



CENTRE DE RECERCA MATEMÀTICA

This is a preprint of: *On the limit cycles of a class of Kukles type differential systems*

Journal Information: *CRM Preprints*,

Author(s): R. Rabanal.

Volume, pages: 1-22,

DOI:[--]

Preprint núm. 1162

June 2013

On the limit cycles of a class of Kukles type differential systems

R. Rabanal

ON THE LIMIT CYCLES OF A CLASS OF KUKLES TYPE DIFFERENTIAL SYSTEMS

ROLAND RABANAL

ABSTRACT. We study the limit cycles of two families of differential systems in the plane. These systems are obtained by polynomial perturbations with arbitrary degree on the second component of the standard linear center. In this context, in both cases, we provide an accurate upper bound of the maximum number of limit cycles that the perturbed system can have bifurcating from the periodic orbits of the linear center, using the averaging theory of first, second and third order.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

One of the main problems in the qualitative theory of real planar differential systems is the determination of limit cycles as defined by Poincaré [18]: existence, number and stability. For instance, the second part of the 16th Hilbert problem [11, 25] wants to find an upper bound on the maximum number of limit cycles that a polynomial vector field of a fixed degree can have. We shall consider a very particular case, and we will try to give a partial answer to this problem for the Kukles type systems: the polynomial differential systems of the form

$$(1) \quad \dot{x} = -y, \quad \dot{y} = x + \lambda y + \sum_{d=2}^n g_d(x, y),$$

where $\lambda \in \mathbb{R}$ and $g_d(x, y)$ is a homogeneous polynomial of degree $d \in \mathbb{Z}^+$.

It was initiated by Kukles [10], giving necessary and sufficient conditions in order that (1) with $n = 3$ has a center at the origin. This cubic system without the term y^3 was also studied in [21] and the authors called it reduced. In [20] appear a description of the bifurcation of its critical period, and [4] presents the existence of reduced Kukles systems with five limit cycles. It was also studied in [27] by using techniques of bifurcation theory and qualitative analysis. In [22]

2000 *Mathematics Subject Classification.* 34C29; 34C25; 47H11.

Key words and phrases. Limit cycle; Kukles systems; Averaging theory.

Partially supported by ‘Pontificia Universidad Católica del Perú’ (DAI: 70242.3020), ‘Institut d’Estudis Catalans’ and ‘Centre de Recerca Matemàtica’ (Lluís Santaló: 2013). This paper was written when the author served as an Associate Fellow at ‘Abdus Salam’, ICTP in Italy, and the author also acknowledges the hospitality of CRM in Catalonia during the preparation of part of the work. I am indebted to J. Llibre and I. Szántó for very important comments on an earlier version of this work.

the author proves that some cubic systems of the form (1) can have seven limit cycles. In [17] we can find some interesting center characterizations of these cubic systems by using symbolic calculations.

System (1) with $\lambda = 0$ has either a center or a weak focus at the origin, and it is closely related with the problem to distinguish a center from a focus. To describe it, we choose one-sided analytic transversal at the origin with a local analytic parameter h , and represent the return map by an expansion $r(h) = h + \sum_{i=0}^{\infty} v_i h^i$. The stability of the origin is clearly given by the sign of the first non-zero v_i . If all the v_i are zero then the origin is a *center*: all the regular orbits in some neighborhood are simple closed curves, surrounding the singularity. If the displacement function $\delta(h) = r(h) - h$ is not flat (i.e. there exists i such that its i th derivative $\delta^{(i)}(0) \neq 0$) we have a *weak focus*. The origin is a *weak focus of order k* if $v_i = 0$ for each $i \leq 2k$ but $v_{2k+1} \neq 0$. Moreover, it is well-known that at most k limit cycles can bifurcate from a weak focus of order k under perturbation of the coefficients of $\sum_{d=2}^n g_d(x, y)$. For more details about the definitions and statements of this paragraph see, for instance [1, 19].

1.1. Kukles type systems. We are particularly interested in studying the maximum number of small amplitude limit cycles of a class of systems as in (1) which can coexist with closed invariant algebraic curves [6, 5, 24, 23, 28]. In this way, our methods are strongly influenced by the methods and ideas from [14, 15, 16], and following [13] we apply the averaging theory in order to study the maximum number of limit cycles which can bifurcate from the linear center $\dot{x} = -y$, $\dot{y} = x$, perturbed in a special class of systems. Specifically, we consider the following system in the plane:

$$(2) \quad \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + (x^2 + y^2) \sum_{\ell \geq 1} \varepsilon^\ell (q_\ell(x, y) - A_\ell), \end{aligned}$$

where $A_\ell > 0$ and the polynomial $q_\ell(x, y)$ has degree $n_\ell - 2 \geq 1$ with $q_\ell(0, 0) = 0$.

