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This is a preprint of: *Description of interpolation spaces for the pair of local Morrey-type spaces and their generalizations*  
Journal Information: *CRM Preprints*,  
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Volume, pages: 1-43, DOI:[--]

Preprint núm. 1160

June 2013

Description of interpolation spaces for  
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# DESCRIPTION OF INTERPOLATION SPACES FOR THE PAIR OF LOCAL MORREY-TYPE SPACES AND THEIR GENERALIZATIONS

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**ABSTRACT.** We consider the real interpolation method and prove that for general local Morrey-type spaces, in the case when they have the same integrability parameter, the interpolation spaces are again general local Morrey-type spaces with appropriately chosen parameters. This result is a particular case of the interpolation theorem for much more general spaces defined with the help of an operator acting from some function space to the cone of non-negative non-decreasing functions on  $(0, \infty)$ . It is also shown how the classical interpolation theorems due to Stein-Weiss, Peetre, Calderón, Gilbert, Lizorkin, Freitag and some of their new variants can be derived from this theorem.

## 1. INTRODUCTION

Let  $0 < p \leq \infty$  and  $0 \leq \lambda \leq \frac{n}{p}$ . The Morrey spaces  $M_p^\lambda$  were defined in [18] as the spaces of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  such that

$$\|f\|_{M_p^\lambda} = \sup_{x \in \mathbb{R}^n} \sup_{r > 0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty,$$

where  $B(x, r)$  is the open ball of radius  $r > 0$  with center at point  $x \in \mathbb{R}^n$ . If  $\lambda = 0$ , then  $M_p^0 = L_p(\mathbb{R}^n)$ , while if  $\lambda = \frac{n}{p}$ , then  $M_p^{\frac{n}{p}} = L_\infty(\mathbb{R}^n)$ . If  $\lambda < 0$  or  $\lambda > \frac{n}{p}$ , then  $M_p^\lambda = \Theta$ , where  $\Theta$  is the set of all functions that are equivalent to zero on  $\mathbb{R}^n$ .

The problem of interpolation for the Morrey spaces was considered in [27, 10, 19, 22, 4, 17]. It follows from the results of [19] that

$$(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta, \infty} \subset M_p^\lambda,$$

where  $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ . In [22, 4] it was established that this inclusion is strict, which raised the problem of giving a complete description of the interpolation spaces for the pair of Morrey spaces. This problem still remains open.

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2010 *Mathematics Subject Classification.* 42B35, 46E30, 47B38, 46B70.

*Key words and phrases.* Morrey spaces, general local Morrey-type spaces, real interpolation method,  $K$ -functional, interpolation theorems.

In [17] a more detailed investigation of the interpolation problem for Morrey spaces was carried out. In particular, it was proved that the inclusion

$$(M_{p_0}^{\lambda_0}, M_{p_1}^{\lambda_1})_{\theta, \infty} \subset M_p^\lambda,$$

where  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  and  $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$  holds if and only if  $p_0 = p_1$ .

In [6] the interpolation problem was considered for a local variant of the Morrey spaces and for their generalizations involving an additional parameter, namely for the local Morrey-type spaces  $LM_{p,q}^\lambda$  that are defined for  $\lambda \geq 0$ , and  $0 < p, q \leq \infty$  as the spaces of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  such that

$$\|f\|_{LM_{p,q}^\lambda} = \left( \int_0^\infty (t^{-\lambda} \|f\|_{L_p(B(0,t))})^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,$$

with the conventional modification for  $q = \infty$ . Note that  $LM_{p,q}^\lambda \neq \Theta$  if and only if  $\lambda > 0$  for  $q < \infty$  and  $\lambda \geq 0$  for  $q = \infty$ . If  $q = \infty$ , then  $LM_{p,\infty}^0 = L_p(\mathbb{R}^n)$ .

It appeared that, in contrast to the scale of the Morrey spaces  $M_p^\lambda$ , the scale of the local Morrey-type spaces  $LM_{p,q}^\lambda$  with fixed  $p$  is closed under the procedure of interpolation. Namely, the following statement was proved in [8].

**Theorem 1.** *Let  $0 < p, q_0, q_1, q \leq \infty$  and  $0 < \theta < 1$ . Suppose, in addition, that  $\lambda_0 \neq \lambda_1$  and  $0 < \lambda_0, \lambda_1 < \frac{n}{p}$  if  $p < \infty$  and at least one of the parameters  $q_0, q_1$  and  $q$  is finite, and  $0 \leq \lambda_0, \lambda_1 \leq \frac{n}{p}$  if  $q_0 = q_1 = q = \infty$ . Then*

$$(LM_{p,q_0}^{\lambda_0}, LM_{p,q_1}^{\lambda_1})_{\theta, q} = LM_{p,q}^\lambda,$$

where  $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$ .

In the last two decades there was a great interest in studying general Morrey-type spaces and classical operators of theory of functions acting in such spaces. See survey papers [5, 6, 14, 16, 21, 25, 26]. In Section 4 we introduce the variant of general local Morrey-type spaces  $LM_{p,q}^\lambda(G, \mu)$  obtained from the definition of the space  $LM_{p,q}^\lambda$  by replacing  $\|f\|_{L_p(B(0,t))}$  by  $\|f\|_{L_p(G_t, \mu)}$ , where  $G = \{G_t\}_{t>0}$  is a family of nested sets  $G_t$  satisfying some natural assumptions, and  $\mu$  is a general measure.

The main result of the paper is that the scale of the spaces  $LM_{p,q}^\lambda(G, \mu)$  with fixed  $p$  is closed, like the scale of the spaces  $LM_{p,q}^\lambda$ , under interpolation (Theorem 5 in Section 5). By this theorem it also follows that Theorem 1 is valid without the restriction  $\lambda_0, \lambda_1 < \frac{n}{p}$  if  $q < \infty$  and  $\lambda_0, \lambda_1 \leq \frac{n}{p}$  if  $q = \infty$  (Corollary 2 in Section 5). Without proof Theorem 5 was formulated in [7].

In fact we consider much more general spaces  $\Phi_{\lambda,q}(F)$  obtained from the definition of the spaces  $LM_{p,q}^\lambda(G, \mu)$  by replacing  $\|f\|_{L_p(G_t, \mu)}$  by an operator  $F$  acting from some function space  $Z$  to the cone  $M^\uparrow$  of non-negative non-decreasing functions on  $(0, \infty)$  and prove that under the appropriate assumptions on the

operators  $F_0, F_1, F: Z \rightarrow M^\uparrow$  and on the numerical parameters

$$(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q} = \Phi_{\lambda, q}(F)$$

(Theorem 4 in Section 3). The proof of Theorem 4 is based on an interpolation theorem for the cones of non-negative non-decreasing functions (Theorem 2 in Section 2).

Theorem 5 is a particular case of Theorem 4. Moreover, by appropriately choosing the operators  $F_0, F_1$  and  $F$ , one can obtain classical interpolation theorems due to Stein-Weiss, Peetre, Calderón, Gilbert, Lizorkin, Freitag and some of their new variants. (Theorems 6–9 in Section 6).

## 2. INTERPOLATION OF CONES OF NON-DECREASING FUNCTIONS

Let  $0 < p \leq \infty$ ,  $\Omega$  be a Lebesgue measurable set in  $\mathbb{R}^n$ , and  $w$  be a weight function on  $\Omega$ , i.e.  $w$  is a non-negative Lebesgue measurable function on  $\Omega$ . By  $L_p(\Omega, w)$  we denote the space of all functions  $f$  Lebesgue measurable on  $\Omega$  for which

$$\|f\|_{L_p(\Omega, w)} = \|fw\|_{L_p(\Omega)} < \infty,$$

where

$$\|fw\|_{L_p(\Omega)} = \left( \int_0^\infty (|f(x)|w(x))^p dx \right)^{\frac{1}{p}}$$

if  $p < \infty$ , and

$$\|fw\|_{L_\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |f(x)|w(x)$$

if  $p = \infty$ . (If  $w \equiv 1$ , then  $L_p(\Omega, 1) \equiv L_p(\Omega)$ ).

Moreover, given  $\lambda \in \mathbb{R}$ , by  $\Phi_{\lambda, p}$  we denote the space  $L_p((0, \infty), t^{-\lambda-\frac{1}{p}})$ , i.e. the space of all functions  $\varphi$  Lebesgue measurable on  $(0, \infty)$  for which

$$\|\varphi\|_{\Phi_{\lambda, p}} = \left( \int_0^\infty (t^{-\lambda}|\varphi(t)|)^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty$$

if  $p < \infty$ , and

$$\|\varphi\|_{\Phi_{\lambda, \infty}} = \text{ess sup}_{x \in (0, \infty)} t^{-\lambda}|\varphi(t)| < \infty$$

if  $p = \infty$ .

Finally, for  $0 < \lambda < \infty$  if  $p < \infty$ , and for  $0 \leq \lambda < \infty$  if  $p = \infty$ , let  $\Phi_{\lambda, p}^\uparrow$  denote the subspace of  $\Phi_{\lambda, p}$  consisting of all functions  $\varphi \in \Phi_{\lambda, p}$  which are non-negative and non-decreasing on  $(0, \infty)$ , i.e. <sup>1</sup>

$$\Phi_{\lambda, p}^\uparrow = \Phi_{\lambda, p} \cap M^\uparrow,$$

<sup>1</sup>In the sequel we shall always assume, without stating this specially, that  $0 < \lambda < \infty$  if  $p < \infty$  and  $0 \leq \lambda < \infty$  if  $p = \infty$  when considering the spaces  $\Phi_{\lambda, p}^\uparrow$ .

where  $M^\uparrow$  is the cone of all functions  $\varphi$  non-negative and non-decreasing on  $(0, \infty)$ .

For  $0 < \lambda_0, \lambda_1 < \infty$  and  $0 < p_0, p_1 \leq \infty$  by  $K(\Phi_{\lambda_0, p_0}, \Phi_{\lambda_1, p_1}, \varphi)$  we denote the standard  $K$ -functional for  $t > 0$ :

$$K(\Phi_{\lambda_0, p_0}, \Phi_{\lambda_1, p_1}, \varphi)(t) = \inf_{\substack{\varphi = \varphi_0 + \varphi_1 \\ \varphi_0 \in \Phi_{\lambda_0, p_0}, \varphi_1 \in \Phi_{\lambda_1, p_1}}} \left( \|\varphi_0\|_{\Phi_{\lambda_0, p_0}} + t \|\varphi_1\|_{\Phi_{\lambda_1, p_1}} \right)$$

and by  $K(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow, \varphi)$  we denote a variant of the standard  $K$ -functional defined for functions  $\varphi \in \Phi_{\lambda_0, p_0}^\uparrow + \Phi_{\lambda_1, p_1}^\uparrow$  and  $t > 0$  by

$$K(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow, \varphi)(t) = \inf_{\substack{\varphi = \varphi_0 + \varphi_1 \\ \varphi_0 \in \Phi_{\lambda_0, p_0}^\uparrow, \varphi_1 \in \Phi_{\lambda_1, p_1}^\uparrow}} \left( \|\varphi_0\|_{\Phi_{\lambda_0, p_0}} + t \|\varphi_1\|_{\Phi_{\lambda_1, p_1}} \right).$$

Note that, in contrast to the standard  $K$ -functional, the infimum is taken with respect to all representations  $\varphi = \varphi_0 + \varphi_1$  of a function  $\varphi$  non-negative and non-decreasing on  $(0, \infty)$  as the sum of functions  $\varphi_0 \in \Phi_{\lambda_0, p_0}^\uparrow$ ,  $\varphi_1 \in \Phi_{\lambda_1, p_1}^\uparrow$ , which are also non-negative and non-decreasing on  $(0, \infty)$ .

Let, for  $s > 0$  and a function  $\varphi \in M^\uparrow$ , the functions  $A_s\varphi$  and  $B_s\varphi$  be defined by

$$A_s\varphi(t) = \begin{cases} \varphi(t), & \text{if } 0 < t \leq s, \\ \varphi(s), & \text{if } t > s, \end{cases}$$

and

$$B_s\varphi(t) = \varphi(t)\chi_{(s, \infty)}(t), \quad t > 0,$$

where  $\chi_{(s, \infty)}$  is the characteristic function of the interval  $(s, \infty)$ .

Clearly that

$$(I - A_s)\varphi \leq B_s\varphi.$$

Note that  $A_s\varphi$  and  $B_s\varphi$  are non-negative and non-decreasing functions.

Furthermore, given  $0 < \theta < 1$ ,  $0 < q \leq \infty$ , by  $(\Phi_{\lambda_0, p_0}, \Phi_{\lambda_1, p_1})_{\theta, q}$  and  $(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow)_{\theta, q}$  we denote the interpolation spaces of all functions  $\varphi$  Lebesgue measurable on  $(0, \infty)$  for which

$$\|\varphi\|_{(\Phi_{\lambda_0, p_0}, \Phi_{\lambda_1, p_1})_{\theta, q}} = \|K(\Phi_{\lambda_0, p_0}, \Phi_{\lambda_1, p_1}, \varphi)\|_{\Phi_{\theta, q}} < \infty,$$

and

$$\|\varphi\|_{(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow)_{\theta, q}} = \|K(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow, \varphi)\|_{\Phi_{\theta, q}} < \infty$$

respectively.

**Lemma 1.** *Let  $0 < p_0 < p_1 \leq \infty$  and  $0 < \lambda < \infty$ . Then*

$$\Phi_{\lambda, p_0}^\uparrow \hookrightarrow \Phi_{\lambda, p_1}^\uparrow.$$

Moreover,

$$(1) \quad \|\varphi\|_{\Phi_{\lambda,p_1}} \leq (\lambda p_0)^{\frac{1}{p_0} - \frac{1}{p_1}} \|\varphi\|_{\Phi_{\lambda,p_0}}$$

for all functions  $\varphi \in \Phi_{\lambda,p_0}^\uparrow$ .

*Proof.* First let  $p_1 = \infty$ . Then by the monotonicity of the function  $\varphi \in \Phi_{\lambda,p_0}^\uparrow$  it follows that

$$\begin{aligned} \|\varphi\|_{\Phi_{\lambda,\infty}} &= \sup_{t>0} t^{-\lambda} \varphi(t) = (\lambda p_0)^{\frac{1}{p_0}} \sup_{t>0} \left( \int_t^\infty \tau^{-\lambda p_0} \frac{d\tau}{\tau} \right)^{\frac{1}{p_0}} \varphi(t) \\ &\leq (\lambda p_0)^{\frac{1}{p_0}} \sup_{t>0} \left( \int_t^\infty (\tau^{-\lambda} \varphi(\tau))^{p_0} \frac{d\tau}{\tau} \right)^{\frac{1}{p_0}} = (\lambda p_0)^{\frac{1}{p_0}} \|\varphi\|_{\Phi_{\lambda,p_0}}. \end{aligned}$$

If  $p_1 < \infty$ , then it suffices to use in addition the interpolation inequality

$$\|\varphi\|_{\Phi_{\lambda,p_1}} \leq (\|\varphi\|_{\Phi_{\lambda,\infty}})^{1-\frac{p_0}{p_1}} (\|\varphi\|_{\Phi_{\lambda,p_0}})^{\frac{p_0}{p_1}},$$

which completes the proof.  $\square$

**Theorem 2.** Let  $0 < p_0, p_1, q \leq \infty$ ,  $0 < \theta < 1$ . Then

$$(2) \quad \left( \Phi_{\lambda_0,p_0}^\uparrow, \Phi_{\lambda_1,p_1}^\uparrow \right)_{\theta,q} = \Phi_{\lambda,q}^\uparrow,$$

where  $\lambda_0 \neq \lambda_1$  and  $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$ . Moreover, there exist  $c_1, c_2 > 0$  depending only on  $p_0, p_1, q, \lambda_0, \lambda_1, \theta$  such that

$$\begin{aligned} (3) \quad c_1 \|\varphi\|_{\Phi_{\lambda,q}} &\leq \|\varphi\|_{(\Phi_{\lambda_0,p_0}^\uparrow, \Phi_{\lambda_1,p_1}^\uparrow)_{\theta,q}} \\ &\leq \min \left\{ \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_0,p_0}} + t \|B_s \varphi\|_{\Phi_{\lambda_1,p_1}} \right) \right\} \|\varphi\|_{\Phi_{\theta,q}}, \end{aligned}$$

$$(4) \quad \left\| \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_1,p_1}} + t \|B_s \varphi\|_{\Phi_{\lambda_0,p_0}} \right) \right\|_{\Phi_{1-\theta,q}} \leq c_2 \|\varphi\|_{\Phi_{\lambda,q}}.$$

*Proof.* 1. Let  $\varphi = \varphi_0 + \varphi_1$ , where  $\varphi_0$  and  $\varphi_1$  are non-negative and non-decreasing on  $(0, \infty)$ . Then

$$\begin{aligned} t^{-\lambda} \varphi(t) &= t^{-\lambda+\lambda_0} (t^{-\lambda_0} \varphi_0(t)) + t^{\lambda_1-\lambda} (t^{-\lambda_1} \varphi_1(t)) \\ &\leq t^{-\lambda+\lambda_0} \sup_{\eta>0} \eta^{-\lambda_0} \varphi_0(\eta) + t^{\lambda_1-\lambda} \sup_{\eta>0} \eta^{-\lambda_1} \varphi_1(\eta) \\ &= t^{-\lambda+\lambda_0} \|\varphi_0\|_{\Phi_{\lambda_0,\infty}} + t^{\lambda_1-\lambda} \|\varphi_1\|_{\Phi_{\lambda_1,\infty}}. \end{aligned}$$

By inequality (1) we get

$$t^{-\lambda} \varphi(t) \leq c_3 \left( t^{-\lambda+\lambda_0} \|\varphi_0\|_{\Phi_{\lambda_0,p_0}} + t^{\lambda_1-\lambda} \|\varphi_1\|_{\Phi_{\lambda_1,p_1}} \right),$$

where  $c_3 = \max \left\{ (\lambda_0 p_0)^{\frac{1}{p_0}}, (\lambda_1 p_1)^{\frac{1}{p_1}} \right\}$ .



