aligned}
f_1 &= 8a_1a_3 - 6a_2^2 + 3a_2a_4 + 24a_2 - 3a_4 - 18,\\
f_2 &= 60a_1^2a_3 - 90a_1a_2^2 + 45a_1a_2a_4 + 15a_2^2a_3 + 560a_1a_2 + 1120a_1a_3 - 145a_1a_4 - 840a_2^2\\
&\quad - 99a_2a_3 + 420a_2a_4 - 870a_1 + 3360a_2 - 168a_3 - 420a_4 - 2520,\\
f_3 &= 750a_1^2a_2 + 4800a_1^2a_3 - 375a_1^2a_4 - 4680a_1a_2^2 + 1653a_1a_2a_3 + 2340a_1a_2a_4 + 1120a_1a_3^2\\
&\quad + 840a_1a_3a_4 - 1212a_2^3 - 480a_2^2a_3 - 24a_2^2a_4 - 98a_2a_3^2 + 420a_2a_3a_4 + 315a_2a_4^2 - 49a_3^2a_4\\
&\quad - 2250a_1^2 + 23520a_1a_2 + 13281a_1a_3 - 4740a_1a_4 - 6012a_2^2 + 984a_2a_3 + 7344a_2a_4\\
&\quad + 294a_3^2 - 420a_3a_4 - 315a_4^2 - 28440a_1 + 39384a_2 - 6552a_3 - 7110a_4 - 31320,\\
f_4 &= 3600a_1^3a_2 + 23040a_1^3a_3 - 1800a_1^3a_4 - 34560a_1^2a_2^2 + 11520a_1^2a_2a_3 + 17280a_1^2a_2a_4
\end{aligned}
$$

$$
\begin{aligned}
&+ 7680a_1^2a_3^2 + 5760a_1^2a_3a_4 - 17280a_1a_2^3 - 5760a_1a_2^2a_3 - 1200a_1a_2a_3^2 + 5760a_1a_2a_3a_4\\
&+ 4320a_1a_2a_4^2 - 600a_1a_3^2a_4 + 3780a_2^3a_3 + 1920a_2^2a_3^2 + 1440a_2^2a_3a_4 - 225a_2a_3a_4^2\\
&- 10800a_1^3 + 323040a_1^2a_2 + 579120a_1^2a_3 - 109680a_1^2a_4 - 426816a_1a_2^2 + 235856a_1a_2a_3\\
&+ 293088a_1a_2a_4 + 144720a_1a_3^2 + 88064a_1a_3a_4 - 13920a_1a_4^2 - 147648a_2^3 - 101144a_2^2a_3\\
&- 6144a_2^2a_4 - 26784a_2a_3^2 + 43416a_2a_3a_4 + 39984a_2a_4^2 + 816a_3^3 - 7056a_3^2a_4 + 1523a_3a_4^2\\
&- 658080a_1^2 + 2311584a_1a_2 + 446352a_1a_3 - 521328a_1a_4 - 5568a_2^2 + 221676a_2a_3\\
&+ 544128a_2a_4 + 20832a_3^2 - 69048a_3a_4 - 39984a_4^2 - 2626848a_1 + 1881216a_2\\
&- 625176a_3 - 507744a_4 - 1607040,
\end{aligned}
$$