Theorem 1.1. *Assume that for $\ell = 1, 2, 3$ the constants $A_\ell > 0$, the polynomials $q_\ell(x, y)$ have degree $n_\ell - 2$ and $q_\ell(0, 0) = 0$. Suppose that $n_\ell \in \{2k_\ell, 2k_\ell - 1\}$ and $k_\ell \geq 2$. Then for $|\varepsilon| \neq 0$, sufficiently small the maximum number of limit cycles of system (2) bifurcating from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ using the averaging theory*

- (a) of first order is $k_1 - 2$.
- (b) of second order is $\max \left\{ k_2 - 2; 2 \left\lfloor \frac{n_1 - 2}{2} \right\rfloor - 2 \right\}$.
- (c) of third order is $\max \left\{ k_3 - 2; \left\lfloor \frac{n_2 - 2}{2} \right\rfloor - 1 \right\}$.

Theorem 1.1 was motivated by the results of [23] and [13]. In [23], the authors introduce the following systems:

$$(3) \quad \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + \varepsilon(x^2 + y^2) \left(\sum_{i=1}^{n_1-2} (q_i x^i + \tilde{q}_i y^i) - A_1 \right), \end{aligned}$$

where $q_i, \tilde{q}_i \in \mathbb{R}$ and $A_1 > 0$. This class is given by using the canonical form of Kukles type systems with an invariant ellipse, presented in [5] (see also [6]). Moreover, some results of [23] are achieved by using a nice application of a theorem of [7]. In this context, (2) extends (3), and Subsection 3.1 also gives the results of [23] (without using [7], but applying the methods of [13]).

1.2. A second family. In order to research the limit cycles of systems of the form (1), we extend the class (2) by considering the system

$$(4) \quad \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + \sum_{\ell \geq 1} \varepsilon^\ell \left((x^2 + y^2) (q_\ell(x, y) - A_\ell) - Q_\ell(x, y) \right); \end{aligned}$$

where

$$Q_\ell(x, y) = E^2 \left(x + \frac{1}{A_\ell} \right) \left(1 + x + \frac{1}{A_\ell} \right) (q_\ell(x, y) - A_\ell)$$

and $0 \leq E < 1$. In particular, when $E = 0$ we obtain (2). Therefore, by applying the methods of the proof of Theorem 1.1 give us the next:

Theorem 1.2. *Assume that for $\ell = 1, 2$ the constants $A_\ell > 0$, the polynomials $q_\ell(x, y)$ have degree $n_\ell - 2$ and $q_\ell(0, 0) = 0$. Suppose that $n_\ell \in \{2k_\ell, 2k_\ell - 1\}$ and $k_\ell \geq 2$. Then for $|\varepsilon| \neq 0$, sufficiently small the maximum number of limit cycles of (4) bifurcating from the periodic orbits of the linear center $\dot{x} = -y, \dot{y} = x$ using the averaging theory*

- (a) of first order is $\max \left\{ k_1 - 2; \left\lfloor \frac{n_1 - 2}{2} \right\rfloor - 1 \right\}$.
- (b) of second order is $\max \left\{ k_2 - 2; \left\lfloor \frac{n_2 - 2}{2} \right\rfloor - 1 \right\}$.
- (c) of third order is $\max \left\{ k_3 - 2; \left\lfloor \frac{n_3 - 2}{2} \right\rfloor - 1 \right\}$.

Both theorems are obtained by using the averaging method. Section 2 describe the averaging method. Section 3 contains the proof of Theorem 1.1 and Section 4 presents Theorem 1.2.

2. THE AVERAGING THEORY OF FIRST, SECOND AND THIRD ORDER

The averaging theory is also an interesting method to research the limit cycles. Essentially, we have to look the zeros of some specific function, associated to the system. It was developed many years ago and can be found in the books [26, 9], and more recently in [12, 2, 3, 8]. It is summarized as follows.

Consider the differential system

$$(5) \quad \dot{x} = \varepsilon F_1(\theta, x) + \varepsilon^2 F_2(\theta, x) + \varepsilon^3 F_3(\theta, x) + \varepsilon^4 R(\theta, x, \varepsilon),$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \rightarrow \mathbb{R}$ and $R : \mathbb{R} \times D \times (\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}$ are continuous functions, T -periodic in the first variable, and D is an open subset of \mathbb{R} . Assume that the following hypotheses (i) and (ii) hold.