Now we note that  $-\lambda + \lambda_0 = -\theta(\lambda_1 - \lambda_0)$  and  $-\lambda + \lambda_1 = (1 - \theta)(\lambda_1 - \lambda_0)$ , hence

$$\begin{aligned} t^{-\lambda}\varphi(t) &\leq c_3 \left( t^{-\theta(\lambda_1 - \lambda_0)} \|\varphi_0\|_{\Phi_{\lambda_0, p_0}} + t^{(1-\theta)(\lambda_1 - \lambda_0)} \|\varphi_1\|_{\Phi_{\lambda_1, p_1}} \right) \\ &= c_3 t^{-\theta(\lambda_1 - \lambda_0)} \left( \|\varphi_0\|_{\Phi_{\lambda_0, p_0}} + t^{\lambda_1 - \lambda_0} \|\varphi_1\|_{\Phi_{\lambda_1, p_1}} \right). \end{aligned}$$

By taking infimum with respect to  $\varphi_0 \in \Phi_{\lambda_0, p_0}^\uparrow$  and  $\varphi_1 \in \Phi_{\lambda_1, p_1}^\uparrow$  we obtain

$$\begin{aligned} t^{-\lambda}\varphi(t) &\leq c_3 t^{-\theta(\lambda_1 - \lambda_0)} \inf_{\substack{\varphi = \varphi_0 + \varphi_1 \\ \varphi_0 \in \Phi_{\lambda_0, p_0}^\uparrow, \varphi_1 \in \Phi_{\lambda_1, p_1}^\uparrow}} \left( \|\varphi_0\|_{\Phi_{\lambda_0, p_0}} + t^{\lambda_1 - \lambda_0} \|\varphi_1\|_{\Phi_{\lambda_1, p_1}} \right) \\ &= c_3 t^{-\theta(\lambda_1 - \lambda_0)} K \left( \Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow, \varphi \right) (t^{\lambda_1 - \lambda_0}). \end{aligned}$$

Hence

$$\begin{aligned} \|\varphi\|_{\Phi_{\lambda, p}} &\leq c_3 \left\| K \left( \Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow, \varphi \right) (t^{\lambda_1 - \lambda_0}) \right\|_{\Phi_{\theta(\lambda_1 - \lambda_0), q}} \\ &= [\tau] = c_3 |\lambda_1 - \lambda_0|^{-\frac{1}{q}} \left\| K \left( \Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow, \varphi \right) (\tau) \right\|_{\Phi_{\theta, q}} \\ &= c_3 |\lambda_1 - \lambda_0|^{-\frac{1}{q}} \|\varphi\|_{(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow)_{\theta, q}}. \end{aligned}$$

2. Now we prove inequality (4). First let  $\lambda_0 < \lambda_1$ . Note that

$$\begin{aligned} \|\varphi\|_{(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow)_{\theta, q}} &= \left\| K \left( \Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow, \varphi \right) (\tau) \right\|_{\Phi_{\theta, q}} \\ &\leq \left\| \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_0, p_0}} + \tau \|(I - A_s)\varphi\|_{\Phi_{\lambda_1, p_1}} \right) \right\|_{\Phi_{\theta, q}} \leq I(\lambda_0, \lambda_1, p_0, p_1, \theta, q), \end{aligned}$$

where<sup>2</sup>

$$\begin{aligned} I(\lambda_0, \lambda_1, p_0, p_1, \theta, q, \varphi) &= \left\| \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_0, p_0}} + \tau \|B_s \varphi\|_{\Phi_{\lambda_1, p_1}} \right) \right\|_{\Phi_{\theta, q}} = [\tau = t^{\lambda_1 - \lambda_0}] \\ &= |\lambda_1 - \lambda_0|^{\frac{1}{q}} \left\| \inf_{s>0} \|A_s \varphi\|_{\Phi_{\lambda_0, p_0}} + t^{\lambda_1 - \lambda_0} \|B_s \varphi\|_{\Phi_{\lambda_1, p_1}} \right\|_{\Phi_{\theta(\lambda_1 - \lambda_0), q}} \\ &\leq |\lambda_1 - \lambda_0|^{\frac{1}{q}} \left\| \|A_t \varphi\|_{\Phi_{\lambda_0, p_0}} + t^{\lambda_1 - \lambda_0} \|B_t \varphi\|_{\Phi_{\lambda_1, p_1}} \right\|_{\Phi_{\theta(\lambda_1 - \lambda_0), q}} \\ &\leq |\lambda_1 - \lambda_0|^{\frac{1}{q}} 2^{(\frac{1}{q}-1)+} \left( \left\| \|A_t \varphi\|_{\Phi_{\lambda_0, p_0}} \right\|_{\Phi_{\theta(\lambda_1 - \lambda_0), q}} + \left\| t^{\lambda_1 - \lambda_0} \|B_t \varphi\|_{\Phi_{\lambda_1, p_1}} \right\|_{\Phi_{\theta(\lambda_1 - \lambda_0), q}} \right) \\ &= |\lambda_1 - \lambda_0|^{\frac{1}{q}} 2^{(\frac{1}{q}-1)+} \left( \left\| \|A_t \varphi\|_{\Phi_{\lambda_0, p_0}} \right\|_{\Phi_{\theta(\lambda_1 - \lambda_0), q}} + \left\| \|B_t \varphi\|_{\Phi_{\lambda_1, p_1}} \right\|_{\Phi_{(1-\theta)(\lambda_0 - \lambda_1), q}} \right). \end{aligned}$$

Therefore,

$$I(\lambda_0, \lambda_1, p_0, p_1, \theta, q, \varphi) \leq |\lambda_1 - \lambda_0|^{\frac{1}{q}} 2^{(\frac{1}{q}-1)+} (I_1 + I_2),$$

<sup>2</sup> As usual,  $a_+$  denotes the positive part of the number  $a \in \mathbb{R}$ .

where

$$I_1 = \left\| \|A_t \varphi\|_{\Phi_{\lambda_0, p_0}} \right\|_{\Phi_{\theta(\lambda_1 - \lambda_0), q}},$$

$$I_2 = \left\| \|B_t \varphi\|_{\Phi_{\lambda_1, p_1}} \right\|_{\Phi_{(1-\theta)(\lambda_0 - \lambda_1), q}}.$$

3. First we estimate  $\|A_t \varphi\|_{\Phi_{\lambda_0, p_0}}$ :

$$\begin{aligned} \|A_t \varphi\|_{\Phi_{\lambda_0, p_0}} &= \left( \int_0^\infty (\eta^{-\lambda_0} A_t \varphi(\eta))^{p_0} \frac{d\eta}{\eta} \right)^{\frac{1}{p_0}} \\ &\leq 2^{(\frac{1}{p_0}-1)+} \left( \left( \int_0^t (\eta^{-\lambda_0} \varphi(\eta))^{p_0} \frac{d\eta}{\eta} \right)^{\frac{1}{p_0}} + \left( \int_t^\infty (\eta^{-\lambda_0} \varphi(t))^{p_0} \frac{d\eta}{\eta} \right)^{\frac{1}{p_0}} \right) \\ &\equiv 2^{(\frac{1}{p_0}-1)+} \left( J_1(t) + \varphi(t) (\lambda_0 p_0)^{-\frac{1}{p_0}} t^{-\lambda_0} \right), \end{aligned}$$

where

$$J_1(t) = \left( \int_0^t (\eta^{-\lambda_0} \varphi(\eta))^{p_0} \frac{d\eta}{\eta} \right)^{\frac{1}{p_0}}.$$

Next

$$\begin{aligned} t^{-\lambda_0} \varphi(t) &= (\lambda_1 p_1)^{\frac{1}{p_1}} t^{-\lambda_0 + \lambda_1} \varphi(t) \left( \int_t^\infty \eta^{-\lambda_1 p_1} \frac{d\eta}{\eta} \right)^{\frac{1}{p_1}} \\ &\leq (\lambda_1 p_1)^{\frac{1}{p_1}} t^{-\lambda_0 + \lambda_1} \left( \int_t^\infty (\eta^{-\lambda_1} \varphi(\eta))^{p_1} \frac{d\eta}{\eta} \right)^{\frac{1}{p_1}} \equiv (\lambda_1 p_1)^{\frac{1}{p_1}} t^{-\lambda_0 + \lambda_1} J_2(t), \end{aligned}$$

where

$$J_2(t) = \left( \int_t^\infty (\eta^{-\lambda_1} \varphi(\eta))^{p_1} \frac{d\eta}{\eta} \right)^{\frac{1}{p_1}}.$$

So,

$$\|A_t \varphi\|_{\Phi_{\lambda_0, p_0}} \leq c_4 (J_1(t) + t^{-\lambda_0 + \lambda_1} J_2(t)),$$

where  $c_4 = 2^{(\frac{1}{p_0}-1)+} \max\{(\lambda_0 p_0)^{-\frac{1}{p_0}} (\lambda_1 p_1)^{\frac{1}{p_1}}, 1\}$ . (If  $p_0 = \infty$  and/or  $p_1 = \infty$ , here and in the sequel, the integrals should be replaced by the appropriate suprema.) Therefore,

$$\begin{aligned} I_1 &\leq c_4 2^{(\frac{1}{q}-1)+} \left( \|J_1(t)\|_{\Phi_{\theta(\lambda_1-\lambda_0),q}} + \|t^{-\lambda_0+\lambda_1} J_2(t)\|_{\Phi_{\theta(\lambda_1-\lambda_0),q}} \right) \\ &= c_4 2^{(\frac{1}{q}-1)+} \left( \|J_1\|_{\Phi_{\theta(\lambda_1-\lambda_0),q}} + \|J_2\|_{\Phi_{(1-\theta)(\lambda_0-\lambda_1),q}} \right). \end{aligned}$$

Next note that

$$\|B_t \varphi\|_{\Phi_{\lambda_1,p_1}} = \left( \int_0^\infty (\eta^{-\lambda_1} B_t \varphi(\eta))^{p_1} \frac{d\eta}{\eta} \right)^{\frac{1}{p_1}} = \left( \int_t^\infty (\eta^{-\lambda_1} \varphi(\eta))^{p_1} \frac{d\eta}{\eta} \right)^{\frac{1}{p_1}} = J_2(t).$$

Hence

$$I_2 = \|J_2\|_{\Phi_{(1-\theta)(\lambda_0-\lambda_1),q}}.$$

So,

$$(5) \quad I(\lambda_0, \lambda_1, p_0, p_1, \theta, q, \varphi) \leq c_5 \left( \|J_1\|_{\Phi_{\theta(\lambda_1-\lambda_0),q}} + \|J_2\|_{\Phi_{(1-\theta)(\lambda_0-\lambda_1),q}} \right),$$

where  $c_5 = |\lambda_1 - \lambda_0|^{\frac{1}{q}} 2^{(\frac{1}{q}-1)+} \left( c_4 2^{(\frac{1}{q}-1)+} + 1 \right)$ .

4. Next we use the Hardy inequalities of the forms

$$(6) \quad \left\| \int_0^t g(\eta) d\eta \right\|_{\Phi_{\mu,\sigma}} \leq \frac{1}{|\mu|} \|g(t)\|_{\Phi_{\mu-1,\sigma}}$$

$$(7) \quad \left\| \int_t^\infty g(\eta) d\eta \right\|_{\Phi_{\mu,\sigma}} \leq \frac{1}{|\mu|} \|g(t)\|_{\Phi_{\mu-1,\sigma}},$$

where  $\sigma \geq 1, \mu > 0$  in the first inequality,  $\mu < 0$  in the second one, and  $g$  is a non-negative measurable function on  $(0, \infty)$ .

4.1. Assume first that  $p_0, p_1 \leq q$ . If  $\lambda_0 < \lambda_1$ , then by applying inequality (6) with  $\mu = \theta(\lambda_1 - \lambda_0)p_0 > 0$ , we get

$$\begin{aligned} \|J_1\|_{\Phi_{\theta(\lambda_1-\lambda_0),q}} &= \left\| \int_0^t (\eta^{-\lambda_0} \varphi(\eta))^{p_0} \frac{d\eta}{\eta} \right\|_{\Phi_{\theta(\lambda_1-\lambda_0)p_0, \frac{q}{p_0}}}^{\frac{1}{p_0}} \\ &\leq (\theta(\lambda_1 - \lambda_0)p_0)^{-\frac{1}{p_0}} \|t^{-\lambda_0 p_0 - 1} (\varphi(t))^{p_0}\|_{\Phi_{\theta(\lambda_1-\lambda_0)p_0-1, \frac{q}{p_0}}}^{\frac{1}{p_0}} \\ &= (\theta(\lambda_1 - \lambda_0)p_0)^{-\frac{1}{p_0}} \left( \int_0^\infty (t^{-(\theta(\lambda_1-\lambda_0)p_0-1)-\lambda_0 p_0-1} (\varphi(t))^{p_0})^{\frac{q}{p_0}} \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= (\theta(\lambda_1 - \lambda_0)p_0)^{-\frac{1}{p_0}} \|\varphi\|_{\Phi_{\lambda,q}}. \end{aligned}$$

Similarly, by applying inequality (7) with  $\mu = (1 - \theta)(\lambda_0 - \lambda_1)p_1 < 0$ , we get

$$\begin{aligned} \|J_2\|_{\Phi_{(1-\theta)(\lambda_0-\lambda_1),q}} &= \left\| \int_t^\infty (\eta^{-\lambda_1} \varphi(\eta))^{p_1} \frac{d\eta}{\eta} \right\|_{\Phi_{(1-\theta)(\lambda_0-\lambda_1)p_1, \frac{q}{p_1}}}^{\frac{1}{p_1}} \\ &\leq ((1 - \theta)(\lambda_1 - \lambda_0)p_1)^{-\frac{1}{p_1}} \|\varphi\|_{\Phi_{\lambda,q}}. \end{aligned}$$

Hence by estimate (5)

$$I(\lambda_0, p_0, \lambda_1, p_1, \theta, q) \leq c_6 \|\varphi\|_{\Phi_{\lambda,q}},$$

where  $c_6 = c_5 \max\{(\theta(\lambda_1 - \lambda_0)p_0)^{-\frac{1}{p_0}}, ((1 - \theta)(\lambda_1 - \lambda_0)p_1)^{-\frac{1}{p_1}}\}$ .

4.2. In the general case  $0 < p_0, p_1 \leq \infty$  we set  $\tau_0 = \min\{p_0, q\}$ ,  $\tau_1 = \min\{p_1, q\}$ . By Lemma 1 and Steps 2, 3 and 4.1

$$\begin{aligned} I(\lambda_0, \lambda_1, p_0, p_1, \theta, q) &= \left\| \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_0, p_0}} + t \|B_s \varphi\|_{\Phi_{\lambda_1, p_1}} \right) \right\|_{\Phi_{\theta, q}} \\ &\leq \left\| \inf_{s>0} \left( (\lambda_0 p_0)^{\frac{1}{\tau_0} - \frac{1}{p_0}} \|A_s \varphi\|_{\Phi_{\lambda_0, \tau_0}} + t (\lambda_1 p_1)^{\frac{1}{\tau_1} - \frac{1}{p_1}} \|B_s \varphi\|_{\Phi_{\lambda_1, \tau_1}} \right) \right\|_{\Phi_{\theta, q}} \\ &\leq \max\{(\lambda_0 p_0)^{\frac{1}{\tau_0} - \frac{1}{p_0}}, (\lambda_1 p_1)^{\frac{1}{\tau_1} - \frac{1}{p_1}}\} I(\lambda_0, \lambda_1, \tau_0, \tau_1, \theta, q) \leq c_7 \|\varphi\|_{\Phi_{\lambda,q}}, \end{aligned}$$

where  $c_7 = \max\{(\lambda_0 p_0)^{\frac{1}{\tau_0} - \frac{1}{p_0}}, (\lambda_1 p_1)^{\frac{1}{\tau_1} - \frac{1}{p_1}}\} \cdot \tilde{c}_6$ , and  $\tilde{c}_6$  is obtained from  $c_6$  by replacing  $p_0, p_1$  by  $\tau_0, \tau_1$ .

5. If  $\lambda_1 < \lambda_0$ , then by Steps 2-4

$$\begin{aligned} \|\varphi\|_{(\Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow)_{\theta, q}} &= \|\varphi\|_{(\Phi_{\lambda_1, p_1}^\uparrow, \Phi_{\lambda_0, p_0}^\uparrow)_{1-\theta, q}} \\ &\leq \left\| \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_1, p_1}} + t \|B_s \varphi\|_{\Phi_{\lambda_0, p_0}} \right) \right\|_{\Phi_{1-\theta, q}} \\ &= I(\lambda_1, \lambda_0, p_1, p_0, 1 - \theta, q, \varphi) \leq c_8 \|\varphi\|_{\Phi_{\lambda,q}}, \end{aligned}$$

where  $c_8$  is obtained by replacing in  $c_7$   $\lambda_0, \lambda_1, p_0, p_1, \theta$  by  $\lambda_1, \lambda_0, p_1, p_0, 1 - \theta$ , because  $(1 - (1 - \theta))\lambda_1 + (1 - \theta)\lambda_0 = \lambda$  and the proof of the theorem is completed.  $\square$

**Remark 1.** By the above proof if  $\lambda_0 < \lambda_1$ , then

$$(8) \quad c_1 \|\varphi\|_{\Phi_{\lambda,q}} \leq \left\| \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_0, p_0}} + t \|B_s \varphi\|_{\Phi_{\lambda_1, p_1}} \right) \right\|_{\Phi_{\theta, q}} \leq c_2 \|\varphi\|_{\Phi_{\lambda,q}}$$

and if  $\lambda_1 < \lambda_0$ , then

$$(9) \quad c_1 \|\varphi\|_{\Phi_{\lambda,q}} \leq \left\| \inf_{s>0} \left( \|A_s \varphi\|_{\Phi_{\lambda_1, p_1}} + t \|B_s \varphi\|_{\Phi_{\lambda_0, p_0}} \right) \right\|_{\Phi_{1-\theta, q}} \leq c_2 \|\varphi\|_{\Phi_{\lambda,q}}.$$

**Remark 2.** *The most important part of Theorem 2 are inequalities (4), (8) and (9) which will be essentially used in the sequel. As for equality (2), in an equivalent form it was proved in [23]. See also [24].*

**Remark 3.** *Recall that by the well-known results on the interpolation of weighted Lebesgue spaces, under the assumptions of Theorem 2 on the numerical parameters, the equality*

$$(10) \quad (\Phi_{\lambda_0, p_0}, \Phi_{\lambda_1, p_1})_{\theta, q} = \Phi_{\lambda, q}$$

*holds only in the “diagonal case”:  $q = p$ , where  $p$  is defined by the equality  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ , whilst equality (2) holds for any  $0 < q \leq \infty$ . In the diagonal case  $q = p$  equality (10) and the result by M.S. Baouendi and Ch. Goulaonic [1] imply equality (2):*

$$\left( \Phi_{\lambda_0, p_0}^\uparrow, \Phi_{\lambda_1, p_1}^\uparrow \right)_{\theta, p} = (\Phi_{\lambda_0, p_0} \cap M^\uparrow, \Phi_{\lambda_1, p_1} \cap M^\uparrow)_{\theta, p} = \Phi_{\lambda, p} \cap M^\uparrow = \Phi_{\lambda, p}^\uparrow.$$

### 3. INTERPOLATION OF SPACES DEFINED WITH THE HELP OF OPERATORS ACTING TO THE CONE OF NON-NEGATIVE NON-DECREASING FUNCTIONS

Let  $X$  be a linear space on which a functional  $\|\cdot\|: X \rightarrow \mathbb{R}^+$  is defined and  $Z_0, Z_1$  be linear subspaces of  $X$ , on which functionals  $\|\cdot\|_0: Z_0 \rightarrow \mathbb{R}^+, \|\cdot\|_1: Z_1 \rightarrow \mathbb{R}^+$  respectively, are defined. Given  $0 < q \leq \infty, 0 < \theta < 1$ , let  $(Z_0, Z_1)_{\theta, q}$  denote the interpolation space of all  $f \in Z_0 + Z_1$  for which

$$\|f\|_{(Z_0, Z_1)_{\theta, q}} = \|K(Z_0, Z_1, f)\|_{\Phi_{\theta, q}} < \infty,$$

where  $K(Z_0, Z_1, f)$  is the standard  $K$ -functional, defined for  $t > 0$  by

$$K(Z_0, Z_1, f)(t) = \inf_{\substack{f=f_0+f_1 \\ f_0 \in Z_0, f_1 \in Z_1}} (\|f_0\|_{Z_0} + t\|f_1\|_{Z_1}).$$

In this section we consider spaces  $\Phi_{\lambda, q}(F)$  of functions  $f$  defined by the finiteness of the expression

$$\|f\|_{\Phi_{\lambda, q}(F)} = \|Ff\|_{\Phi_{\lambda, q}},$$

where  $f$  belongs to a subspace  $Z \subset X$  and  $F: Z \rightarrow M^\uparrow$ , for various choices of  $Z$  and  $F$ , and interpolation spaces  $(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}$ .