$$
\begin{aligned}
f_5 = {}& 720000a_1^3a_2 + 2386800a_1^3a_3 - 360000a_1^3a_4 - 2438640a_1^2a_2^2 + 1869360a_1^2a_2a_3\\
&+ 1354320a_1^2a_2a_4 + 1147200a_1^2a_3^2 + 772080a_1^2a_3a_4 - 67500a_1^2a_4^2 - 1621440a_1a_2^3\\
&- 526920a_1a_2^2a_3 - 28800a_1a_2^2a_4 + 266160a_1a_2a_3^2 + 855540a_1a_2a_3a_4 + 419760a_1a_2a_4^2\\
&+ 147600a_1a_3^3 + 136320a_1a_3^2a_4 + 92040a_1a_3a_4^2 - 221400a_2^4 - 121080a_2^3a_3 - 107460a_2^3a_4\\
&- 60240a_2^2a_3^2 + 58560a_2^2a_3a_4 + 46890a_2^2a_4^2 - 23520a_2a_3^3 + 20340a_2a_3^2a_4 + 64350a_2a_3a_4^2\\
&+ 31095a_2a_4^3 - 11760a_3^3a_4 - 8820a_3^2a_4^2 - 2160000a_1^3 + 16558560a_1^2a_2 + 12839120a_1^2a_3\\
&- 5026320a_1^2a_4 - 4939840a_1a_2^2 + 7959200a_1a_2a_3 + 9117760a_1a_2a_4 + 3502960a_1a_3^2\\
&- 377600a_2a_3^2 + 1230980a_1a_3a_4 - 848560a_1a_4^2 - 2428520a_2^3 - 1888240a_2^2a_3\\
&+ 489400a_2^2a_4 + 1309488a_2a_3a_4 + 1108880a_2a_4^2 + 111360a_3^3 - 191740a_3^2a_4 + 550a_3a_4^2\\
&- 33295a_4^3 - 27727920a_1^2 + 47743680a_1a_2 - 2597880a_1a_3 - 14256960a_1a_4\\
&+ 3526680a_2^2 + 5056920a_2a_3 + 9864060a_2a_4 - 199440a_3^2 - 2398224a_3a_4 - 1126770a_4^2\\
&- 54993600a_1 + 24634440a_2 - 13051440a_3 - 9252000a_4 - 22140000.
\end{aligned}
$$

We remark that the terms O_2 are polynomial in u_7 and rational in a. We claim that there exists at least a transversal intersection point a^* of the zero level sets of $f_1, \ldots, f_4$, where f_5 is nonvanishing. Once we prove the claim the result will follow because near this point a^*, thanks to the implicit function theorem, we can write (28) as $X_j = u_7 v_{j-6}$, being $v_1, \ldots, v_4$ are new independent variables. Hence, the existence of a weak focus of maximal order 11 is clear and also its unfolding provides only 9 nonzero limit cycles because the displacement map (25) starts with degree 2 terms. The extra limit cycle for proving the last statement follows from the fact that $\rho = 0$ is an isolated solution.

Let us prove finally that the claim is true. To this end we note that the system of equations $\{f_1 = f_2 = f_3 = f_4 = 0\}$ has solutions that are written as

$$a^* = (a_1^*, a_2^*, a_3^*, a_4^*) = (\beta, \alpha, \beta p_3(\alpha), p_4(\alpha)),$$

where $\beta^2 = p_2(\alpha)$ with p_2, p_3, and p_4 some polynomials of degree 13 with rational coefficients, and α is a simple real solution of the polynomial

$$
\begin{aligned}
p_1(x) = {}& 14352187500x^{14} - 657776700000x^{13} + 11284736929875x^{12}\\
&- 42406416759825x^{11} - 1391899076716315x^{10} + 23023762786511909x^9
\end{aligned}
$$

$$- 123393106663586826x^8 - 218236503571470586x^7 + 575636088434736349 4x^6$$
$$- 27272611466754481126x^5 + 55780715677026807263x^4 - 48094920597945273157x^3$$
$$+ 9193962961957763353x^2 + 5105496738368043633x - 1609769302079739192$$

verifying $p_2(\alpha) > 0$. Straightforward computations show that there exist p_5 and p_6, polynomials of degree 13 with rational coefficients such that $f_5(a^*) = p_5(\alpha)$ and the determinant of the Jacobian matrix $\det \mathrm{Jac}_{(f_1,f_2,f_3,f_4)}(a^*) = p_6(\alpha)$. Moreover, the polynomials p_k for $k = 2, \ldots, 6$ do not vanish at α, because the respective resultants with p_1, with respect to α, are nonzero rational numbers. It only remains to prove that there exists α such that also $p_2(\alpha) > 0$. This follows just computing the real zeros of the polynomials p_1 and p_2, ordering them in the real line, and comparing their plots. From the 8 simple real zeros of p_1 only 4 satisfy the condition $p_2 > 0$. They are located near $-12.079846278, -6.6037190290, 1.81965668348, 1.84169431112$. This concludes the proof of the result. ■