- (i) $F_1(\theta, \cdot) \in C^2(D)$, $F_2(\theta, \cdot) \in C^1(D)$ for all $\theta \in \mathbb{R}$; and $F_1, F_2, F_3, R, D_x^2 F_1, D_x F_2$ are locally Lipschitz with respect to x , and R is twice differentiable with respect to ε . We define $F_{k,0} : D \rightarrow \mathbb{R}$ for $k = 1, 2, 3$ as

$$\begin{aligned} F_{1,0}(z) &= \frac{1}{T} \int_0^T F_1(s, z) ds, \\ F_{2,0}(z) &= \frac{1}{T} \int_0^T [D_z F_1(s, z) \cdot y_1(s, z) + F_2(s, z)] ds, \\ F_{3,0}(z) &= \frac{1}{T} \int_0^T \left[\left(y_1(s, z) \right)^2 \frac{\partial^2}{\partial r^2} F_1(s, z) \cdot y_1(s, z) + \frac{1}{2} \frac{\partial}{\partial r} F_1(s, z) \cdot y_2(s, z) \right. \\ &\quad \left. + \frac{\partial}{\partial r} F_2(s, z) \cdot y_1(s, z) + F_3(s, z) \right] ds. \end{aligned}$$

where

$$\begin{aligned} y_1(s, z) &= \int_0^s F_1(t, z) dt, \\ y_2(s, z) &= \int_0^s \left[\frac{\partial}{\partial z} F_1(t, z) \int_0^t F_1(r, z) dr + F_2(t, z) \right] dt. \end{aligned}$$

- (ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (\varepsilon_0, \varepsilon_0) \setminus \{0\}$, there exists $a_{\varepsilon,3} \in V$ such that $F_{1,0}(a_{\varepsilon,3}) + \varepsilon F_{2,0}(a_{\varepsilon,3}) + \varepsilon^2 F_{3,0}(a_{\varepsilon,3}) = 0$ and $d_B(F_{1,0} + \varepsilon F_{2,0} + \varepsilon^2 F_{3,0}, V, a_{\varepsilon,3}) \neq 0$.

Then for $|\varepsilon| > 0$, sufficiently small there exists a T -periodic solution $\varphi(\cdot, a_\varepsilon)$ of system such that $\varphi(0, a_\varepsilon) \rightarrow a_{\varepsilon,3}$ when $\varepsilon \rightarrow 0$.

The expression $d_B(F_{1,0} + \varepsilon F_{2,0} + \varepsilon^2 F_{3,0}, V, a_{\varepsilon,3}) \neq 0$ means that the Brouwer degree of the function $F_{1,0} + \varepsilon F_{2,0} + \varepsilon^2 F_{3,0} : V \rightarrow \mathbb{R}$ at the point $a_{\varepsilon,3}$ is not zero.

If $F_{1,0}$ is not identically zero, then the zeros of $F_{1,0} + \varepsilon F_{2,0} + \varepsilon^2 F_{3,0}$ are mainly the zeros of $F_{1,0}$ for ε sufficiently small. In this case the previous result provides the *averaging theory of first order*.

If $F_{1,0}$ is identically zero and $F_{2,0}$ is not identically zero, then the zeros of $F_{1,0} + \varepsilon F_{2,0} + \varepsilon^2 F_{3,0}$ are mainly the zeros of $F_{2,0}$ for ε sufficiently small. In this case the previous result provides the *averaging theory of second order*.

If $F_{1,0}$ and $F_{2,0}$ are identically zero and $F_{3,0}$ is not identically zero, then the zeros of $F_{1,0} + \varepsilon F_{2,0} + \varepsilon^2 F_{3,0}$ are mainly the zeros of $F_{3,0}$ for ε sufficiently small. In this case the previous result provides the *averaging theory of third order*.

3. PROOF OF THEOREM 1.1

Theorem 1.1 is obtained by using the results of Section 2. So, the next formulae will be needed.

$$(6) \quad \int_0^{2\pi} \cos^{2p+1} \theta \sin^q \theta d\theta = \int_0^{2\pi} \cos^p \theta \sin^{2q+1} \theta d\theta = 0, \quad \forall p, q \in \mathbb{N} \cup \{0\}.$$