Let, for  $\sigma > 0$ , the operator  $F^\sigma: Z \rightarrow M^\uparrow$  be defined by

$$(F^\sigma f)(t) = ((Ff)(t))^\sigma, \quad f \in Z, \quad t > 0.$$

Given  $\sigma > 0$ , by  $X^\sigma$  we denote the same space  $X$  equipped with the functional  $\|\cdot\|^\sigma$ .

It is easy to verify that

$$(11) \quad \Phi_{\lambda, q}(F) = \left( \Phi_{\lambda\sigma, \frac{q}{\sigma}}(F^\sigma) \right)^{\frac{1}{\sigma}}.$$

Let  $Z_0, Z_1, Z$  be subspaces of  $X$  satisfying

$$Z \subset Z_0 + Z_1$$

and let operators  $F_0, F_1$  and  $F$  be such that

$$(12) \quad F_0: Z_0 \rightarrow M^\uparrow, \quad F_1: Z_1 \rightarrow M^\uparrow, \quad F: Z \rightarrow M^\uparrow.$$

Recall that an operator  $F: Z \rightarrow M^\uparrow$  is called quasi-additive if there exists  $c \geq 1$  such that for each  $f_0, f_1 \in Z$

$$(13) \quad F(f_0 + f_1)(t) \leq c(Ff_0(t) + Ff_1(t)), \quad t > 0.$$

If  $c = 1$ , then  $F$  is subadditive.

**Definition 1.** We say that  $(F, F_0, F_1)$  is a triple of quasi-additive type if there exists  $c \geq 1$  such that for each  $f_0, f_1 \in Z$

$$(14) \quad F(f_0 + f_1)(t) \leq c(F_0f_0(t) + F_1f_1(t)), \quad t > 0.$$

If  $c = 1$ , then  $(F, F_0, F_1)$  is the triple of subadditive type.

We say that  $F$  is weakly quasi-additive if there exist  $c, \alpha \geq 1$  such that for each  $f_0, f_1 \in Z$

$$(15) \quad F(f_0 + f_1)(t) \leq c(Ff_0(\alpha t) + Ff_1(\alpha t)), \quad t > 0.$$

If  $c = 1$ , then  $F$  is weakly subadditive.

We also say that  $(F, F_0, F_1)$  is a triple of weak quasi-additive type if there exist  $c, \alpha \geq 1$  such that for each  $f_0, f_1 \in Z$

$$(16) \quad F(f_0 + f_1)(t) \leq c(F_0f_0(\alpha t) + F_1f_1(\alpha t)), \quad t > 0.$$

If  $c = 1$ , then  $(F, F_0, F_1)$  is the triple of weak subadditive type.

Consider the families  $A = \{A_s\}_{s>0}$  and  $B = \{B_s\}_{s>0}$  of the operators  $A_s, B_s$  defined in Section 2.

**Definition 2.** We say that  $F$  admits an  $A$ - $B$  majorizable decomposition if there exists  $c \geq 1$  and for each  $f \in Z$  and  $s > 0$  there exist  $f_{0,s}, f_{1,s} \in Z$  such that

$$(17) \quad f = f_{0,s} + f_{1,s}$$

and

$$(18) \quad Ff_{0,s}(t) \leq cA_sFf(t), \quad Ff_{1,s}(t) \leq cB_sFf(t), \quad t > 0.$$

We say that  $F$  admits an weakly  $A$ - $B$  majorizable decomposition if there exist  $c, \alpha \geq 1$ , and for each  $f \in Z$  and  $s > 0$  there exist  $f_{0,s}, f_{1,s} \in Z$  such that equality (17) holds and

$$(19) \quad Ff_{0,s}(t) \leq cA_sFf(\alpha t), \quad Ff_{1,s}(t) \leq cB_sFf(\alpha t), \quad t > 0.$$

We also say that a triple  $(F_0, F_1, F)$  admits an  $A$ - $B$  majorizable decomposition if there exists  $c \geq 1$ , and for each  $f \in Z$  and  $s > 0$  there exist  $f_{0,s} \in Z_0, f_{1,s} \in Z_1$  such that equality (17) holds and

$$(20) \quad F_0f_{0,s}(t) \leq cA_sFf(t), \quad F_1f_{1,s}(t) \leq cB_sFf(t), \quad t > 0.$$

Finally, we say that a triple  $(F_0, F_1, F)$  admits an weakly  $A$ - $B$  majorizable decomposition if there exist  $c, \alpha \geq 1$ , and for each  $f \in Z$  and  $s > 0$  there exist  $f_{0,s} \in Z_0$ ,  $f_{1,s} \in Z_1$  such that equality (17) holds and

$$(21) \quad F_0 f_{0,s}(t) \leq c A_s F f(\alpha t), \quad F_1 f_{1,s}(t) \leq c B_s F f(\alpha t), \quad t > 0.$$

**Remark 4.** Let  $L_0, L_1: Z \rightarrow M^\uparrow$ . We shall say that operator  $L_0$  is majorized by operator  $L_1$  and write  $L_0 \prec L_1$  if there exists  $c \geq 1$  such that  $L_0 f(t) \leq c L_1 f(t)$  for all  $f \in Z$  and  $t > 0$ . We shall also say that operator  $L_0$  is weakly majorized by operator  $L_1$  and write  $L_0 \preceq L_1$  if there exist  $c, \alpha \geq 1$  such that  $L_0 f(t) \leq c L_1 f(\alpha t)$  for all  $f \in Z$  and  $t > 0$ . Moreover, let in Definition 2  $H_{0,s}, H_{1,s}: Z \rightarrow M^\uparrow$  be operators defined by

$$H_{0,s} f = f_{0,s}, \quad H_{1,s} f = f_{1,s}, \quad f \in Z.$$

Then

$$F = H_{0,s} + H_{1,s}, \quad s > 0$$

and conditions (20) and (21) respectively can be rewritten as

$$F_0 H_{0,s} \prec A_s F, \quad F_1 H_{1,s} \prec B_s F, \quad s > 0,$$

$$F_0 H_{0,s} \preceq A_s F, \quad F_1 H_{1,s} \preceq B_s F, \quad s > 0,$$

respectively, uniformly in  $s > 0$ , which means that the quantities  $c, \alpha \geq 1$  introduced above are independent of  $s$ . This explains the terminology used in Definition 2.

In the sequel we shall need the following statement.

**Theorem 3** ([2]). Let  $0 < q \leq \infty$ ,  $0 < \theta < 1$ ,  $0 < \beta_0, \beta_1, \beta < \infty$  and

$$\frac{1}{\beta} = \frac{1 - \theta}{\beta_0} + \frac{\theta}{\beta_1}.$$

Then

$$(22) \quad \left( Z_0^{\beta_0}, Z_1^{\beta_1} \right)_{\frac{\theta\beta}{\beta_1}, \frac{q}{\beta}}^{\frac{1}{\beta}} = (Z_0, Z_1)_{\theta, q}.$$

Moreover, there exist  $c_1, c_2 > 0$  depending only on  $q, \theta, \beta_0$  and  $\beta_1$  such that

$$c_1 \|f\|_{(Z_0, Z_1)_{\theta, q}} \leq \|f\|_{\left( Z_0^{\beta_0}, Z_1^{\beta_1} \right)_{\frac{\theta\beta}{\beta_1}, \frac{q}{\beta}}^{\frac{1}{\beta}}} \leq c_2 \|f\|_{(Z_0, Z_1)_{\theta, q}}$$

for all  $f \in (Z_0, Z_1)_{\theta, q}$ .

Statement (22) is the so-called ‘‘Theorem on powers’’, formulated in [2] in an equivalent form for the case in which  $\|\cdot\|_0$  and  $\|\cdot\|_1$  are quasi-norms. The proof of statement (22) for the case of arbitrary functionals  $\|\cdot\|_0: Z_0 \rightarrow \mathbb{R}^+$  and  $\|\cdot\|_1: Z_1 \rightarrow \mathbb{R}^+$  is the same as the proof of statement (22) given in [2] for the case of quasi-norms.

**Theorem 4.** Let  $0 < q_0, q_1, q \leq \infty$ ,  $0 < \theta < 1$ , and let  $Z_0, Z_1, Z$  be subspaces of  $X$  and  $F_0, F_1, F$  be operators satisfying (12).

1. If for some  $0 < \sigma_0, \sigma_1, \sigma < \infty$  satisfying the conditions

$$(23) \quad \frac{1}{\sigma} = \frac{1-\theta}{\sigma_0} + \frac{\theta}{\sigma_1}$$

and

$$(24) \quad \lambda_0 \sigma_0 \neq \lambda_1 \sigma_1,$$

$(F^\sigma, F_0^{\sigma_0}, F_1^{\sigma_1})$  is the triple of weak quasi-additive type, then

$$(25) \quad (\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q} \subset \Phi_{\lambda, q}(F),$$

where  $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$ .

Moreover, there exists  $c_3 > 0$ , depending only on  $q_0, q_1, q, \lambda_0, \lambda_1, \theta, \sigma_0, \sigma_1$  and on the parameters  $c$  and  $\alpha$  in Definition 1, such that

$$(26) \quad c_3 \|f\|_{\Phi_{\lambda, q}(F)} \leq \|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}}.$$

2. If for some  $0 < \sigma_0, \sigma_1, \sigma < \infty$  satisfying conditions (23) and (24), the triple  $(F_0^{\sigma_0}, F_1^{\sigma_1}, F^\sigma)$  admits an weakly  $A$ - $B$  majorizable decomposition, then

$$(27) \quad \Phi_{\lambda, q}(F) \subset (\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}.$$

Moreover, there exists  $c_4 > 0$ , depending only on  $q_0, q_1, q, \lambda_0, \lambda_1, \theta, \sigma_0, \sigma_1$  and on the parameters  $c$  and  $\alpha$  in Definition 1, such that

$$(28) \quad \|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} \leq c_4 \|f\|_{\Phi_{\lambda, q}(F)}.$$

3. If the assumptions of both Part 1 and Part 2 are satisfied, then <sup>3</sup>

$$(29) \quad (\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q} = \Phi_{\lambda, q}(F)$$

and

$$(30) \quad c_3 \|f\|_{\Phi_{\lambda, q}(F)} \leq \|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} \leq c_4 \|f\|_{\Phi_{\lambda, q}(F)}.$$

*Proof.* 1. Let the assumptions of Part 1 be satisfied. Moreover, let  $f \in Z$ ,  $f_0 \in Z_0$ ,  $f_1 \in Z_1$ ,  $f = f_0 + f_1$  and  $\|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} < \infty$ . Then by inequality

$$(16) \quad F^\sigma f(t) = F^\sigma(f_0 + f_1)(t) \leq c(F_0^{\sigma_0} f_0(\alpha t) + F_1^{\sigma_1} f_1(\alpha t)).$$

Hence

$$t^{-\lambda\sigma} F^\sigma f(t) \leq ct^{-\lambda\sigma} \left( (\alpha t)^{\lambda_0\sigma_0} \sup_{s>0} s^{-\lambda_0\sigma_0} F_0^{\sigma_0} f_0(s) + (\alpha t)^{\lambda_1\sigma_1} \sup_{s>0} s^{-\lambda_1\sigma_1} F_1^{\sigma_1} f_1(s) \right)$$

<sup>3</sup> The parameters  $\sigma_0, \sigma_1, \sigma$  in Part 2 may be different from these parameters in Part 1.



$$\begin{aligned} &\leq c_1 \left( t^{\lambda_0 \sigma_0 - \lambda \sigma} \sup_{s>0} s^{-\lambda_0 \sigma_0} F_0^{\sigma_0} f_0(s) + t^{\lambda_1 \sigma_1 - \lambda \sigma} \sup_{s>0} s^{-\lambda_1 \sigma_1} F_1^{\sigma_1} f_1(s) \right) \\ &= c_1 \left( t^{\lambda_0 \sigma_0 - \lambda \sigma} \|F_0^{\sigma_0} f_0\|_{\Phi_{\lambda_0 \sigma_0, \infty}} + t^{\lambda_1 \sigma_1 - \lambda \sigma} \|F_1^{\sigma_1} f_1\|_{\Phi_{\lambda_1 \sigma_1, \infty}} \right), \end{aligned}$$

where  $c_1 = c \max\{\alpha^{\lambda_0 \sigma_0}, \alpha^{\lambda_1 \sigma_1}\}$ . By inequality (1)

$$t^{-\lambda \sigma} F^\sigma f(t) \leq c_2 \left( t^{\lambda_0 \sigma_0 - \lambda \sigma} \|F_0^{\sigma_0} f_0\|_{\Phi_{\lambda_0 \sigma_0, \frac{q_0}{\sigma_0}}} + t^{\lambda_1 \sigma_1 - \lambda \sigma} \|F_1^{\sigma_1} f_1\|_{\Phi_{\lambda_1 \sigma_1, \frac{q_1}{\sigma_1}}} \right),$$

where  $c_2 = c_1 \max\{(\lambda_0 q_0)^{\frac{\sigma_0}{q_0}}, (\lambda_1 q_1)^{\frac{\sigma_1}{q_1}}\}$ .

By condition (23) it follows that

$$\lambda_0 \sigma_0 - \lambda \sigma = -\eta(\lambda_1 \sigma_1 - \lambda_0 \sigma_0), \quad \lambda_1 \sigma_1 - \lambda \sigma = (1 - \eta)(\lambda_1 \sigma_1 - \lambda_0 \sigma_0),$$

where  $\eta = \theta \frac{\sigma}{\sigma_1}$ . Hence

$$\begin{aligned} t^{-\lambda \sigma} F^\sigma f(t) &\leq c_2 \left( t^{-\eta(\lambda_1 \sigma_1 - \lambda_0 \sigma_0)} \|f_0\|_{\Phi_{\lambda_0, q_0}(F_0)}^{\sigma_0} + t^{(1-\eta)(\lambda_1 \sigma_1 - \lambda_0 \sigma_0)} \|f_1\|_{\Phi_{\lambda_1, q_1}(F_1)}^{\sigma_1} \right) \\ &= c_2 t^{-\eta(\lambda_1 \sigma_1 - \lambda_0 \sigma_0)} \left( \|f_0\|_{\Phi_{\lambda_0, q_0}(F_0)}^{\sigma_0} + t^{\lambda_1 \sigma_1 - \lambda_0 \sigma_0} \|f_1\|_{\Phi_{\lambda_1, q_1}(F_1)}^{\sigma_1} \right). \end{aligned}$$

By taking infimum with respect to all  $f_0 \in Z_0, f_1 \in Z_1$  we get

$$t^{-\lambda \sigma} F^\sigma f(t) \leq c_2 t^{-\eta(\lambda_1 \sigma_1 - \lambda_0 \sigma_0)} K((\Phi_{\lambda_0, q_0}(F_0))^{\sigma_0}, (\Phi_{\lambda_1, q_1}(F_1))^{\sigma_1}, f)(t^{\lambda_1 \sigma_1 - \lambda_0 \sigma_0}).$$

Therefore, by (11)

$$\begin{aligned} \|f\|_{\Phi_{\lambda, q}(F)} &= \|f\|_{\Phi_{\lambda \sigma, \frac{q}{\sigma}}(F^\sigma)}^{\frac{1}{\sigma}} = \left( \int_0^\infty (t^{-\lambda \sigma} F^\sigma f(t))^{\frac{q}{\sigma}} \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq c_2^{\frac{1}{\sigma}} \left( \int_0^\infty (t^{-\eta(\lambda_1 \sigma_1 - \lambda_0 \sigma_0)} K((\Phi_{\lambda_0, q_0}(F_0))^{\sigma_0}, (\Phi_{\lambda_1, q_1}(F_1))^{\sigma_1}, f)(t^{\lambda_1 \sigma_1 - \lambda_0 \sigma_0}))^{\frac{q}{\sigma}} \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

After the change of the variable  $t = \tau^{\frac{1}{\lambda_1 \sigma_1 - \lambda_0 \sigma_0}}$  we have

$$\begin{aligned} \|f\|_{\Phi_{\lambda, q}(F)} &\leq c_3 \left[ \left( \int_0^\infty (\tau^{-\eta} K((\Phi_{\lambda_0, q_0}(F_0))^{\sigma_0}, (\Phi_{\lambda_1, q_1}(F_1))^{\sigma_1}, f)(\tau))^{\frac{q}{\sigma}} \frac{d\tau}{\tau} \right)^{\frac{\sigma}{q}} \right]^{\frac{1}{\sigma}} \\ &= c_3 \|f\|_{((\Phi_{\lambda_0, q_0}(F_0))^{\sigma_0}, (\Phi_{\lambda_1, q_1}(F_1))^{\sigma_1})_{\eta, \frac{q}{\sigma}}}^{\frac{1}{\sigma}} = c_3 \|f\|_{((\Phi_{\lambda_0, q_0}(F_0))^{\sigma_0}, (\Phi_{\lambda_1, q_1}(F_1))^{\sigma_1})_{\frac{\theta \sigma}{\sigma_1}, \frac{q}{\sigma}}}^{\frac{1}{\sigma}}, \end{aligned}$$

where  $c_3 = c_2^{\frac{1}{\sigma}} |\lambda_1 \sigma_1 - \lambda_0 \sigma_0|^{-\frac{1}{q}}$ .