We remark that working on the last proof with the complete Lyapunov constants instead of the linear developments we have not obtained more limit cycles, and the computations to provide the weak focus of maximum order are even worse. Moreover, although from the proof it seems that we are computing only linear developments, from the final writing $X_j = u_7 v_{j-6}$ it is clear that a second order bifurcation mechanism is used, as we have shown throughout the present paper.

We finish the appendix by making some considerations about numerical simulations regarding other values of m and n. Theorem 1.2 improves the general lower bounds of $\mathcal{H}_{3,2}(n,m)$ that appear in the literature, in particular the ones in [6]. The new general lower bound is $2(n+m)-1$ and, as we have commented before, it is very close to the total number of parameters $2(n+m)+2$ in the system. Since we can rescale x and do a translation in θ, two of these parameters can be removed and only $2(n+m)$ remain. Hence it is reasonable to conjecture that $\mathcal{H}_{3,2}(n,m) = 2(n+m)$. Nevertheless it can be checked that with the degenerated Hopf bifurcation explained above we cannot get such a number of limit cycles when $n + m \leq 4$ except for $(n,m) = (1,3)$; see Proposition A.2. Some of these computations were done in [6] by studying the maximum weak-focus order. With regard to the segment $n+m = 5$, Proposition A.3 gets this value of limit cycles for $(n,m) = (4,1)$ whereas, numerically, we can get (also using the technique explained in the last section) that $\mathcal{H}_{3,2}(2,3) \geq 10^*$ and $\mathcal{H}_{3,2}(1,4) \geq 10^*$. But we have not been able to improve, not even numerically, the lower bound $\mathcal{H}_{3,2}(3,2) \geq 9$. (Here and below the superscript $*$ means that we have not an analytic proof but only numerical evidence.) The main difficulty is not in finding the system of equations to solve but in solving it. The numerical solutions that we have found seem to be values to have a center at the origin and not a weak focus of the order that we look for.

For $(n,m) = (1,4)$, as we have done in Proposition A.2, we fix $a_1 = 1$ and $b_1 = 0$. The next step is the computation of the first linearized Lyapunov constants in the form (26) that are similar to (27). Then, again with the implicit function theorem, we write $X_k = u_k$ for $k = 2, \ldots, 6$ and we obtain for $X_7, \ldots, X_{11}$ the functions $f_1, \ldots, f_5$ similarly as the ones in (28), depending only on $c = (c_1, c_2, c_3, c_4)$. Here the polynomials f_k have degree $k+5$ for $k = 1, \ldots, 5$. Hence, an equivalent Proposition A.3 could be conjectured for this case because, working with enough precision to see the stabilization of the digits, we have found numerically a weak focus of order 11 at $c^* = (0.8339985012, 3.0122982805, 1.9052668985, -5.4437166429)$ that unfolds 9 nonzero limit cycles of small amplitude and yields to $\mathcal{H}_{3,2}(1,4) \geq 10^*$.

For $(n,m) = (2,3)$ we can get a similar numerical result for $\mathcal{H}_{3,2}(2,3) \geq 10^*$. The main difference with the latter case is that the functions f_k depend on (a_1, a_2, c_1, c_2) and have degree $3k+7$ for $k = 1, \ldots, 5$. Here the numerical approximation of the weak focus point is $(a_1^*, a_2^*, c_1^*, c_2^*) = (-0.05247784623, 0.6187352312, -0.2084251822, 0.3470405002)$.

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