It follows because the integrand functions are odd. Moreover, for a polynomial of the form

$$q_1(x, y) = \sum_{d=1}^{n_1-2} \left(\sum_{j=0}^d a_{j,d} x^{d-j} y^j \right),$$

we consider

$$(7) \quad \begin{aligned} q_1(x, y) = & \overbrace{\sum_{\tilde{i}=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{\tilde{j}=1}^{\tilde{i}} a_{2\tilde{i}-1, 2\tilde{j}} x^{2\tilde{i}-2\tilde{j}+1} y^{2\tilde{j}-1} \right)}^{A_{q_1}(x,y)} + \overbrace{\sum_{\tilde{i}=1}^{k_1-1} \left(\sum_{\tilde{j}=0}^{\tilde{i}} a_{2\tilde{i}, 2\tilde{j}-1} x^{2\tilde{i}-1-2\tilde{j}} y^{2\tilde{j}} \right)}^{B_{q_1}(x,y)} \\ & + \overbrace{\sum_{\hat{i}=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{\hat{j}=0}^{\hat{i}} a_{2\hat{i}, 2\hat{j}} x^{2\hat{i}-2\hat{j}} y^{2\hat{j}} \right)}^{C_{q_1}(x,y)} + \overbrace{\sum_{\hat{i}=1}^{k_1-1} \left(\sum_{\hat{j}=1}^{\hat{i}} a_{2\hat{i}-1, 2\hat{j}-1} x^{2\hat{i}-2\hat{j}} y^{2\hat{j}-1} \right)}^{D_{q_1}(x,y)}. \end{aligned}$$

3.1. Proof of statement (a) of Theorem 1.1. This proof is based on the first order averaging theory, by using the coordinates (r, θ) with

$$(8) \quad x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad r > 0.$$

In this context, system (2), with $\ell = 1$ can be written as

$$(9) \quad \begin{aligned} \dot{r} &= \varepsilon r^2 \left(q_1(r \cos \theta, r \sin \theta) - A_1 \right) \sin \theta, \\ \dot{\theta} &= 1 + \varepsilon r \left(q_1(r \cos \theta, r \sin \theta) - A_1 \right) \cos \theta. \end{aligned}$$

Now taking θ as the new independent variable and using that $q_1(0, 0) = 0$, the system becomes

$$\frac{dr}{d\theta} = \varepsilon r^2 \left(q_1(r \cos \theta, r \sin \theta) - A_1 \right) \sin \theta + \mathcal{O}(\varepsilon^2),$$

which is in the standard form for applying the averaging theory. Therefore

$$F_{1,0}(r) = \frac{1}{2\pi} \int_0^{2\pi} r^2 (q_1(r \cos \theta, r \sin \theta) - A_1) \sin \theta d\theta$$

satisfies

$$(10) \quad F_{1,0}(r) = \frac{r^2}{2\pi} \int_0^{2\pi} q_1(r \cos \theta, r \sin \theta) \sin \theta d\theta.$$

By using (7), the formulae in (6) show that

$$(11) \quad F_{1,0}(r) = \frac{r^3}{2\pi} \sum_{\iota=1}^{k_1-1} \left(\sum_{i=1}^{\iota} a_{2i-1, 2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota-2},$$

for every $n_1 \in \{2k_1, 2k_1 - 1\}$.

The polynomial (11) has at most $\frac{2(k_1-1)-2}{2} = k_1 - 2$ positive roots, and we can select the coefficients in such a way that $F_{1,0}(r)$ has exactly $k_1 - 2$ simple positive roots. Therefore, Statement (a) in Theorem 1.1 holds.

Corollary 3.1. *There are $\varepsilon > 0$ and some $q_1(x, y)$ such that (2), with $\ell = 1$ and $n_1 \in \{2k_1, 2k_1 - 1\}$ has a weak focus of order $k_1 - 1$, located at the origin.*

Proof. The result is directly obtained from (9) and (11) because $2(k_1 - 1) - 2 + 3 = 2(k_1 - 1) + 1$. We conclude from the definition of order of a weak focus and $F_{1,0}(r)$. \square

Remark 3.2. In particular, when $n_1 = 3$ and $q_1(x, y) = a_{0,1}x + a_{1,1}y$ with $a_{1,1} \neq 0$ we obtain that $F_{1,0}(r) = \frac{a_{1,1}}{2}r^3$. Therefore, (2) induces a system with a focus of order one, located at the origin. Notice that $\frac{\partial}{\partial y}q_1(0, 0) = a_{1,1}$.

3.2. Proof of Statement (b) of Theorem 1.1. As before we apply the change of variables (8). In this context, if we write

$$q_1(x, y) = \sum_{d=1}^{n_1-2} \left(\sum_{j=0}^d a_{j,d} x^{d-j} y^j \right) \quad \text{and} \quad q_2(x, y) = \sum_{d=1}^{n_2-2} \left(\sum_{j=0}^d b_{j,d} x^{d-j} y^j \right),$$

then system (2), with $\ell = 1, 2$ take the form

$$(12) \quad \begin{aligned} \dot{r} &= -\varepsilon \left(A_1 + \varepsilon A_2 - q_1(r, \theta) - \varepsilon q_2(r, \theta) \right) r^2 \sin \theta \\ \dot{\theta} &= 1 - \varepsilon \left(A_1 + \varepsilon A_2 - q_1(r, \theta) - \varepsilon q_2(r, \theta) \right) r \cos \theta, \end{aligned}$$

where $q_j(r, \theta) = q_j(r \cos \theta, r \sin \theta)$, for every $j \in \{1, 2\}$.