By Theorem 3 with  $\beta_0 = \sigma_0, \beta_1 = \sigma_1, \beta = \sigma$  there exists  $\tilde{c}_2 > 0$  depending only on  $q, \theta, \sigma_0, \sigma_1$  such that

$$\|f\|_{\Phi_{\lambda, q}(F)} \leq c_3 \tilde{c}_2 \|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} = c_4 \|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}},$$

where  $c_4$  depends only on  $q_0, q_1, q, \lambda_0, \lambda_1, \theta, \sigma_0, \sigma_1$  and on the parameters  $c$  and  $\alpha$  in Definition 1.

2. Let the assumptions of Part 2 with  $\lambda_0\sigma_0 < \lambda_1\sigma_1$  be satisfied. Applying Theorem 3 with  $\beta_0 = \frac{\sigma_0}{\sigma}$ ,  $\beta_1 = \frac{\sigma_1}{\sigma}$ , hence by (23) with  $\beta = 1$ , we get

$$(31) \quad \begin{aligned} \|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} &\leq \tilde{c}_1^{-1} \|f\|_{(\Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0), \Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1))_{\frac{\theta\sigma}{\sigma_1}, q}} \\ &= \tilde{c}_1^{-1} \left\| K(\Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0), \Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1), f) \right\|_{\Phi_{\frac{\theta\sigma}{\sigma_1}, q}}, \end{aligned}$$

where  $\tilde{c}_1 > 0$  depends only on  $q, \theta, \sigma_0, \sigma_1$ . By inequality (18) and equality (11) we have

$$\begin{aligned} &K\left(\Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0), \Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1), f\right)(t) \\ &= \inf_{f=f_0+f_1} \left( \|f_0\|_{\Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0)} + t \|f_1\|_{\Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1)} \right) \\ &\leq \inf_{s>0} \left( \|f_{0,s}\|_{\Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0)} + t \|f_{1,s}\|_{\Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1)} \right) \\ &= \inf_{s>0} \left( \|f_{0,s}\|_{\Phi_{\lambda_0\sigma_0, \frac{q_0}{\sigma_0}}^{\frac{1}{\sigma}}(F_0^{\sigma_0})} + t \|f_{1,s}\|_{\Phi_{\lambda_1\sigma_1, \frac{q_1}{\sigma_1}}^{\frac{1}{\sigma}}(F_1^{\sigma_1})} \right) \\ &= \inf_{s>0} \left( \|F_0^{\sigma_0} f_{0,s}\|_{\Phi_{\lambda_0\sigma_0, \frac{q_0}{\sigma_0}}^{\frac{1}{\sigma}}(F_0^{\sigma_0})} + t \|F_1^{\sigma_1} f_{1,s}\|_{\Phi_{\lambda_1\sigma_1, \frac{q_1}{\sigma_1}}^{\frac{1}{\sigma}}(F_1^{\sigma_1})} \right) \\ &= \inf_{s>0} \left( \left( \int_0^\infty (\rho^{-\lambda_0\sigma_0} F_0^{\sigma_0} f_{0,s}(\rho))^{\frac{q_0}{\sigma_0}} \frac{d\rho}{\rho} \right)^{\frac{\sigma_0}{q_0\sigma}} \right. \\ &\quad \left. + t \left( \int_0^\infty (\rho^{-\lambda_1\sigma_1} F_1^{\sigma_1} f_{1,s}(\rho))^{\frac{q_1}{\sigma_1}} \frac{d\rho}{\rho} \right)^{\frac{\sigma_1}{q_1\sigma}} \right) \\ &\leq c \inf_{s>0} \left( \left( \int_0^\infty (\rho^{-\lambda_0\sigma_0} A_s F^\sigma f(\alpha\rho))^{\frac{q_0}{\sigma_0}} \frac{d\rho}{\rho} \right)^{\frac{\sigma_0}{q_0\sigma}} \right. \\ &\quad \left. + t \left( \int_0^\infty (\rho^{-\lambda_1\sigma_1} B_s F^\sigma f(\alpha\rho))^{\frac{q_1}{\sigma_1}} \frac{d\rho}{\rho} \right)^{\frac{\sigma_1}{q_1\sigma}} \right) \\ &= c \inf_{s>0} \left( \alpha^{\frac{\lambda_0}{\sigma}} \left( \int_0^\infty (\tau^{-\lambda_0\sigma_0} A_s F^\sigma f(\tau))^{\frac{q_0}{\sigma_0}} \frac{d\tau}{\tau} \right)^{\frac{\sigma_0}{q_0\sigma}} \right. \\ &\quad \left. + \alpha^{\frac{\lambda_1}{\sigma}} t \left( \int_0^\infty (\tau^{-\lambda_1\sigma_1} B_s F^\sigma f(\tau))^{\frac{q_1}{\sigma_1}} \frac{d\tau}{\tau} \right)^{\frac{\sigma_1}{q_1\sigma}} \right) \\ &\leq c_5 \inf_{s>0} \left( \left( \int_0^\infty (\tau^{-\lambda_0\sigma_0} A_s F^\sigma f(\tau))^{\frac{q_0}{\sigma_0}} \frac{d\tau}{\tau} \right)^{\frac{\sigma_0}{q_0\sigma}} \right. \end{aligned}$$

$$\begin{aligned}
& +t \left( \int_0^\infty (\tau^{-\lambda_1 \sigma_1} B_s F^\sigma f(\tau))^{\frac{q_1}{\sigma_1}} \frac{d\tau}{\tau} \right)^{\frac{\sigma_1}{q_1 \sigma}} \\
& = c_5 \inf_{s>0} \left( \|A_s F^\sigma f\|_{\Phi_{\lambda_0 \sigma_0, \frac{q_0}{\sigma_0}}}^{\frac{1}{\sigma}} + t \|B_s F^\sigma f\|_{\Phi_{\lambda_1 \sigma_1, \frac{q_1}{\sigma_1}}}^{\frac{1}{\sigma}} \right),
\end{aligned}$$

where  $c_5 = c\alpha^{\max\{\frac{\lambda_0}{\sigma}, \frac{\lambda_1}{\sigma}\}}$ .

Since

$$(32) \quad A_s F^\sigma f = (A_s F f)^\sigma, \quad B_s F^\sigma f = (B_s f)^\sigma,$$

by (11) we get

$$K \left( \Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0), \Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1), f \right) (t) \leq c_5 \inf_{s>0} \left( \|A_s F f\|_{\Phi_{\lambda_0 \sigma_0, \frac{q_0}{\sigma_0}}} + t \|B_s F f\|_{\Phi_{\lambda_1 \sigma_1, \frac{q_1}{\sigma_1}}} \right).$$

Therefore, by inequality (31) and by inequality (8) of Theorem 2 with  $\lambda_0, \lambda_1, p_0, p_1$  replaced by  $\frac{\lambda_0 \sigma_0}{\sigma}, \frac{\lambda_1 \sigma_1}{\sigma}, \frac{q_0 \sigma}{\sigma_0}, \frac{q_1 \sigma}{\sigma_1}$  and with  $\varphi$  replaced by  $Ff$  we have

$$\begin{aligned}
\|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} & \leq \tilde{c}_1^{-1} c_5 \left\| \inf_{s>0} \left( \|A_s F f\|_{\Phi_{\lambda_0 \sigma_0, \frac{q_0}{\sigma_0}}} + t \|B_s F f\|_{\Phi_{\lambda_1 \sigma_1, \frac{q_1}{\sigma_1}}} \right) \right\|_{\Phi_{\frac{\theta \sigma}{\sigma_1}, q}} \\
& \leq c_6 \|Ff\|_{\Phi_{\lambda, q}} = c_6 \|f\|_{\Phi_{\lambda, q}(F)},
\end{aligned}$$

where  $c_6 > 0$  depends only on  $q_0, q_1, q, \lambda_0, \lambda_1, \theta, \sigma_0, \sigma_1$  and on the parameters  $c$  and  $\alpha$  in Definition 1, because by (23)

$$\left(1 - \frac{\theta \sigma}{\sigma_1}\right) \frac{\lambda_0 \sigma_0}{\sigma} + \frac{\theta \sigma}{\sigma_1} \frac{\lambda_1 \sigma_1}{\sigma} = (1 - \theta) \lambda_0 + \theta \lambda_1 = \lambda.$$

3. Let the assumptions of Part 2 with  $\lambda_1 \sigma_1 < \lambda_0 \sigma_0$  be satisfied. Then by Theorem 3 with  $\lambda_0, \lambda_1, q_0, q_1, \theta$  replaced by  $\lambda_1, \lambda_0, q_1, q_0, 1 - \theta$  and with  $\beta_0 = \frac{\sigma_1}{\sigma}, \beta_1 = \frac{\sigma_0}{\sigma}$ , hence by (23) again with  $\beta = 1$ , we get

$$\begin{aligned}
\|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} & = \|f\|_{(\Phi_{\lambda_1, q_1}(F_1), \Phi_{\lambda_0, q_0}(F_0))_{1-\theta, q}} \\
& \leq \tilde{c}_1^{-1} \|f\|_{\left(\Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1), \Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0)\right)_{\frac{(1-\theta)\sigma}{\sigma_0}, q}} \\
& = \tilde{c}_1^{-1} \left\| K \left( \Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1), \Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0), f \right) \right\|_{\Phi_{\frac{(1-\theta)\sigma}{\sigma_0}, q}},
\end{aligned}$$

where  $\tilde{c}_1^{-1} > 0$  depends only on  $q, \theta, \sigma_0, \sigma_1$ .

Similarly to Step 2

$$\begin{aligned}
K \left( \Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1), \Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0), f \right) (t) & \leq \inf_{s>0} \left( \|f_{1, s}\|_{\Phi_{\lambda_1, q_1}^{\frac{\sigma_1}{\sigma}}(F_1)} + t \|f_{0, s}\|_{\Phi_{\lambda_0, q_0}^{\frac{\sigma_0}{\sigma}}(F_0)} \right) \\
& \leq c_7 \inf_{s>0} \left( \|A_s F f\|_{\Phi_{\lambda_1 \sigma_1, \frac{q_1}{\sigma_1}}} + t \|B_s F f\|_{\Phi_{\lambda_0 \sigma_0, \frac{q_0}{\sigma_0}}} \right),
\end{aligned}$$

hence by inequality (9) of Theorem 2

$$\begin{aligned}
& \|f\|_{(\Phi_{\lambda_0, q_0}(F_0), \Phi_{\lambda_1, q_1}(F_1))_{\theta, q}} \\
& \leq \tilde{c}_1^{-1} c_7 \left\| \inf_{s>0} \left( \|A_s F f\|_{\Phi_{\frac{\lambda_1 \sigma_1}{\sigma}, \frac{q_1 \sigma}{\sigma_1}}} + t \|B_s F f\|_{\Phi_{\frac{\lambda_0 \sigma_0}{\sigma}, \frac{q_0 \sigma}{\sigma_0}}} \right) \right\|_{\Phi_{\frac{(1-\theta)\sigma}{\sigma_0}, q}} \\
& \leq c_8 \|f\|_{\Phi_{\lambda, q}(F)},
\end{aligned}$$

where  $c_7$  and  $c_8$  are obtained from  $c_5$  and from  $c_6$  respectively by replacing  $\lambda_0, \lambda_1, q_0, q_1, q, \theta, \sigma_0, \sigma_1$  by  $\lambda_1, \lambda_1, q_1, q_0, q, 1 - \theta, \sigma_1, \sigma_0$ , because by (23) again we have

$$\left(1 - \frac{(1-\theta)\sigma}{\sigma_0}\right) \frac{\lambda_1 \sigma_1}{\sigma} + \frac{(1-\theta)\sigma}{\sigma_0} \frac{\lambda_0 \sigma_0}{\sigma} = \theta \lambda_1 + (1-\theta) \lambda_0 = \lambda,$$

and the proof of the theorem is completed.  $\square$

**Remark 5.** In contrast to the equality  $B_s F^\sigma f = (B_s f)^\sigma$  in formula (32), which is valid for any  $\sigma > 0$ , the equality  $(I - A_s) F^\sigma f = ((I - A_s) f)^\sigma$  holds only if  $\sigma = 1$ . If  $\sigma < 1$ , then  $(I - A_s) F^\sigma f \leq ((I - A_s) f)^\sigma$  for all  $f \in Z$ . However, if  $\sigma > 1$ , then the inequality  $(I - A_s) F^\sigma f \leq c((I - A_s) f)^\sigma$  with any  $c \geq 1$  independent of  $f \in Z$  does not hold. This is one of the reasons why in Definition 2 the operator  $B_s$  is used rather than the operator  $I - A_s$ .

Next we formulate a particular case of Theorem 4 which will mostly be used in the sequel.

**Theorem 4'.** Let  $0 < q_0, q_1, q \leq \infty$ ,  $\lambda_0 \neq \lambda_1$ ,  $0 < \theta < 1$ , and  $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$ . Moreover, let  $Z$  be a subspace of  $X$  and  $F: Z \rightarrow M^\uparrow$ .

1. If  $F$  is weakly quasi-additive, then

$$\Phi_{\lambda, q}(F) \subset (\Phi_{\lambda_0, q_0}(F), \Phi_{\lambda_1, q_1}(F))_{\theta, q}.$$

2. If  $F$  admits an weakly  $A$ - $B$  majorizable decomposition, then

$$(\Phi_{\lambda_0, q_0}(F), \Phi_{\lambda_1, q_1}(F))_{\theta, q} \subset \Phi_{\lambda, q}(F).$$

3. If the assumptions of both Part 1 and Part 2 are satisfied, then

$$(\Phi_{\lambda_0, q_0}(F), \Phi_{\lambda_1, q_1}(F))_{\theta, q} = \Phi_{\lambda, q}(F).$$

The above inclusions and the equality are accompanied by inequalities (26), (28) and (30) respectively with  $F_0 = F_1 = F$ .

#### 4. EXAMPLES OF OPERATORS $F$ AND SPACES $\Phi_{\lambda, q}(F)$

##### 4.1. Local Morrey spaces and general local Morrey-type spaces. Ex-

**ample 1.** Let  $0 < p, q \leq \infty$  and  $0 < \lambda < \infty$ . Let  $Z = L_p^{loc}(\mathbb{R}^n)$  and

$Ff(t) = \|f\|_{L_p(B(0,t))}$ ,  $f \in Z, t > 0$ , then the space  $\Phi_{\lambda,q}(F)$  coincides with the local Morrey-type space  $LM_{p,q}^\lambda$  considered in [8], and

$$\|f\|_{\Phi_{\lambda,q}(F)} = \|f\|_{LM_{p,q}^\lambda} \equiv \left( \int_0^\infty (t^{-\lambda} \|f\|_{L_p(B(0,t))})^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

**Example 2.** Let  $0 < p, q \leq \infty$  and  $0 < \lambda < \infty$  if  $q < \infty$  and  $0 \leq \lambda < \infty$  if  $q = \infty$ . Let  $\Omega \subset \mathbb{R}^n$  and let  $\mu$  be a  $\sigma$ -finite Borel measure on  $\Omega$  and  $\mu(\Omega) > 0$ . Let  $G = \{G_t\}_{t>0}$  be a family of  $\mu$ -measurable subsets of  $\Omega$  for which

$$(33) \quad G_t \neq \Omega \text{ for some } t > 0, \quad G_{t_1} \subset G_{t_2} \text{ if } 0 < t_1 < t_2 < \infty \text{ and } \bigcup_{t>0} G_t = \Omega.$$

Let  $Z$  be the space of all functions  $f$   $\mu$ -measurable on  $\Omega$  and such that  $\|f\|_{L_p(G_t,\mu)} < \infty$  for all  $t > 0$ , where

$$\|f\|_{L_p(G_t,\mu)} = \left( \int_{G_t} |f(x)|^p d\mu \right)^{\frac{1}{p}},$$

if  $p < \infty$ , and

$$\|f\|_{L_\infty(G_t,\mu)} = \text{ess sup}_{x \in G_t} |f(x)| \equiv \inf_{g \subset G_t, \mu(g)=0} \sup_{x \in G_t \setminus g} |f(x)|$$

if  $p = \infty$ . Moreover, let

$$(34) \quad Ff(t) = \|f\|_{L_p(G_t,\mu)}, \quad f \in Z, \quad t > 0.$$

We denote the space  $\Phi_{\lambda,q}(F)$  by  $LM_{p,q}^\lambda(G, \mu)$ . It is the space of all functions  $f$   $\mu$ -measurable on  $\Omega$  with finite quasi-norm

$$\|f\|_{\Phi_{\lambda,q}(F)} = \|f\|_{LM_{p,q}^\lambda(G,\mu)} \equiv \left( \int_0^\infty (t^{-\lambda} \|f\|_{L_p(G_t,\mu)})^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

The space  $LM_{p,q}^\lambda(G, \mu)$  is a one of the variants of general variant of local Morrey-type spaces. The spaces  $LM_{p,q}^\lambda$  clearly correspond to the case in which  $G_t = B(0, t)$  and  $\mu$  is the Lebesgue measure on  $\mathbb{R}^n$ .