Taking θ as the new independent variable, system (12) becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + \mathcal{O}(\varepsilon^3)$$

where

$$(13) \quad \begin{cases} F_1(r, \theta) = (q_1(r, \theta) - A_1)r^2 \sin \theta; \\ F_2(r, \theta) = (q_2(r, \theta) - A_2 - (q_1(r, \theta) - A_1)^2 r \cos \theta)r^2 \sin \theta. \end{cases}$$

In order to compute $F_{2,0}(r)$, we need that $F_{1,0}(r)$ be identically zero. Then from equation (11) we consider:

$$(14) \quad a_{2i-1, 2\ell-1} = 0, \quad \forall 1 \leq i, \ell \leq k_1 - 1.$$

First we compute

$$(15) \quad \begin{aligned} \frac{\partial}{\partial r} F_1(r, \theta) &= \sum_{\ell=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\ell} a_{2i-1, 2\ell} \cos^{2\ell-2i+1} \theta \sin^{2i} \theta \right) (2\ell + 2)r^{2\ell+1} - 2A_1 r \sin \theta \\ &+ \sum_{\ell=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=0}^{\ell} a_{2i, 2\ell} \cos^{2\ell-2i} \theta \sin^{2i+1} \theta \right) (2\ell + 2)r^{2\ell+1} \\ &+ \sum_{\ell=1}^{k_1-1} \left(\sum_{i=0}^{\ell-1} a_{2i, 2\ell-1} \cos^{2\ell-1-2i} \theta \sin^{2i+1} \theta \right) (2\ell + 1)r^{2\ell}. \end{aligned}$$

and $y_1(r, \theta) = \int_0^\theta F_1(r, \phi) d\phi$. To this end, we rewrite

$$(16) \quad \begin{aligned} F_1(r, \theta) &= \sum_{\ell=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\ell} a_{2i-1, 2\ell} \sum_{a=0}^{\ell-i} \binom{\ell-i}{a} (-1)^a \sin^{2a+2i} \theta \cos \theta \right) r^{2\ell+2} - A_1 r^2 \sin \theta \\ &+ \sum_{\ell=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=0}^{\ell} a_{2i, 2\ell} \sum_{a=0}^i \binom{i}{a} (-1)^a \cos^{2\ell-2i+2a} \theta \sin \theta \right) r^{2\ell+2} \\ &+ \sum_{\ell=1}^{k_1-1} \left(\sum_{i=0}^{\ell-1} a_{2i, 2\ell-1} \sum_{a=0}^i \binom{i}{a} (-1)^a \cos^{2\ell-2i+2a-1} \theta \sin \theta \right) r^{2\ell+1}. \end{aligned}$$

Then

$$(17) \quad \begin{aligned} y_1(r, \theta) &= \sum_{\ell=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\ell} a_{2i-1, 2\ell} \sum_{a=0}^{\ell-i} \binom{\ell-i}{a} \frac{\sin^{2a+2i+1} \theta}{2a+2i+1} (-1)^a \right) r^{2\ell+2} \\ &+ A_1 (\cos \theta - 1) r^2 \\ &+ \sum_{\ell=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=0}^{\ell} a_{2i, 2\ell} \sum_{a=0}^i \binom{i}{a} \frac{1 - \cos^{2\ell-2i+2a+1} \theta}{2\ell-2i+2a+1} (-1)^a \right) r^{2\ell+2} \end{aligned}$$

$$+ \sum_{\iota=1}^{k_1-1} \left(\sum_{i=0}^{\iota-1} a_{2i,2\iota-1} \sum_{a=0}^i \binom{i}{a} \frac{1 - \cos^{2\iota-2i+2a} \theta}{2\iota + 2i + 2a} (-1)^a \right) r^{2\iota+1}.$$

By using (6) we obtain that

$$\begin{aligned} 2\pi F_{2,0}(r) &= \int_0^{2\pi} \left(\frac{\partial}{\partial r} F_1(r, \theta) \cdot y_1(r, \theta) + F_2(r, \theta) \right) d\theta, \\ (18) \quad &= 2A_1 \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\iota} a_{2i-1,2\iota} \sum_{a=0}^{\iota-i} \binom{\iota-i}{a} \frac{\int_0^{2\pi} \sin^{2a+2i+2} \theta d\theta}{2a + 2i + 1} (-1)^{a+1} \right) r^{2\iota+3} \\ &+ A_1 \sum_{\tilde{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{\iota}} a_{2\tilde{i}-1,2\tilde{\iota}} \int_0^{2\pi} \cos^{2\tilde{\iota}-2\tilde{i}+2} \theta \sin^{2\tilde{i}} \theta d\theta \right) (2\tilde{\iota} + 2) r^{2\tilde{\iota}+3} \\ &+ \sum_{\tilde{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\tilde{i}=1}^{\tilde{\iota}} \sum_{i=0}^{\iota} a_{2\tilde{i}-1,2\tilde{\iota}} a_{2i,2\iota} (*) (2\tilde{\iota} + 2) r^{2\tilde{\iota}+2\iota+3} \\ &+ \sum_{\tilde{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\tilde{i}=0}^{\tilde{\iota}} \sum_{i=1}^{\iota} a_{2i-1,2\iota} a_{2\tilde{i},2\tilde{\iota}} (**) (2\tilde{\iota} + 2) r^{2\tilde{\iota}+2\iota+3} + \int_0^{2\pi} F_2(r, \theta) d\theta, \end{aligned}$$

where

$$(*) = \sum_{a=0}^i \binom{i}{a} \frac{\int_0^{2\pi} \cos^{2\iota-2i+2a+2\tilde{\iota}-2\tilde{i}+2} \theta \sin^{2\tilde{i}} \theta d\theta}{2\iota - 2i + 2a + 1} (-1)^{a+1}$$

and

$$(**) = \sum_{a=0}^{\iota-i} \binom{\iota-i}{a} \frac{\int_0^{2\pi} \cos^{2\tilde{\iota}-2\tilde{i}} \theta \sin^{2\tilde{i}+2a+2i+2} \theta d\theta}{2a + 2i + 1} (-1)^a.$$

But, (13) implies that

$$\begin{aligned} (19) \quad \int_0^{2\pi} F_2(r, \theta) d\theta &= r^2 \int_0^{2\pi} q_2(r, \theta) \sin \theta d\theta + \left(\overbrace{r^3 \int_0^{2\pi} [q_1(r, \theta) \sin \theta] \cos \theta d\theta}^{\circ} \right) 2A_1 \\ &- r^3 \int_0^{2\pi} (q_1(r, \theta))^2 \cos \theta \sin \theta d\theta. \end{aligned}$$

More precisely, (10) joint to (11) imply

$$(20) \quad r^2 \int_0^{2\pi} q_2(r, \theta) \sin \theta d\theta = \sum_{\iota=1}^{k_2-1} \left(\sum_{i=1}^{\iota} b_{2i-1,2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota+1}.$$

Similarly, (16) and (6) give

$$(21) \quad \circ = \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\iota} a_{2i-1,2\iota} \int_0^{2\pi} \cos^{2\iota-2i+2} \theta \sin^{2i} \theta d\theta \right) r^{2\iota+3}.$$

Moreover, since q_1 satisfies (7) with $D_{q_1} \equiv 0$ (see (14)) and

$$\begin{aligned} (q_1(x, y))^2 &= A_{q_1}^2 + 2B_{q_1}A_{q_1} + B_{q_1}^2 + 2B_{q_1}C_{q_1} + C_{q_1}^2 \\ &\quad + 2 \sum_{\tilde{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\tilde{i}=1}^{\tilde{\iota}} \sum_{\hat{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\hat{i}=0}^{\hat{\iota}} a_{2\hat{i},2\tilde{\iota}} a_{2\tilde{i}-1,2\tilde{\iota}} x^{2\tilde{\iota}-2\tilde{i}+2\hat{\iota}-2\hat{i}+1} y^{2\hat{i}+2\tilde{i}-1}, \end{aligned}$$

we have that

$$r^3 \int_0^{2\pi} (q_1(r, \theta))^2 \cos \theta \sin \theta d\theta = 2 \sum_{\tilde{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\tilde{i}=1}^{\tilde{\iota}} \sum_{\hat{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\hat{i}=0}^{\hat{\iota}} a_{2\hat{i},2\tilde{\iota}} a_{2\tilde{i}-1,2\tilde{\iota}} (* * *) r^{2\tilde{\iota}+2\hat{\iota}+3}$$

where

$$(* * *) = \int_0^{2\pi} \cos^{2\tilde{\iota}-2\tilde{i}+2\hat{\iota}-2\hat{i}+2} \theta \sin^{2\hat{i}+2\tilde{i}} \theta d\theta.$$