Note that, for any family  $G$  of subsets of set  $\Omega$  satisfying (33) and any measure  $\mu$  on  $\Omega$  the space  $LM_{p,q}^\lambda(G, \mu)$  is non-trivial, i.e. consists not only on functions  $\mu$ -equivalent to 0 on  $\Omega$ .

Indeed, if  $f \in L_p(G, \mu)$ ,  $f$  is not  $\mu$ -equivalent to 0 on  $\Omega$  and  $\tau > 0$ , then the function  $f\chi_{\mathbb{R}^n \setminus G_\tau} \in LM_{p,q}^\lambda(G, \mu)$ , because

$$\|f\chi_{\mathbb{R}^n \setminus G_\tau}\|_{LM_{p,q}^\lambda(G,\mu)} \leq \begin{cases} (\lambda q)^{-\frac{1}{q}} \|f\|_{L_p(G,\mu)}, & \text{if } q < \infty, \\ \|f\|_{L_p(G,\mu)}, & \text{if } q = \infty. \end{cases}$$

Moreover, for some  $\tau$ ,  $f\chi_{\mathbb{R}^n \setminus G_\tau}$  is not  $\mu$ -equivalent to 0 on  $\Omega$ . (Otherwise  $f$  is  $\mu$ -equivalent to 0 on  $\Omega$ .)

Spaces of such type with slightly different definitions and various operators acting in these spaces were intensively studied in the last three decades. See survey papers [5, 6, 14, 16, 21, 25, 26].

Next we discuss the relationship between this definition and some other close definitions.

**Example 3.** Let  $0 < p, q \leq \infty$  and  $a \geq 0$ . Let  $v$  be a function positive, locally absolutely continuous, strictly increasing on  $(a, \infty)$  and such that  $v = 0$  on  $(0, a]$ ,  $\lim_{t \rightarrow a^+} v(t) = \alpha$ ,  $\lim_{t \rightarrow +\infty} v(t) = \infty$ . Moreover, let  $\mu$  be the Lebesgue measure on  $\mathbb{R}^n$ . If in Example 2 we take  $\lambda = 1$ ,  $\Omega = \mathbb{R}^n$ ,  $Z = L_p^{loc}(\mathbb{R}^n)$ ,

$$G_t = \begin{cases} \emptyset, & \text{if } 0 < t \leq a, \\ B(0, v^{(-1)}(t)), & \text{if } \alpha < t < \infty, \end{cases}$$

and

$$Ff(t) = \|f\|_{L_p(G_t)} = \begin{cases} 0, & \text{if } 0 < t \leq a, \\ \|f\|_{L_p(B(0, v^{(-1)}(t)))}, & \text{if } \alpha < t < \infty, \end{cases}$$

then the space  $\Phi_{1,q}(F)$  is the space  $LM_{pq}^{v(\cdot)}$  of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  with finite quasi-norm

$$\begin{aligned} \Phi_{1,q}(F) &= \left( \int_0^\infty (t^{-1} \|f\|_{L_p(G_t)})^q \frac{dt}{t} \right)^{1/q} = \left( \int_\alpha^\infty (t^{-1} \|f\|_{L_p(B(0, v^{(-1)}(t)))})^q \frac{dt}{t} \right)^{1/q} \\ &= (v^{(-1)}(t) = r) = \|f\|_{LM_{pq}^{v(\cdot)}} \equiv \left( \int_a^\infty \left( \frac{\|f\|_{L_p(B(0, r))}}{v(r)} \right)^q \frac{dv(r)}{v(r)} \right)^{1/q}. \end{aligned}$$

A similar argument shows that for any  $0 < \lambda < \infty$ ,  $\Phi_{\lambda,q}(F) = LM_{pq}^{v^\lambda(\cdot)}$  and

$$\|f\|_{\Phi_{\lambda,q}(F)} = \lambda^{-\frac{1}{q}} \|f\|_{LM_{pq}^{v^\lambda(\cdot)}}.$$

Note also that if  $F_\lambda f(t) = 0$  for  $0 < t \leq a$  and  $F_\lambda f(t) = \|f\|_{L_p(B(0, (v^\frac{1}{\lambda})^{(-1)}(t)))}$  for  $t > a$ , then  $\Phi_{\frac{1}{\lambda},q}(F_\lambda) = LM_{pq}^{v(\cdot)}$  and

$$\|f\|_{\Phi_{\frac{1}{\lambda},q}(F_\lambda)} = \lambda^{-\frac{1}{q}} \|f\|_{LM_{pq}^{v(\cdot)}}.$$

**Example 4.** Let  $0 < p, q \leq \infty$ , and let  $w$  be a positive measurable function on  $(0, \infty)$  such that  $\|w\|_{L_q(t, \infty)} < \infty$  for some  $t > 0$ . Set  $a = \inf\{t > 0 : \|w\|_{L_q(t, \infty)} < \infty\}$ , and let in Example 3

$$v(t) = \begin{cases} 0, & \text{if } 0 < t \leq a, \\ q^{-\frac{1}{q}} \|w\|_{L_q(t, \infty)}^{-1}, & \text{if } a < t < \infty, \end{cases}$$

and  $\alpha = \lim_{t \rightarrow a^+} q^{-\frac{1}{q}} \|w\|_{L_q(t, \infty)}^{-1}$ . Clearly, if  $q < \infty$ , then  $\lim_{t \rightarrow +\infty} v(t) = \infty$ . If  $q = \infty$ , then, in order that this equality also hold, we shall assume, in addition, that  $\lim_{t \rightarrow +\infty} \|w\|_{L_\infty(t, \infty)} = 0$ .

In this case  $Z = L_p^{loc}(\mathbb{R}^n)$  and  $\Phi_{1,q}(F) = LM_{pq,w(\cdot)}$  – the general local Morrey-type space of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  with finite quasi-norm

$$\|f\|_{\Phi_{1,q}(F)} = \|f\|_{LM_{pq,w(\cdot)}} \equiv \|w(r)\|_{L_p(B(0,r))} \|f\|_{L_q(0,\infty)},$$

because  $v^{-q-1}(r)v'(r) = w^q(r)$  for almost all  $r > a$ .

Usually this space is considered under slightly weaker assumptions on  $w$ , namely it is assumed that  $w \in \Omega_q \Leftrightarrow w$  is non-negative measurable on  $(0, \infty)$ , not equivalent to 0 and such that  $\|w\|_{L_q(t, \infty)} < \infty$  for some  $t > 0$ . Recall that, if a function  $w$  non-negative measurable on  $(0, \infty)$  and not equivalent to 0, then the space  $LM_{pq,w(\cdot)}$  is non-trivial if and only if  $w \in \Omega_q$ . ( See [5].)

**Lemma 2.** *Let  $0 < p, q \leq \infty$ , and  $F: L_p^{loc}(\mathbb{R}^n) \rightarrow M^\uparrow$  be defined by (34). Then  $F$  is quasi-additive and admits an  $A$ - $B$  majorizable decomposition.*

*Proof.* 1. The operator  $F$  is quasi-additive because for all  $f_0, f_1 \in L_p^{loc}(\mathbb{R}^n)$  for all  $t > 0$

$$\begin{aligned} F(f_0 + f_1)(t) &= \|f_0 + f_1\|_{L_p(G_t)} \leq 2^{(\frac{1}{p}-1)+} (\|f_0\|_{L_p(G_t)} + \|f_1\|_{L_p(G_t)}) \\ &= 2^{(\frac{1}{p}-1)+} (Ff_0(t) + Ff_1(t)). \end{aligned}$$

2. Given  $f \in L_p^{loc}(\mathbb{R}^n)$  and  $s > 0$ . Set  $f_{0,s} = f\chi_{G_s}$ ,  $f_{1,s} = f\chi_{\mathbb{R}^n \setminus G_s}$ . Then  $f = f_{0,s} + f_{1,s}$ . Moreover, for all  $t > 0$

$$\begin{aligned} Ff_{0,s}(t) &= \|f\chi_{G_s}\|_{L_p(G_t)} = \begin{cases} \|f\|_{L_p(G_t)}, & \text{if } 0 < t \leq s, \\ \|f\|_{L_p(G_s)}, & \text{if } t > s \end{cases} \\ &= \begin{cases} Ff(t), & \text{if } 0 < t \leq s, \\ Ff(s), & \text{if } t > s \end{cases} = A_s Ff(t) \end{aligned}$$

and

$$Ff_{1,s}(t) = \|f\chi_{G_s^c}\|_{L_p(G_t)} = 0 \text{ if } t \leq s, \quad Ff_{1,s}(t) \leq \|f\|_{L_p(G_t)} \text{ if } t > s,$$

hence

$$(35) \quad Ff_{1,s}(t) \leq B_s Ff(t).$$

Hence operator  $F$  admits an  $A$ - $B$  majorizable decomposition.  $\square$

**Remark 6.** If  $p = 1$ , then  $Ff_{1,s}(t) = (I - A_s)Ff(t)$  for  $t > s$ . If  $0 < p < 1$ , then for  $t > s$

$$Ff_{1,s}(t) = \left( \int_{G_t} |f(x)|^p dx - \int_{G_s} |f(x)|^p dx \right)^{\frac{1}{p}} \leq \|f\|_{L_p(G_t)} - \|f\|_{L_p(G_s)}.$$

Hence, for  $0 < p \leq 1$ ,  $Ff_{1,s}(t) \leq (I - A_s)Ff(t)$ ,  $t > 0$ . However, for  $p > 1$ , the inequality  $Ff_{1,s}(t) \leq c(I - A_s)Ff(t)$ ,  $t > 0$  does not hold for any  $c \geq 1$  independent of  $t$  and  $f$ . Importantly, inequality (35) holds. This is one more reason why in Definition 2 the operator  $B_s$  is used rather than the operator  $I - A_s$ . (See also Remark 5.)

**Remark 7.** In (34)  $G$  and  $\mu$  do not depend on a function  $f$ , however one can assume that in the definition of the spaces  $LM_{p,q}^\lambda(G, \mu)$  the family  $G$  and the measure  $\mu$  depend on  $f \in Z$  with the assumption that condition (33) is satisfied. In this case

$$Ff(t) = \|f\|_{L_p(G_t(f), \mu(f))}, \quad t > 0$$

and

$$\|f\|_{\Phi_{\lambda,q}(F)} = \|f\|_{LM_{p,q}^\lambda(G(f), \mu(f))} \equiv \left( \int_0^\infty (t^{-\lambda} \|f\|_{L_p(G_t(f), \mu(f))})^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

See Examples 7, 8 and 9 below for more details. In this case one can apply Theorem 2 but there is no guarantee that one can apply Theorem 4, since there is no guarantee that Lemma 2 will be true or that  $F$  will be a weakly quasi-additive and admitting a weakly  $A$ - $B$  majorizable decomposition.

**4.2. Weighted Lebesgue spaces.** In this subsection  $\Omega \subset \mathbb{R}^n$ ,  $\mu$  is a  $\sigma$ -finite Borel measure on  $\Omega$ , and  $w$  is a  $\mu$ -measurable positive function on  $\Omega$  (weight function). By  $L_p(\Omega, w, \mu)$ , where  $0 < p \leq \infty$ , we denote the space of all  $\mu$ -measurable functions on  $\Omega$  for which

$$\|f\|_{L_p(\Omega, w, \mu)} = \left( \int_{\Omega} (w(x)|f(x)|)^p d\mu \right)^{\frac{1}{p}} < \infty.$$

If  $w \equiv 1$ , then  $L_p(\Omega, 1, \mu) = L_p(\Omega, \mu)$ ; if  $\mu$  is the Lebesgue measure, then  $L_p(\Omega, w, \mu) \equiv L_p(\Omega, w)$  and  $L_p(\Omega, \mu) \equiv L_p(\Omega)$ .

**Example 5.** If  $0 < p \leq \infty$  and  $q = \infty$ , then, for any family of  $\mu$ -measurable subsets  $G_t$  of set  $\Omega$  satisfying (33),  $LM_{p,\infty}^0(G, \mu) = L_p(\Omega, \mu)$  and

$$\|f\|_{LM_{p,\infty}^0(G, \mu)} = \|f\|_{L_p(\Omega, \mu)}.$$



**Example 6.** Let  $G = \{G_t\}_{t>0}$ , where

$$G_t = \left\{x \in \Omega : w(x) > \frac{1}{t}\right\}.$$

Then condition (33) is satisfied and for any  $0 < p \leq \infty$   $L_p(\Omega, w, \mu) = LM_{p,p}^1(G, \mu)$ . Moreover, for any  $0 < \lambda < \infty$ ,  $LM_{p,p}^\lambda(G, \mu) = L_p(\Omega, w^\lambda, \mu)$  and

$$\|f\|_{LM_{p,p}^\lambda(G, \mu)} = (\lambda p)^{-\frac{1}{p}} \|f\|_{L_p(\Omega, w^\lambda, \mu)}.$$

Indeed, since the measure  $\mu$  is  $\sigma$ -finite, then by applying Fubini's theorem we get

$$\begin{aligned} \|f\|_{LM_{p,p}^\lambda(G, \mu)} &= \left( \int_0^\infty \left( t^{-\lambda} \|f\|_{L_p(G_t)} \right)^p \frac{dt}{t} \right)^{1/p} \\ &= \left( \int_0^\infty t^{-\lambda p - 1} \left( \int_{G_t} |f(x)|^p d\mu \right) dt \right)^{1/p} \\ &= \left( \int_\Omega \left( \int_{w(x)^{-1}}^\infty t^{-\lambda p - 1} dt \right) |f(x)|^p d\mu \right)^{1/p} \\ &= (\lambda p)^{-\frac{1}{p}} \left( \int_\Omega (w^\lambda(x) |f(x)|)^p d\mu \right)^{1/p} \\ &= (\lambda p)^{-\frac{1}{p}} \|f\|_{L_p(\Omega, w^\lambda, \mu)}. \end{aligned}$$

**Example 7.** If in Example 6  $w \equiv 1$ , then  $G_t = \emptyset$  if  $t \leq 1$  and  $G_t = \Omega$  if  $t > 1$ . Therefore, for any  $0 < p, q \leq \infty$  and  $0 < \lambda < \infty$ ,  $LM_{p,q}^\lambda(G, \mu) = L_p(\Omega, \mu)$  and

$$\|f\|_{LM_{p,q}^\lambda(G, \mu)} = (\lambda q)^{-\frac{1}{q}} \|f\|_{L_p(\Omega, \mu)}.$$

**Example 8.** Let  $0 < p \leq \infty$ ,  $0 < \lambda_0, \lambda_1 < \infty$ ,  $\lambda_0 \neq \lambda_1$ . Moreover, let  $w_0, w_1$  be positive  $\mu$ -measurable functions on  $\Omega$ ,  $G_{\lambda_0, \lambda_1} = \{G_{t, \lambda_0, \lambda_1}\}_{t>0}$ ,

$$G_{t, \lambda_0, \lambda_1} = \{x \in \Omega : w_0^{\alpha_0}(x) w_1^{\alpha_1}(x) < t\}, \quad t > 0,$$

and for  $p < \infty$

$$d\nu_{\lambda_0, \lambda_1} = \left( w_0^{\beta_0}(x) w_1^{\beta_1}(x) \right)^p d\mu,$$

where

$$\alpha_0 = \frac{1}{\lambda_1 - \lambda_0}, \quad \alpha_1 = \frac{1}{\lambda_0 - \lambda_1}, \quad \beta_0 = \frac{\lambda_1}{\lambda_1 - \lambda_0}, \quad \beta_1 = \frac{\lambda_0}{\lambda_0 - \lambda_1}.$$

Then

$$LM_{p,p}^{\lambda_0}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1}) = L_p(\Omega, w_0, \mu), \quad LM_{p,p}^{\lambda_1}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1}) = L_p(\Omega, w_1, \mu)$$

and

$$\begin{aligned} \|f\|_{LM_{p,p}^{\lambda_0}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})} &= (\lambda_0 p)^{-\frac{1}{p}} \|f\|_{L_p(\Omega, w_0, \mu)}, \quad \|f\|_{LM_{p,p}^{\lambda_1}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})} \\ &= (\lambda_1 p)^{-\frac{1}{p}} \|f\|_{L_p(\Omega, w_1, \mu)}. \end{aligned}$$

Indeed, if  $p < \infty$ , then

$$\begin{aligned} \|f\|_{LM_{p,p}^{\lambda_0}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})}^p &= \int_0^\infty t^{-\lambda_0 p} \left( \int_{x \in \Omega: w_0^{\alpha_0}(x) w_1^{\alpha_1}(x) < t} \left( |f(x)| w_0^{\beta_0}(x) w_1^{\beta_1}(x) \right)^p d\mu \right) \frac{dt}{t} \\ &= \int_\Omega |f(x)|^p w_0^{\beta_0 p}(x) w_1^{\beta_1 p}(x) \left( \int_{w_0^{\alpha_0}(x) w_1^{\alpha_1}(x)}^\infty t^{-\lambda_0 p - 1} dt \right) d\mu \\ &= \frac{1}{\lambda_0 p} \int_\Omega (|f(x)| w_0(x))^p d\mu, \end{aligned}$$

because  $\beta_0 - \lambda_0 \alpha_0 = 1$ ,  $\beta_1 - \alpha_1 \lambda_0 = 0$ , and

$$\begin{aligned} \|f\|_{LM_{p,p}^{\lambda_1}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})}^p &= \int_0^\infty t^{-\lambda_1 p} \left( \int_{x \in \Omega: w_0^{\alpha_0}(x) w_1^{\alpha_1}(x) < t} (|f(x)| w_0^{\beta_0}(x) w_1^{\beta_1}(x))^p d\mu \right) \frac{dt}{t} \\ &= \int_\Omega w_0^{\beta_0 p}(x) w_1^{\beta_1 p}(x) |f(x)|^p \left( \int_{w_0^{\alpha_0}(x) w_1^{\alpha_1}(x)}^\infty t^{-\lambda_1 p - 1} dt \right) d\mu \\ &= \frac{1}{\lambda_1 p} \int_\Omega (|f(x)| w_1(x))^p d\mu, \end{aligned}$$

because  $\beta_0 - \lambda_1 \alpha_0 = 0$ ,  $\beta_1 - \alpha_1 \lambda_1 = 1$ .

If  $p = \infty$ , then we assume that

$$(36) \quad \|f\|_{LM_{\infty,\infty}^{\lambda_k}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})} = \|f w_0^{\beta_0} w_1^{\beta_1}\|_{LM_{\infty,\infty}^{\lambda_k}(G_{\lambda_0,\lambda_1}, \mu)}, \quad k = 1, 2.$$

Then

$$LM_{\infty,\infty}^{\lambda_0}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1}) = L_\infty(\Omega, w_0, \mu), \quad LM_{\infty,\infty}^{\lambda_1}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1}) = L_\infty(\Omega, w_1, \mu)$$

and

$$\|f\|_{LM_{\infty,\infty}^{\lambda_0}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})} = \|f\|_{L_\infty(\Omega, w_0, \mu)}, \quad \|f\|_{LM_{\infty,\infty}^{\lambda_1}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})} = \|f\|_{L_\infty(\Omega, w_1, \mu)}.$$

Indeed, by (36)

$$\|f\|_{LM_{\infty,\infty}^{\lambda_0}(G_{\lambda_0,\lambda_1}, \nu_{\lambda_0,\lambda_1})} = \left\| t^{-\lambda_0} \|f w_0^{\beta_0} w_1^{\beta_1}\|_{L_\infty(\{x \in \Omega: w_0^{\alpha_0}(x) w_1^{\alpha_1}(x) < t\}, \mu)} \right\|_{L_\infty(0, \infty)}$$

$$\begin{aligned}
&= \left\| f w_0^{\beta_0} w_1^{\beta_1} \| t^{-\lambda_0} \|_{L_\infty(w_0^{\alpha_0}(x) w_1^{\alpha_1}(x), \infty)} \right\|_{L_\infty(\Omega, \mu)} \\
&= \| f w_0 \|_{L_\infty(\Omega, \mu)}.
\end{aligned}$$

The proof of the second equality is similar.

**Example 9.** Let  $w_0, w_1$  be positive  $\mu$ -measurable functions on  $\Omega$ ,  $0 < p_0, p_1 \leq \infty$ ,  $p_0 \neq p_1$ , and

$$h_1(x) = \left( \frac{w_0(x)}{w_1(x)} \right)^{\frac{p_0 p_1}{p_1 - p_0}}, \quad h_2(x) = \left( \frac{w_0^{p_0}(x)}{w_1^{p_1}(x)} \right)^{\frac{1}{p_1 - p_0}}, \quad x \in \Omega.$$

Moreover, let  $Z$  be the space of all functions  $\mu$ -measurable on  $\Omega$  and the operator  $H: Z \rightarrow M^\uparrow$  be defined by

$$Hf(t) = \int_{G_t(f)} h_1 d\mu, \quad f \in Z, \quad t > 0,$$

where

$$G_t(f) = \left\{ x \in \Omega : |f(x)| > \frac{h_2(x)}{t} \right\}.$$

**Lemma 3.** *The operator  $H$  is weakly subadditive and admits an weakly  $A$ - $B$  majorizable decomposition.*

*Proof.* 1. Note that for any  $f_0, f_1 \in Z$  and  $t > 0$

$$G_t(f_0 + f_1) \subset G_{2t}(f_0) \cup G_{2t}(f_1).$$

Indeed, if  $x \in \Omega$  and  $|f_0(x) + f_1(x)| > \frac{h_2(x)}{t}$ , then either  $|f_0(x)| > \frac{h_2(x)}{2t}$  or  $|f_1(x)| > \frac{h_2(x)}{2t}$ , because otherwise  $|f_0(x) + f_1(x)| \leq |f_0(x)| + |f_1(x)| \leq \frac{h_2(x)}{t}$ . Therefore,

$$\begin{aligned}
H(f_0 + f_1)(t) &= \int_{G_t(f_0 + f_1)} h_1(x) d\mu \leq \int_{G_{2t}(f_0)} h_1(x) d\mu + \int_{G_{2t}(f_1)} h_1(x) d\mu \\
&= Hf_0(2t) + Hf_1(2t).
\end{aligned}$$

2. Given  $f \in Z$  and  $s > 0$ , set

$$f_{0,s} = f \chi_{G_s(f)}, \quad f_{1,s} = f \chi_{\mathbb{R}^n \setminus G_s(f)}.$$

Then  $f = f_{0,s} + f_{1,s}$  and

$$\begin{aligned}
G_t(f_{0,s}) &= \left\{ x \in \Omega : |f(x)| \chi_{G_s(f)}(x) > \frac{h_2(x)}{t} \right\} \\
&= \left\{ x \in G_s(f) : |f(x)| > \frac{h_2(x)}{t} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ x \in \Omega : |f(x)| > \max \left\{ \frac{1}{s}, \frac{1}{t} \right\} h_2(x) \right\} \\
&= \begin{cases} G_t(f), & \text{if } 0 < t \leq s, \\ G_s(f), & \text{if } s < t < \infty. \end{cases}
\end{aligned}$$

Hence for all  $s, t > 0$

$$\begin{aligned}
Hf_{0,s}(t) &= \begin{cases} \int h_1(x) d\mu, & \text{if } 0 < t \leq s, \\ \int_{G_t(f)} h_1(x) d\mu, & \text{if } s < t \leq \infty \end{cases} \\
&= \begin{cases} Hf(t), & \text{if } 0 < t \leq s, \\ Hf(s), & \text{if } s < t \leq \infty \end{cases} = A_s Hf(t).
\end{aligned}$$

Moreover,

$$\begin{aligned}
G_t(f_{1,s}) &= \left\{ x \in \Omega : |f(x)| \chi_{c_{G_s(f)}}(x) > \frac{h_2(x)}{t} \right\} \\
&= \left\{ x \in \Omega \setminus G_s(f) : |f(x)| > \frac{h_2(x)}{t} \right\} \\
&= \left\{ x \in \Omega : \frac{h_2(x)}{t} < |f(x)| \leq \frac{h_2(x)}{s} \right\} \\
&= \begin{cases} \emptyset, & \text{if } 0 < t \leq s, \\ G_t(f) \setminus G_s(f), & \text{if } s < t < \infty. \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
Hf_{1,s}(t) &= \begin{cases} 0, & \text{if } 0 < t \leq s, \\ \int_{G_t(f)} h_1(x) d\mu - \int_{G_s(f)} h_1(x) d\mu, & \text{if } s < t \leq \infty \end{cases} \\
&= \begin{cases} 0, & \text{if } 0 < t \leq s, \\ Hf(t) - Hf(s), & \text{if } s < t < \infty \end{cases} \\
&= (I - A_s)Hf(t) \leq B_s Hf(t).
\end{aligned}$$

□

**Remark 8.** A stronger property than (18) was proved, namely,

$$Hf_{0,s}(t) = A_s Hf(t), \quad Hf_{1,s}(t) = (I - A_s)Hf(t), \quad t > 0.$$

Next, let for  $f \in Z$  the operators  $F_0, F_1, F$  be defined by

$$(37) \quad F_0 = H^{\frac{1}{p_0}}, \quad F_1 = H^{\frac{1}{p_1}}, \quad F = H^{\frac{1}{p}}.$$

**Lemma 4.** Let  $0 < p_0, p_1, p \leq \infty$ ,  $p_0 \neq p_1$ ,  $0 < \theta < 1$ , and

$$(38) \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$(39) \quad \|f\|_{\Phi_{1,p_0}(F_0)} = p_0^{-\frac{1}{p_0}} \|f\|_{L_{p_0}(\Omega, w_0, \mu)},$$

$$(40) \quad \|f\|_{\Phi_{1,p_1}(F_1)} = p_1^{-\frac{1}{p_1}} \|f\|_{L_{p_1}(\Omega, w_1, \mu)}$$

and

$$(41) \quad \|f\|_{\Phi_{1,p}(F)} = p^{-\frac{1}{p}} \|f\|_{L_p(\Omega, w_0^{1-\theta} w_1^\theta, \mu)}.$$

*Proof.* 1. Let  $\Omega_0(f) = \{x \in \Omega : f(x) \neq 0\}$ . Then  $\bigcup_{t>0} G_t(f) = \Omega_0(f)$  and

$$\begin{aligned} \|f\|_{L_{p_0}(\Omega, w_0, \mu)}^{p_0} &= \int_{\Omega_0(f)} (|f(x)| w_0(x))^{p_0} d\mu \\ &= \int_{\Omega_0(f)} \left( \frac{w_0(x)}{w_1(x)} \right)^{\frac{p_0 p_1}{p_1 - p_0}} \left( |f(x)| \left( \frac{w_0^{p_0}(x)}{w_1^{p_1}(x)} \right)^{\frac{1}{p_0 - p_1}} \right)^{p_0} d\mu \\ &= \int_{\Omega_0(f)} h_1(x) (|f(x)| h_2^{-1}(x))^{p_0} d\mu \\ &= p_0 \int_{\Omega_0(f)} h_1(x) \left( \int_{|f(x)|^{-1} h_2(x)}^{\infty} t^{-p_0-1} dt \right) d\mu \\ &= p_0 \int_0^{\infty} t^{-p_0} \left( \int_{|f(x)|^{-1} h_2(x) < t} h_1(x) d\mu \right) \frac{dt}{t} \\ &= p_0 \int_0^{\infty} t^{-p_0} \left( \int_{G_t(f)} h_1(x) d\mu \right) \frac{dt}{t} \\ &= p_0 \int_0^{\infty} t^{-p_0} H f(t) \frac{dt}{t} \\ &= p_0 \int_0^{\infty} (t^{-1} F_0 f(t))^{p_0} \frac{dt}{t} \\ &= p_0 \|F_0 f\|_{\Phi_{1,p_0}}^{p_0} = p_0 \|f\|_{\Phi_{1,p_0}(F_0)}^{p_0}, \end{aligned}$$

hence equality (39) follows.

2. Similarly,

$$\begin{aligned}
\|f\|_{L_{p_1}(\Omega, w_1, \mu)}^{p_1} &= \int_{\Omega_0(f)} (|f(x)| w_1(x))^{p_1} d\mu \\
&= \int_{\Omega_0(f)} \left( \frac{w_0(x)}{w_1(x)} \right)^{\frac{p_0 p_1}{p_1 - p_0}} \left( |f(x)| \left( \frac{w_0^{p_0}(x)}{w_1^{p_1}(x)} \right)^{\frac{1}{p_0 - p_1}} \right)^{p_1} d\mu \\
&= \int_{\Omega_0(f)} h_1(x) (|f(x)| h_2^{-1}(x))^{p_1} d\mu \\
&= p_1 \int_0^\infty t^{-p_1} H f(t) \frac{dt}{t} \\
&= p_1 \int_0^\infty (t^{-1} F_1 f(t))^{p_1} \frac{dt}{t} = p_1 \|f\|_{\Phi_{1, p_1}(F_1)}^{p_1},
\end{aligned}$$

and equality (40) follows.

3. Finally, by (39)

$$\begin{aligned}
\|f\|_{L_p(\Omega, w_0^{1-\theta} w_1^\theta, \mu)}^p &= \int_{\Omega_0(f)} (|f(x)| w_0^{1-\theta}(x) w_1^\theta(x))^p d\mu \\
&= \int_{\Omega_0(f)} \left( \frac{w_0(x)}{w_1(x)} \right)^{\frac{p_0 p_1}{p_1 - p_0}} \left( |f(x)| \left( \frac{w_0^{p_0}(x)}{w_1^{p_1}(x)} \right)^{\frac{1}{p_0 - p_1}} \right)^p d\mu \\
&= \int_{\Omega_0(f)} h_1(x) (|f(x)| h_2^{-1}(x))^p d\mu,
\end{aligned}$$

since by (38)

$$\frac{p_0 p_1}{p_1 - p_0} + \frac{p_0 p}{p_0 - p_1} = (1 - \theta)p, \quad -\frac{p_0 p_1}{p_1 - p_0} - \frac{p_1 p}{p_0 - p_1} = \theta p.$$

Hence as above

$$\|f\|_{L_p(\Omega, w_0^{1-\theta} w_1^\theta)}^p = p \|f\|_{\Phi_{1, p}(F)}^p,$$

and equality (41) follows.  $\square$

**Remark 9.** If, given  $f \in Z$  such that  $f(x) \neq 0$  for almost all  $x \in \Omega$ , we introduce the measure  $\mu_{p_0}(f)$  defined by  $d\mu_{p_0}(f) = |f(x)|^{-p_0} h_1(x) d\mu$ , then

$$\|f\|_{LM_{p_0, p_0}^1(G(f), \mu_{p_0}(f))} = \left( \int_0^\infty t^{-p_0} \|f\|_{L_{p_0}(G_t(f), \mu_{p_0}(f))}^{p_0} \frac{dt}{t} \right)^{\frac{1}{p_0}}$$

$$\begin{aligned}
&= \left( \int_0^\infty t^{-p_0} \left( \int_{G_t(f)} |f(x)|^{p_0} |f(x)|^{-p_0} h_1(x) d\mu \right) \frac{dt}{t} \right)^{\frac{1}{p_0}} \\
&= \|f\|_{\Phi_{1,p_0}(F_0)} = p_0^{-\frac{1}{p_0}} \|f\|_{L_{p_0}(\Omega, w_0)}
\end{aligned}$$

and similar equalities hold if  $p_0$  is replaced by  $p_1$  or  $p$ .

**Corollary 1.** Let  $0 < p, p_0, p_1 \leq \infty$ ,  $p_0 \neq p_1$ ,  $0 < \theta < 1$  and condition (37) be satisfied. Then

$$\begin{aligned}
1) \quad &\Phi_{1,p_0}(F_0) = L_{p_0}(\Omega, w_0, \mu), \quad \Phi_{1,p_1}(F_1) = L_{p_1}(\Omega, w_1, \mu), \\
&\Phi_{1,p}(F) = L_p(\Omega, w_0^{1-\theta} w_1^\theta, \mu).
\end{aligned}$$

2)  $(F^p, F_0^{p_0}, F_1^{p_1})$  is the triple of weak subadditive type.

3) The triple  $(F_0^{p_0}, F_1^{p_1}, F^p)$  admits an weakly  $A$ - $B$  majorizable decomposition.

*Proof.* Property 1) follows by Lemma 4. Properties 2), 3) follow by Lemma 3, since by (37)  $F_0^{p_0} = F_1^{p_1} = F^p = H$ .  $\square$

**4.3. Lorentz spaces.** In this subsection  $\Omega \subset \mathbb{R}^n$  and  $\mu$  is a  $\sigma$ -finite Borel measure on  $\Omega$ .

**Example 10.** Let  $Z$  be the space of all functions  $f$   $\mu$ -measurable on  $\Omega$  and such that  $m_f(t) = \mu(\{x \in \Omega : |f(x)| > t\}) < \infty$  for all  $t > 0$ . ( $m_f$  is the distribution function of  $f$ .) Let the operator  $H: Z \rightarrow M^\uparrow$  be defined by

$$Hf(t) = m_f\left(\frac{1}{t}\right), \quad f \in Z, \quad t > 0,$$

hence  $Hf(t) = \mu(G_t(f))$ , where  $G_t(f) = \{x \in \Omega : |f(x)| > \frac{1}{t}\}$ . Then  $\Phi_{1,q}(H^{\frac{1}{p}})$  is the Lorentz space  $L_{p,q}(\Omega, \mu)$  and

$$\begin{aligned}
\|f\|_{\Phi_{1,q}(H^{\frac{1}{p}})} &= \|H^{\frac{1}{p}} f\|_{\Phi_{1,q}} = \|(Hf)^{\frac{1}{p}}\|_{\Phi_{1,q}} \\
&= \left( \int_0^\infty \left( t^{-1} (Hf(t))^{\frac{1}{p}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= \left( \int_0^\infty \left( t^{-1} \left( m_f\left(\frac{1}{t}\right) \right)^{\frac{1}{p}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \left( \frac{1}{t} = \tau \right) \\
&= \left( \int_0^\infty \left( \tau (m_f(\tau))^{\frac{1}{p}} \right)^q \frac{d\tau}{\tau} \right)^{\frac{1}{q}} \equiv \|f\|_{L_{p,q}(\Omega, \mu)}.
\end{aligned}$$

**Lemma 5.** *The operator  $H: Z \rightarrow M^+$  is weakly subadditive and admits an A-B majorizable decomposition.*

*Proof.* 1. By the properties of distribution functions, given  $f_0, f_1 \in Z$ , we have

$$H(f_0 + f_1)(t) = m_{f_0+f_1} \left( \frac{1}{t} \right) \leq m_{f_0} \left( \frac{1}{2t} \right) + m_{f_1} \left( \frac{1}{2t} \right) = Hf_0(2t) + Hf_1(2t).$$

2. Furthermore, given  $f \in Z$ , we set

$$f_{0,s} = f\chi_{G_s(f)}, \quad f_{1,s} = f - f_{0,s} = f\chi_{\mathbb{R}^n \setminus G_s(f)}.$$

Then for all  $t > 0$

$$\begin{aligned} Hf_{0,s}(t) &= \mu \left( \left\{ x \in \Omega : |f(x)|\chi_{G_s(f)}(x) > \frac{1}{t} \right\} \right) \\ &= \mu \left( \left\{ x \in G_s(f) : |f(x)| > \frac{1}{t} \right\} \right) \\ &= \mu(G_t(f) \cap G_s(f)) \\ &= \begin{cases} \mu(G_t(f)), & \text{if } 0 < t \leq s, \\ \mu(G_s(f)), & \text{if } s < t < \infty \end{cases} \\ &= \begin{cases} Hf(t), & \text{if } 0 < t \leq s, \\ Hf(s), & \text{if } s < t < \infty \end{cases} = A_s Hf(t) \end{aligned}$$

and

$$\begin{aligned} Hf_{1,s}(t) &= \mu \left( \left\{ x \in \Omega : |f(x)|\chi_{\mathbb{R}^n \setminus G_s(f)}(x) > \frac{1}{t} \right\} \right) \\ &= \mu \left( \left\{ x \in \Omega \setminus G_s(f) : |f(x)| > \frac{1}{t} \right\} \right) \\ &= \mu(G_t(f) \setminus G_s(f)) \\ &= \begin{cases} 0, & \text{if } 0 < t \leq s, \\ \mu(G_t(f)) - \mu(G_s(f)), & \text{if } s < t < \infty \end{cases} \\ &= (I - A_s)Hf(t) \leq B_s Hf(t). \end{aligned} \quad \square$$