Therefore, (18), (19), (20), (21) give that

$$\begin{aligned} (22) \quad 2\pi F_{2,0}(r) &= 2A_1 \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\iota} a_{2i-1,2\iota} \sum_{a=0}^{\iota-i} \binom{\iota-i}{a} \frac{\int_0^{2\pi} \sin^{2a+2i+2} \theta d\theta}{2a+2i+1} (-1)^{a+1} \right) r^{2\iota+3} \\ &\quad + A_1 \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\iota} a_{2i-1,2\iota} \int_0^{2\pi} \cos^{2\iota+2i+2} \theta \sin^{2i} \theta d\theta \right) (2\iota+2) r^{2\iota+3} \\ &\quad + \sum_{\tilde{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\tilde{i}=1}^{\tilde{\iota}} \sum_{i=0}^{\iota} a_{2\tilde{i}-1,2\tilde{\iota}} a_{2i,2\iota} (*) (2\tilde{\iota}+2) r^{2\tilde{\iota}+2\iota+3} \\ &\quad + \sum_{\tilde{\iota}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\tilde{i}=0}^{\tilde{\iota}} \sum_{i=1}^{\iota} a_{2i-1,2\iota} a_{2\tilde{i},2\tilde{\iota}} (**) (2\tilde{\iota}+2) r^{2\tilde{\iota}+2\iota+3} \\ &\quad + \sum_{\iota=1}^{k_2-1} \left(\sum_{i=1}^{\iota} b_{2i-1,2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota+1} \\ &\quad + 2A_1 \sum_{\iota=1}^{\left[\frac{n_1-2}{2}\right]} \left(\sum_{i=1}^{\iota} a_{2i-1,2\iota} \int_0^{2\pi} \cos^{2\iota-2i+2} \theta \sin^{2i} \theta d\theta \right) r^{2\iota+3} \end{aligned}$$

$$- 2 \sum_{\tilde{i}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\tilde{i}=1}^{\tilde{i}} \sum_{\hat{i}=1}^{\left[\frac{n_1-2}{2}\right]} \sum_{\hat{i}=0}^{\hat{i}} a_{2\hat{i},2\tilde{i}} a_{2\tilde{i}-1,2\hat{i}} (***) r^{2\tilde{i}+2\hat{i}+3}.$$

Consequently, we can choose the coefficients $a_{2i-1,2\ell}$ and $b_{2i-1,2\ell-1}$ in such a way that $F_{2,0}(r)$ has exactly $\max\{k_2 - 2; 2\left[\frac{n_1-2}{2}\right] - 2\}$ real positive roots. Thus, statement (b) of Theorem 1.1 holds.

3.3. Proof of statement (c) of Theorem 1.1. By using (8) and $q_3(x, y) = \sum_{d=1}^{n_3-2} \left(\sum_{j=0}^d \tilde{a}_{j,d} x^{d-j} y^j \right)$, system (2) take the form

$$(23) \quad \begin{aligned} \dot{r} &= -\varepsilon \left(A_1 + \varepsilon A_2 + \varepsilon^2 A_3 - q_1(r, \theta) - \varepsilon q_2(r, \theta) - \varepsilon^2 q_3(r, \theta) \right) r^2 \sin \theta, \\ \dot{\theta} &= 1 - \varepsilon \left(A_1 + \varepsilon A_2 + \varepsilon^2 A_3 - q_1(r, \theta) - \varepsilon q_2(r, \theta) - \varepsilon^2 q_3(r, \theta) \right) r \sin \theta; \end{aligned}$$

where $q_j(r, \theta) = q_j(r \cos \theta, r \sin \theta)$, for every $j \in \{1, 2, 3\}$. Thus, taking θ as the independent variable, system (23) becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + \varepsilon^3 F_3(r, \theta) + \mathcal{O}(\varepsilon^4)$$

where

$$(24) \quad \begin{cases} F_1(r, \theta) = [q_1(r, \theta) - A_1] r^2 \sin \theta; \\ F_2(r, \theta) = [q_2(r, \theta) - A_2 - (A_1 - q_1(r, \theta))^2 r \cos \theta] r^2 \sin \theta; \\ F_3(r, \theta) = [(q_1(r, \theta) - A_1)^3 r^2 \cos^2 \theta - A_3 + q_3(r, \theta) \\ \quad - 2(q_1(r, \theta) - A_1)(q_2(r, \theta) - A_2) r \cos \theta] r^2 \sin \theta. \end{cases}$$