**Remark 10.** *A stronger property than (18) was proved, namely,*

$$Hf_{0,s}(t) = A_s Hf(t), \quad Hf_{1,s}(t) = (I - A_s)Hf(t), \quad t > 0.$$

**Remark 11.** *If, given  $f \in Z$  such that  $f(x) \neq 0$  for almost all  $x \in \Omega$ , we introduce the measure  $\mu(f)$  defined by  $d\mu(f) = |f(x)|^{-p}d\mu$ , then*

$$\|f\|_{LM_{p,q}^1(G(f), \mu(f))} = \left( \int_0^\infty t^{-q} \|f\|_{L_p(G_t(f), \mu(f))}^q \frac{dt}{t} \right)^{\frac{1}{q}}$$



$$\begin{aligned}
&= \left( \int_0^\infty t^{-q} \left( \int_{G_t(f)} |f(x)|^p |f(x)|^{-p} d\mu \right)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= \left( \int_0^\infty t^{-q} (Hf(t))^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= \left( \int_0^\infty \left( t^{-1} (Hf(t))^{\frac{1}{p}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \|f\|_{L_{p,q}(\Omega, \mu)}.
\end{aligned}$$

**Example 11.** Given  $f \in Z$ , next we set

$$Ff(t) = f^* \left( \frac{1}{t} \right), \quad t > 0,$$

where  $f^*$  is the non-increasing rearrangement of  $f$ , i.e.  $f^*(t) = \inf\{\tau \in (0, \infty) : m_f(\tau) \leq t\}$ ,  $t > 0$ . Then, for any  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $\Phi_{\frac{1}{p}, q}(F) = L_{p,q}(\Omega, \mu)$  and

$$\|f\|_{\Phi_{\frac{1}{p}, q}(F)} = \left( \int_0^\infty \left( t^{-\frac{1}{p}} f^* \left( \frac{1}{t} \right) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \left( \int_0^\infty (\tau^{\frac{1}{p}} f^*(\tau))^q \frac{d\tau}{\tau} \right)^{\frac{1}{q}} \equiv \|f\|_{L_{p,q}(\Omega, \mu)}.$$

**Lemma 6.** *The operator  $F: Z \rightarrow M^\uparrow$  is weakly subadditive and admits an  $A$ - $B$  majorizable decomposition.*

*Proof.* 1. By the properties of rearrangements, given  $f_0, f_1 \in Z$

$$F(f_0 + f_1)(t) = (f_0 + f_1)^* \left( \frac{1}{t} \right) \leq f_0^* \left( \frac{1}{2t} \right) + f_1^* \left( \frac{1}{2t} \right) = Ff_0(2t) + Ff_1(2t).$$

2. Furthermore, given  $f \in Z$ , we set

$$f_{0,s}(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq f^*\left(\frac{1}{s}\right), \\ f^*\left(\frac{1}{s}\right), & \text{if } f(x) > f^*\left(\frac{1}{s}\right), \\ -f^*\left(\frac{1}{s}\right), & \text{if } f(x) < -f^*\left(\frac{1}{s}\right) \end{cases}$$

and  $f_{1,s}(x) = f(x) - f_{0,s}$ . Then

$$|f_{0,s}(x)| = \begin{cases} |f(x)|, & \text{if } |f(x)| \leq f^*\left(\frac{1}{s}\right), \\ f^*\left(\frac{1}{s}\right), & \text{if } |f(x)| > f^*\left(\frac{1}{s}\right). \end{cases}$$

Hence for all  $t > 0$

$$m_{f_{0,s}}(t) = \mu(\{x \in \Omega : |f_{0,s}| > t\})$$

$$\begin{aligned}
&= \begin{cases} \mu(\{x \in \Omega : t < |f(x)| \leq f^*(\frac{1}{s})\}), & \text{if } t < f^*(\frac{1}{s}), \\ 0, & \text{if } t \geq f^*(\frac{1}{s}) \end{cases} \\
&= \begin{cases} \mu(\{x \in \Omega : |f(x)| > t\}) - \mu(\{x \in \Omega : |f(x)| > f^*(\frac{1}{s})\}), & \text{if } t < f^*(\frac{1}{s}), \\ 0, & \text{if } t \geq f^*(\frac{1}{s}) \end{cases} \\
&= \begin{cases} m_f(t) - m_f(f^*(\frac{1}{s})), & \text{if } t < f^*(\frac{1}{s}), \\ 0, & \text{if } t \geq f^*(\frac{1}{s}). \end{cases}
\end{aligned}$$

Next consider the function  $g_s$  of one variable defined by

$$g_s(x) = \begin{cases} f^*(\frac{1}{s}), & \text{if } f^*(x) > f^*(\frac{1}{s}), \quad x > 0, \\ f^*(x), & \text{if } f^*(x) \leq f^*(\frac{1}{s}), \quad x > 0. \end{cases}$$

Note that<sup>4</sup>

$$\begin{aligned}
m_{g_s}(t) &= |\{x \in (0, \infty) : g_s(x) > t\}| \\
&= \begin{cases} |\{x \in (0, \infty) : t < f^*(x) \leq f^*(\frac{1}{s})\}|, & \text{if } t < f^*(\frac{1}{s}), \\ 0, & \text{if } t \geq f^*(\frac{1}{s}) \end{cases} \\
&= \begin{cases} |\{x \in (0, \infty) : f^*(x) > t\}| - |\{x \in (0, \infty) : f^*(x) > f^*(\frac{1}{s})\}|, & \text{if } t < f^*(\frac{1}{s}), \\ 0, & \text{if } t \geq f^*(\frac{1}{s}) \end{cases} \\
&= \begin{cases} m_{f^*}(t) - m_{f^*}(f^*(\frac{1}{s})), & \text{if } t < f^*(\frac{1}{s}), \\ 0, & \text{if } t \geq f^*(\frac{1}{s}). \end{cases}
\end{aligned}$$

Since  $m_f(t) = m_{f^*}(t)$  for all  $t > 0$  it follows that  $m_{f_{0,s}}(t) = m_{g_s}(t)$  for all  $t > 0$ , hence follows that  $f_{0,s}^*(t) = g_s^*(t)$  and

$$\begin{aligned}
Ff_{0,s}(t) &= f_{0,s}^*\left(\frac{1}{t}\right) \\
&= g_s^*\left(\frac{1}{t}\right) = g_s\left(\frac{1}{t}\right) \\
&= \begin{cases} f^*(\frac{1}{t}), & \text{if } t \leq s, \\ f^*(\frac{1}{s}), & \text{if } t > s \end{cases} = A_s Ff(t).
\end{aligned}$$

<sup>4</sup> For a set  $\Omega \subset \mathbb{R}$ ,  $|\Omega|$  denotes the one-dimensional Lebesgue measure of  $\Omega$ .

Next for all  $x \in \Omega$

$$f_{1,s}(x) = f(x) - f_{0,s} = \begin{cases} 0, & \text{if } |f(x)| \leq f^*\left(\frac{1}{s}\right), \\ f(x) - f^*\left(\frac{1}{s}\right), & \text{if } f(x) > f^*\left(\frac{1}{s}\right), \\ f(x) + f^*\left(\frac{1}{s}\right), & \text{if } f(x) < -f^*\left(\frac{1}{s}\right), \end{cases}$$

hence

$$\begin{aligned} |f_{1,s}(x)| &= \begin{cases} 0, & \text{if } |f(x)| \leq f^*\left(\frac{1}{s}\right), \\ f(x) - f^*\left(\frac{1}{s}\right), & \text{if } f(x) > f^*\left(\frac{1}{s}\right), \\ -f(x) - f^*\left(\frac{1}{s}\right), & \text{if } f(x) < -f^*\left(\frac{1}{s}\right) \end{cases} \\ &= \begin{cases} 0, & \text{if } |f(x)| \leq f^*\left(\frac{1}{s}\right), \\ |f(x)| - f^*\left(\frac{1}{s}\right), & \text{if } |f(x)| > f^*\left(\frac{1}{s}\right). \end{cases} \end{aligned}$$

Therefore, for all  $t > 0$

$$\begin{aligned} m_{f_{1,s}}(t) &= \mu(\{x \in \Omega : |f_{1,s}| > t\}) \\ &= \mu\left(\left\{x \in \Omega : |f(x)| > f^*\left(\frac{1}{s}\right), |f_{1,s}(x)| > t\right\}\right) \\ &= \mu\left(\left\{x \in \Omega : |f(x)| > f^*\left(\frac{1}{s}\right), |f(x)| - f^*\left(\frac{1}{s}\right) > t\right\}\right) \\ &= \mu\left(\left\{x \in \Omega : |f(x)| > t + f^*\left(\frac{1}{s}\right)\right\}\right) \\ &= m_f\left(t + f^*\left(\frac{1}{s}\right)\right). \end{aligned}$$

Since for all  $\tau > 0$   $m_f(\tau + f^*(\frac{1}{s})) \leq m_f(f^*(\frac{1}{s})) \leq \frac{1}{s}$  it follows that

$$\begin{aligned} f_{1,s}^*(t) &= \inf\{\tau \in (0, \infty) : m_{f_{1,s}}(\tau) \leq t\} \\ &= \begin{cases} \inf\{\tau \in (0, \infty) : m_f(\tau + f^*(\frac{1}{s})) \leq t\}, & \text{if } t < \frac{1}{s}, \\ 0, & \text{if } t \geq \frac{1}{s} \end{cases} \\ &= \begin{cases} \inf\{\eta - f^*(\frac{1}{s}) : \eta > f^*(\frac{1}{s}), m_f(\eta) \leq t\}, & \text{if } t < \frac{1}{s}, \\ 0, & \text{if } t \geq \frac{1}{s} \end{cases} \\ &= \begin{cases} \inf\{\eta > f^*(\frac{1}{s}) : m_f(\eta) \leq t\} - f^*(\frac{1}{s}), & \text{if } t < \frac{1}{s}, \\ 0, & \text{if } t \geq \frac{1}{s} \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \max\{f^*(\frac{1}{s}), f^*(t)\} - f^*(\frac{1}{s}), & \text{if } t < \frac{1}{s}, \\ 0, & \text{if } t \geq \frac{1}{s} \end{cases} \\
&= \begin{cases} f^*(t) - f^*(\frac{1}{s}), & \text{if } t < \frac{1}{s}, \\ 0, & \text{if } t \geq \frac{1}{s}. \end{cases}
\end{aligned}$$

Thus for all  $t > 0$

$$\begin{aligned}
Ff_{1,s}(t) &= f_{1,s}^* \left( \frac{1}{t} \right) \\
&= \begin{cases} 0, & \text{if } t \leq s, \\ f^*(\frac{1}{t}) - f^*(\frac{1}{s}), & \text{if } t > s \end{cases} \\
&= \begin{cases} 0, & \text{if } t \leq s, \\ Ff(t) - Ff(s), & \text{if } t > s \end{cases} = (I - A_s)Ff(t) \leq B_s Ff(t). \quad \square
\end{aligned}$$

**Remark 12.** A stronger property than (18) was proved, namely,

$$Ff_{0,s}(t) = A_s Ff(t), \quad Ff_{1,s}(t) = (I - A_s)Ff(t), \quad t > 0.$$

## 5. INTERPOLATION THEOREM FOR GENERAL LOCAL MORREY-TYPE SPACES

**Theorem 5.** Let  $0 < p, q_0, q_1, q \leq \infty$ ,  $0 < \lambda_0, \lambda_1 < \infty$ ,  $\lambda_0 \neq \lambda_1$ , and  $0 < \theta < 1$ ,  $\Omega \subset \mathbb{R}^n$ , and let  $\mu$  be a  $\sigma$ -finite Borel measure on  $\Omega$  and  $G = \{G_t\}_{t>0}$  be a family of  $\mu$ -measurable sets  $G_t$ , satisfying (33). Then

$$(42) \quad (LM_{p,q_0}^{\lambda_0}(G, \mu), LM_{p,q_1}^{\lambda_1}(G, \mu))_{\theta,q} = LM_{p,q}^{\lambda}(G, \mu),$$

where  $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ . Moreover, there exist  $c_1, c_2 > 0$ , depending only on  $p, q_0, q_1, q, \lambda_0, \lambda_1$  and  $\theta$ , such that

$$(43) \quad c_1 \|f\|_{LM_{p,q}^{\lambda}(G, \mu)} \leq \|f\|_{(LM_{p,q_0}^{\lambda_0}(G, \mu), LM_{p,q_1}^{\lambda_1}(G, \mu))_{\theta,q}} \leq c_2 \|f\|_{LM_{p,q}^{\lambda}(G, \mu)}$$

for all  $f \in LM_{p,q}^{\lambda}(G, \mu)$ .

*Proof.* Let the space  $Z$  and operator  $F: Z \rightarrow M^{\dagger}$  be as in Example 2. Then by Lemma 2 and Theorem 4' it follows that

$$\begin{aligned}
(LM_{p,q_0}^{\lambda_0}(G, \mu), LM_{p,q_1}^{\lambda_1}(G, \mu))_{\theta,q} &= (\Phi_{\lambda_0,q_0}(F), \Phi_{\lambda_1,q_1}(F))_{\theta,q} \\
&= \Phi_{\lambda,q}(F) = LM_{p,q}^{\lambda}(G, \mu),
\end{aligned}$$

hence it follows that equality (42) equipped with inequality (43) holds.  $\square$

**Corollary 2.** Let  $0 < p, q_0, q_1, q \leq \infty$ ,  $0 < \lambda_0, \lambda_1 < \infty$ ,  $\lambda_0 \neq \lambda_1$ , and  $0 < \theta < 1$ . Then

$$(LM_{p,q_0}^{\lambda_0}, LM_{p,q_1}^{\lambda_1})_{\theta,q} = LM_{p,q}^{\lambda},$$

where  $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ .

*Proof.* Let  $G = \{B(0, t)\}_{t>0}$  and  $\mu$  be the Lebesgue measure on  $\mathbb{R}^n$ . Then

$$\begin{aligned} (LM_{p,q_0}^{\lambda_0}, LM_{p,q_1}^{\lambda_1})_{\theta,q} &= (LM_{p,q_0}^{\lambda_0}(G, \mu), LM_{p,q_1}^{\lambda_1}(G, \mu))_{\theta,q} \\ &= LM_{p,q}^{\lambda}(G, \mu) = LM_{p,q}^{\lambda}. \end{aligned} \quad \square$$

**Remark 13.** Compared with Corollary 2 in Theorem 1 there are additional assumptions on  $\lambda_0$  and  $\lambda_1$ :  $\lambda_0, \lambda_1 < \frac{n}{p}$  if  $q < \infty$  and  $\lambda_0, \lambda_1 \leq \frac{n}{p}$  if  $q = \infty$ . They appeared because in [8] in the proof of Theorem 1 the equality  $LM_{p,q}^{\lambda} = \widetilde{LM}_{p,q}^{\lambda}$  was used where  $\widetilde{LM}_{p,q}^{\lambda}$  is the space of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  such that

$$\|f\|_{\widetilde{LM}_{p,q}^{\lambda}} = \left( \int_0^\infty \left( r^{-\lambda} \|f\|_{\tilde{L}_p(B(0,r))} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}} < \infty,$$

where

$$\|f\|_{\tilde{L}_p(B(x,r))} = |B(x, r)|^{\frac{1}{p}} \sup_{\rho \geq r} \left( \frac{1}{|B(x, \rho)|} \int_{B(x, \rho)} |f|^p dy \right)^{\frac{1}{p}},$$

and  $|B(x, t)|$  is the Lebesgue measure of the ball  $B(x, t)$ , and this equality holds only under the additional assumptions on  $\lambda_0$  and  $\lambda_1$  mentioned above.

**Corollary 3.** Let  $0 < p, q_0, q_1, q \leq \infty$ ,  $0 < \lambda_0, \lambda_1 < \infty$ ,  $\lambda_0 \neq \lambda_1$ ,  $0 < \theta < 1$ , and function  $v$  be as in Example 3. Then

$$(LM_{p,q_0}^{v^{\lambda_0}(\cdot)}, LM_{p,q_1}^{v^{\lambda_1}(\cdot)})_{\theta,q} = LM_{p,q}^{v^{\lambda}(\cdot)},$$

where  $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ .

*Proof.* Let the family  $G$  and measure  $\mu$  be as in Example 3. Then by Theorem 5

$$\begin{aligned} (LM_{p,q_0}^{v^{\lambda_0}(\cdot)}, LM_{p,q_1}^{v^{\lambda_1}(\cdot)})_{\theta,q} &= (LM_{p,q_0}^{\lambda_0}(G, \mu), LM_{p,q_1}^{\lambda_1}(G, \mu))_{\theta,q} \\ &= LM_{p,q}^{\lambda}(G, \mu) = LM_{p,q}^{v^{\lambda}(\cdot)}. \end{aligned} \quad \square$$

**Remark 14.** Note that the equality

$$(44) \quad (LM_{pq_0, w^{\lambda_0}(\cdot)}, LM_{pq_1, w^{\lambda_1}(\cdot)})_{\theta,q} = LM_{pq, w^{\lambda}(\cdot)},$$

where  $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ , may not hold even for the case of the power function  $w(r) = r^{-s}$ ,  $s > 0$ . In this case equality (44) holds if  $\lambda$  is replaced by

$$\nu = \lambda + \frac{1}{s} \left( \frac{1}{q} - \frac{1 - \theta}{q_0} - \frac{\theta}{q_1} \right),$$

hence equality (44) holds only if

$$(45) \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

Indeed, by Corollary 2

$$\begin{aligned} (LM_{pq_0, (r^{-s})^{\lambda_0}(\cdot)}, LM_{pq_1, (r^{-s})^{\lambda_1}(\cdot)})_{\theta, q} &= \left( LM_{p, q_0}^{s\lambda_0 - \frac{1}{q_0}}, LM_{p, q_1}^{s\lambda_1 - \frac{1}{q_1}} \right)_{\theta, q} \\ &= LM_{p, q}^{\nu s - \frac{1}{q}} = LM_{pq, (r^{-s})^\nu}. \end{aligned}$$

However, equality (44) holds for one special choice of  $\lambda_0$  and  $\lambda_1$ .

**Corollary 4.** Let  $0 < p, q_0, q_1, q < \infty$ ,  $q_0 \neq q_1$ ,  $0 < \theta < 1$ , and (45) be satisfied, and let  $w$  be a such positive measurable function on  $(0, \infty)$ , that  $w \in L_1(t, \infty)$  for some  $t > 0$ . Then <sup>5</sup>

$$\left( LM_{p, q_0, w^{\frac{1}{q_0}}(\cdot)}, LM_{p, q_1, w^{\frac{1}{q_1}}(\cdot)} \right)_{\theta, q} = LM_{p, q, w^{\frac{1}{q}}(\cdot)}.$$

*Proof.* By Example 4 for functions  $w^{\frac{1}{q_0}}$  and  $w^{\frac{1}{q_1}}$  functions  $v_0$  and  $v_1$  have the form  $v_0 = v^{\frac{1}{q_0}}$  and  $v_1 = v^{\frac{1}{q_1}}$ , where

$$v(t) = \begin{cases} 0, & \text{if } 0 < t \leq a, \\ ||w||_{L_1(t, \infty)}^{-1}, & \text{if } a < t < \infty. \end{cases}$$

Hence by Corollary 3

$$\begin{aligned} \left( LM_{p, q_0, w^{\frac{1}{q_0}}(\cdot)}, LM_{p, q_1, w^{\frac{1}{q_1}}(\cdot)} \right)_{\theta, q} &= \left( LM_{p, q_0}^{v^{\frac{1}{q_0}}(\cdot)}, LM_{p, q_1}^{v^{\frac{1}{q_1}}(\cdot)} \right)_{\theta, q} \\ &= LM_{p, q}^{v^{\frac{1}{q}}(\cdot)} = LM_{p, q, w^{\frac{1}{q}}(\cdot)}. \quad \square \end{aligned}$$

## 6. DERIVING CLASSICAL INTERPOLATION THEOREMS AND SOME OF THEIR NEW VARIANTS

In this section we shall prove several interpolation theorems by using the results of the previous sections. We start with the statement which is a direct corollary of Theorem 5.

**Theorem 6.** Let  $\Omega \subset \mathbb{R}^n$ ,  $\mu$  be a  $\sigma$ -finite Borel measure on  $\Omega$ , and  $w$  be a positive  $\mu$ -measurable function on  $\Omega$ . Moreover, let  $0 < p, q \leq \infty$ ,  $0 < \lambda_0, \lambda_1 < \infty$ ,  $\lambda_0 \neq \lambda_1$  and  $0 < \theta < 1$ . Then

$$(L_p(\Omega, w^{\lambda_0}, \mu), L_p(\Omega, w^{\lambda_1}, \mu))_{\theta, q} = LM_{p, q}^\lambda(G, \mu),$$

where  $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$  and  $G = \{G_t\}_{t>0}$ ,  $G_t = \{x \in \Omega : w(x) > \frac{1}{t}\}$ . If  $q = p$ , then

$$(L_p(\Omega, w^{\lambda_0}, \mu), L_p(\Omega, w^{\lambda_1}, \mu))_{\theta, p} = L_p(\Omega, w^\lambda, \mu).$$

<sup>5</sup>In [28] this result is extended to all functions  $w \in \Omega_1$ .

*Proof.* By Example 6 and Theorem 5

$$\begin{aligned} (L_p(\Omega, w^{\lambda_0}, \mu), L_p(\Omega, w^{\lambda_1}, \mu))_{\theta, q} &= (LM_{p,p}^{\lambda_0}(G, \mu), LM_{p,p}^{\lambda_1}(G, \mu))_{\theta, q} \\ &= LM_{p,q}^{\lambda}(G, \mu). \end{aligned}$$

If  $q = p$ , then  $LM_{p,p}^{\lambda}(G, \mu) = L_p(\Omega, w^{\lambda}, \mu)$ .  $\square$

**Remark 15.** By Example 6  $f \in (L_p(\Omega, w^{\lambda_0}, \mu), L_p(\Omega, w^{\lambda_1}, \mu))_{\theta, q}$  with  $q < \infty$  if  $f$  is  $\mu$ -measurable on  $\Omega$  and

$$\int_0^\infty t^{\lambda q} \left( \int_{x \in \Omega: w(x) > t} |f(x)|^p d\mu \right)^{\frac{q}{p}} \frac{dt}{t} < \infty.$$

If  $p = q$ , then this condition is equivalent to the condition  $f \in L_p(\Omega, w^{\lambda}, \mu)$ . If  $q = \infty$ , then  $f \in (L_p(\Omega, w^{\lambda_0}, \mu), L_p(\Omega, w^{\lambda_1}, \mu))_{\theta, \infty}$  if  $f$  is  $\mu$ -measurable on  $\Omega$  and there exists  $c = c(f) > 0$  such that for all  $t > 0$

$$\left( \int_{x \in \Omega: w(x) > t} |f(x)|^p d\mu \right)^{\frac{1}{p}} \leq ct^{-\lambda}.$$

**Theorem 7.** let  $\Omega \subset \mathbb{R}^n$ ,  $\mu$  be a  $\sigma$ -finite Borel measure on  $\Omega$ , and  $w_0, w_1$  be positive  $\mu$ -measurable functions on  $\Omega$ . Moreover, let  $0 < p, q \leq \infty, 0 < \lambda_0, \lambda_1 < \infty, \lambda_0 \neq \lambda_1$ , and  $0 < \theta < 1$ . Then

$$(L_p(\Omega, w_0, \mu), L_p(\Omega, w_1, \mu))_{\theta, q} = LM_{p,q}^{\lambda}(G_{\lambda_0, \lambda_1}, \nu_{\lambda_0, \lambda_1}),$$

where  $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$ , and the family of nested sets  $\{G_{t, \lambda_0, \lambda_1}\}_{t>0}$  and the measure  $\nu_{\lambda_0, \lambda_1}$  are such as in Example 8. If  $q = p$ , then

$$(L_p(\Omega, w_0, \mu), L_p(\Omega, w_1, \mu))_{\theta, p} = L_p(\Omega, w_0^{1-\theta} w_1^{\theta}, \mu).$$

For  $q = p$  this is the Stein-Weiss interpolation theorem [29], for  $q \neq p$  this is a new description of the interpolation space in the non-diagonal case. A similar description for the case  $\lambda_0 = 0, \lambda_1 > 0$  was given by J.E. Gilbert [13]. See also survey paper [3].

*Proof.* By Example 8 and Theorem 5

$$\begin{aligned} (L_p(\Omega, w_0, \mu), L_p(\Omega, w_1, \mu))_{\theta, q} &= (LM_{p,p}^{\lambda_0}(G_{\lambda_0, \lambda_1}, \nu_{\lambda_0, \lambda_1}), LM_{p,p}^{\lambda_1}(G_{\lambda_0, \lambda_1}, \nu_{\lambda_0, \lambda_1}))_{\theta, q} \\ &= LM_{p,q}^{\lambda}(G_{\lambda_0, \lambda_1}, \nu_{\lambda_0, \lambda_1}). \end{aligned}$$

Moreover, if  $q = p < \infty$ , then

$$\|f\|_{LM_{p,p}^{\lambda}(G_{\lambda_0, \lambda_1}, \nu_{\lambda_0, \lambda_1})}^p = \int_0^\infty t^{-\lambda p} \left( \int_{x \in \Omega: w_0^{\alpha_0}(x) w_1^{\alpha_1}(x) < t} (|f(x)| w_0^{\beta_0}(x) w_1^{\beta_1}(x))^p d\mu \right) \frac{dt}{t}$$

$$\begin{aligned}
&= \int_{\Omega} w_0^{\beta_0 p}(x) w_1^{\beta_1 p}(x) |f(x)|^p \left( \int_{w_0^{\alpha_0}(x) w_1^{\alpha_1}(x)} t^{-\lambda p-1} dt \right) d\mu \\
&= \frac{1}{\lambda p} \int_{\Omega} (|f(x)| w_0^{1-\theta}(x) w_1^{\theta}(x))^p d\mu,
\end{aligned}$$

because  $\beta_0 - \lambda\alpha_0 = 1 - \theta$ ,  $\beta_1 - \lambda\alpha_1 = \theta$ . Hence it follows that

$$\|f\|_{LM_{p,p}^{\lambda}(G_{\lambda_0,\lambda_1},\nu_{\lambda_0,\lambda_1})} = (\lambda p)^{-\frac{1}{p}} \|f\|_{L_p(\Omega, w_0^{1-\theta} w_1^{\theta}, \mu)}.$$

If  $q = p = \infty$ , then we have

$$\|f\|_{LM_{\infty,\infty}^{\lambda}(G_{\lambda_0,\lambda_1},\nu_{\lambda_0,\lambda_1})} = \|f\|_{L_{\infty}(\Omega, w_0^{1-\theta} w_1^{\theta}, \mu)}$$

if we take into account the convention (36) made in Example 8 according to which  $LM_{\infty,\infty}^{\lambda}(G_{\lambda_0,\lambda_1},\nu_{\lambda_0,\lambda_1}) = \|f w_0^{\beta_0} w_1^{\beta_1}\|_{LM_{\infty,\infty}^{\lambda}(\Omega, \mu)}$ .  $\square$

**Remark 16.** By Example 8  $f \in (L_p(\Omega, w_0, \mu), L_p(\Omega, w_1, \mu))_{\theta,q}$  with  $q < \infty$  if  $f$  is  $\mu$ -measurable on  $\Omega$  and

$$\int_0^{\infty} t^{-\lambda q} \left( \int_{x \in \Omega: w_0^{\alpha_0}(x) w_1^{\alpha_1}(x) < t} (|f(x)| w_0^{\beta_0}(x) w_1^{\beta_1}(x))^p d\mu \right)^{\frac{q}{p}} \frac{dt}{t} < \infty.$$

If  $q = p$ , then this condition is equivalent to the condition  $f \in L_p(\Omega, w_0^{1-\theta} w_1^{\theta}, \mu)$ . If  $q = \infty$ , then  $f \in (L_p(\Omega, w_0, \mu), L_p(\Omega, w_1, \mu))_{\theta,\infty}$  if  $f$  is  $\mu$ -measurable on  $\Omega$  and there exists  $c = c(f) > 0$  such that for all  $t > 0$

$$\left( \int_{x \in \Omega: w_0^{\alpha_0}(x) w_1^{\alpha_1}(x) < t} (|f(x)| w_0^{\beta_0}(x) w_1^{\beta_1}(x))^p d\mu \right)^{\frac{1}{p}} \leq ct^{\lambda}.$$

By Theorem 9 the quasi-norms  $\|f\|_{LM_{p,q}^{\lambda}(G_{\lambda_0,\lambda_1},\nu_{\lambda_0,\lambda_1})}$  are equivalent for different choices of  $\lambda_0, \lambda_1 \in (0, \infty)$ ,  $\lambda_0 \neq \lambda_1$ . (It is not clear how to verify this directly.)

**Remark 17.** Theorem 6 is a particular case of Theorem 7. Indeed, if  $w$  is a positive  $\mu$ -measurable function on  $\Omega$  and  $w_0 = w^{\lambda_0}$ ,  $w_1 = w^{\lambda_1}$ , then it is enough to take into account  $w_0^{\alpha_0} w_1^{\alpha_1} = w^{-1}$  and  $w_0^{\beta_0} w_1^{\beta_1} = 1$ .

**Theorem 8.** Let  $\Omega \subset \mathbb{R}^n$  be a Lebesgue measurable set,  $w_0, w_1$  be positive Lebesgue measurable functions on  $\Omega$ ,  $0 < q \leq \infty$ ,  $0 < \theta < 1$ ,  $0 < p_0, p_1, p < \infty$ ,  $p_0 \neq p_1$ , and

$$(46) \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$(L_{p_0}(\Omega, w_0), L_{p_1}(\Omega, w_1))_{\theta,q} = \Phi_{1,q}(H^{\frac{1}{p}}),$$



where the operator  $H$  is defined in Example 9. If  $q = p$ , then

$$(L_{p_0}(\Omega, w_0), L_{p_1}(\Omega, w_1))_{\theta, p} = L_p(\Omega, w_0^{1-\theta}, w_1^\theta).$$

For  $q = p$  this is the Peetre interpolation theorem [20]. In the non-diagonal case  $q \neq p$  this is a new description of the interpolation space. Another descriptions were given by P.I. Lizorkin [15] for  $p_0, p_1 \geq 1$  and by D. Freitag [11, 12] for  $p_0, p_1 > 0$ . See also survey paper [3].

*Proof.* By Corollary 1

$$(L_{p_0}(\Omega, w_0), L_{p_1}(\Omega, w_1))_{\theta, q} = (\Phi_{1, p_0}(F_0), \Phi_{1, p_1}(F_1))_{\theta, q}.$$

Also by Corollary 1  $(F_0^{p_0}, F_1^{p_1}, F^p)$  is the triple of weak subadditive type and the triple  $(F^p, F_0^{p_0}, F_1^{p_1})$  admits an weakly  $A$ - $B$  majorizable decomposition. Moreover, by equality (46), it follows that for  $\sigma_0 = p_0$ ,  $\sigma_1 = p_1$  and  $\sigma = p$  conditions (23) and (24) are satisfied. Therefore, by Theorem 4

$$(\Phi_{1, p_0}(F_0), \Phi_{1, p_1}(F_1))_{\theta, q} = \Phi_{1, q}(F) = \Phi_{1, q}(H^{\frac{1}{p}}).$$

Moreover, if  $q = p$ , then by Lemma 4

$$\Phi_{1, p}(F) = L_p(\Omega, w_0^{1-\theta}, w_1^\theta). \quad \square$$

**Remark 18.** By Example 9  $f \in (L_{p_0}(\Omega, w_0), L_{p_1}(\Omega, w_1))_{\theta, q}$  with  $q < \infty$  if  $f$  is Lebesgue measurable on  $\Omega$  and

$$\int_0^\infty t^q \left( \int_{x \in \Omega: |f(x)| > h_2(x)t} h_1(x) dx \right)^{\frac{q}{p}} \frac{dt}{t} < \infty.$$

If  $p = q$ , then this condition is equivalent to the condition  $\|f\|_{L_p(\Omega, w_0^{1-\theta} w_1^\theta)} < \infty$ . If  $q = \infty$ , then  $f \in (L_{p_0}(\Omega, w_0), L_{p_1}(\Omega, w_1))_{\theta, \infty}$  if  $f$  is Lebesgue measurable on  $\Omega$  and there exists  $c = c(f) > 0$  such that for all  $t > 0$

$$\int_{x \in \Omega: |f(x)| > h_2(x)t} h_1(x) dx \leq ct^{-p}.$$

Finally we derive the Calderón interpolation theorem [9] by using Theorems 4, 4' and Examples 10, 11.

**Theorem 9.** Let  $\Omega \subset \mathbb{R}^n$  and  $\mu$  be a  $\sigma$ -finite Borel measure on  $\Omega$ . Moreover, let  $0 < q_0, q_1, q \leq \infty$ ,  $0 < p_0, p_1 < \infty$ ,  $p_0 \neq p_1$ ,  $0 < \theta < 1$ . Then

$$(L_{p_0, q_0}(\Omega, \mu), L_{p_1, q_1}(\Omega, \mu))_{\theta, q} = L_{p, q}(\Omega, \mu),$$

where  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ .

*First proof.* By Example 10

$$(L_{p_0, q_0}(\Omega, \mu), L_{p_1, q_1}(\Omega, \mu))_{\theta, q} = \left( \Phi_{1, q_0}(H^{\frac{1}{p_0}}), \Phi_{1, q_1}(H^{\frac{1}{p_1}}) \right)_{\theta, q}.$$

Since by Lemma 5  $\left( \left( H^{\frac{1}{p}} \right)^p, \left( H^{\frac{1}{p_0}} \right)^{p_0}, \left( H^{\frac{1}{p_1}} \right)^{p_1} \right) = (H, H, H)$  is the triple of weak subadditive type, the triple  $\left( \left( H^{\frac{1}{p_0}} \right)^{p_0}, \left( H^{\frac{1}{p_1}} \right)^{p_1}, \left( H^{\frac{1}{p}} \right)^p \right)$  admits an  $A$ - $B$  majorizable decomposition. Moreover,  $p_0 \neq p_1$ . Hence, by Theorem 4

$$\left( \Phi_{1, q_0}(H^{\frac{1}{p_0}}), \Phi_{1, q_1}(H^{\frac{1}{p_1}}) \right)_{\theta, q} = \Phi_{1, q}(H^{\frac{1}{p}}) = L_{p, q}(\Omega, \mu). \quad \square$$

*Second proof.* By Example 11

$$(L_{p_0, q_0}(\Omega, \mu), L_{p_1, q_1}(\Omega, \mu))_{\theta, q} = \left( \Phi_{\frac{1}{p_0}, q_0}(F), \Phi_{\frac{1}{p_1}, q_1}(F) \right)_{\theta, q}.$$

Since by Lemma 6 the operator  $F$  is weakly subadditive, admits an  $A$ - $B$  majorizable decomposition. Moreover,  $p_0 \neq p_1$ . Hence, by Theorem 4'

$$\left( \Phi_{\frac{1}{p_0}, q_0}(F), \Phi_{\frac{1}{p_1}, q_1}(F) \right)_{\theta, q} = \Phi_{\frac{1}{p}, q}(F) = L_{p, q}(\Omega, \mu). \quad \square$$

**Open problem.** The description of the interpolation space  $(LM_{p_0, q_0}^{\lambda_0}(G^{(0)}, \mu_0), LM_{p_1, q_1}^{\lambda_1}(G^{(0)}, \mu_1))_{\theta, q}$  for arbitrary  $0 < p_0, p_1, q_0, q_1, q \leq \infty$ ,  $0 < \lambda_0, \lambda_1 < \infty$ ,  $0 < \theta < 1$ . (Recall that in Theorem 5  $G^{(0)} = G^{(1)}$ ,  $\mu_0 = \mu_1$ ,  $p_0 = p_1$  and  $\lambda_0 \neq \lambda_1$ .)

**Acknowledgements.** The authors are thankful to Professor E.I. Bereznoi for useful comments. This work was partially supported by the grants of the Russian Foundation for Basic Research (projects no. 11-01-0074a, no. 12-01-00554a) and of the Ministry of Education and Science of the Republic of Kazakhstan (projects 1080/ГФ МОХ РК, 1412/ГФ МОХ РК, 1834/ГФ МОХ РК and 0744/ГФ МОХ РК).

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