To compute $F_{3,0}(r)$, we need that $F_{1,0}(r) = F_{2,0}(r) = 0$. By (22), we assume (14) and

$$(25) \quad \begin{aligned} a_{2i-1,2\ell-1} &= 0, & 1 \leq i, \ell \leq k_1 - 1; \\ a_{2i-1,2\ell} &= 0, & 1 \leq i, \ell \leq \left[\frac{n_1-2}{2}\right]; \\ a_{2i,2\ell} &= 0, & 0 \leq i \leq \left[\frac{n_1-2}{2}\right], \quad 1 \leq \ell \leq \left[\frac{n_1-2}{2}\right]; \\ b_{2i-1,2\ell-1} &= 0, & 1 \leq i, \ell \leq k_2 - 1. \end{aligned}$$

Thus (17) implies that

$$(26) \quad \begin{aligned} y_1(r, \theta) &= A_1(\cos \theta - 1)r^2 \\ &+ \sum_{\ell=1}^{k_1-1} \left(\sum_{i=0}^{\ell-1} a_{2i,2\ell-1} \sum_{a=0}^i \binom{i}{a} \frac{1 - \cos^{2\ell-2i+2a} \theta}{2\ell + 2i + 2a} (-1)^a \right) r^{2\ell+1}. \end{aligned}$$

Similarly, (15) gives

$$(27) \quad \frac{\partial}{\partial r} F_1(r, \theta) = \sum_{\iota=1}^{k_1-1} \sum_{i=0}^{\iota-1} a_{2i, 2\iota-1} \cos^{2\iota-2i-1} \theta \sin^{2i+1} \theta (2\iota+1) r^{2\iota} - 2A_1 r \sin \theta$$

and

$$\frac{\partial^2}{\partial r^2} F_1(r, \theta) = \sin \theta \left[\sum_{\iota=1}^{k_1-1} \left(\sum_{i=0}^{\iota-1} a_{2i, 2\iota-1} \cos^{2\iota-2i-1} \theta \sin^{2i} \theta \right) 2\iota(2\iota+1) r^{2\iota-1} - 2A_1 \right].$$

Consequently, there are $a_j \in \mathbb{R}$ and $m \in \mathbb{N}$ such that

$$\left(y_1(r, \theta) \right)^2 \frac{\partial^2}{\partial r^2} F_1(r, \theta) = \sin \theta \sum_{j=0}^m a_j \sin^{2m_j} \theta \cos^{\tilde{m}_j} \theta, \quad m_j, \tilde{m}_j \in \mathbb{N} \cup \{0\},$$

because $y_1(r, \theta)$ does not includes powers of the sine function, see (26). Therefore,

$$\int_0^{2\pi} \frac{1}{2} \left(y_1(r, \theta) \right)^2 \frac{\partial^2}{\partial r^2} F_1(r, \theta) \cdot y_1(r, \theta) d\theta = 0,$$

and so

$$(28) \quad F_{3,0}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2} \frac{\partial}{\partial r} F_1(r, \theta) y_2(r, \theta) + \frac{\partial}{\partial r} F_2(r, \theta) y_1(r, \theta) + F_3(r, \theta) \right] d\theta.$$

To compute $\frac{\partial}{\partial r} F_2(r, \theta) \cdot y_1(r, \theta)$, with $F_2(r, \theta)$ as in (24) we rewrite

$$F_2(r, \theta) = \overbrace{\left(q_2(r, \theta) - A_2 \right) r^2 \sin \theta}^{**} - \overbrace{\left(A_1 - q_1(r, \theta) \right)^2 r^3 \cos \theta \sin \theta}^*$$

As the expresion $(**)$ has the form of $F_1(r, \theta)$, then (15) and (25) imply

$$\begin{aligned} \frac{\partial}{\partial r} (**) &= \sum_{\iota=1}^{\left[\frac{n_2-2}{2} \right]} \left(\sum_{i=1}^{\iota} b_{2i-1, 2\iota} \cos^{2\iota-2i+1} \theta \sin^{2i} \theta \right) (2\iota+2) r^{2\iota+1} - 2A_2 r \sin \theta \\ &\quad + \sum_{\iota=1}^{\left[\frac{n_2-2}{2} \right]} \left(\sum_{i=0}^{\iota} b_{2i, 2\iota} \cos^{2\iota-2i} \theta \sin^{2i+1} \theta \right) (2\iota+2) r^{2\iota+1} \\ &\quad + \sum_{\iota=1}^{k_2-1} \left(\sum_{i=0}^{\iota-1} b_{2i, 2\iota-1} \cos^{2\iota-2i-1} \theta \sin^{2i+1} \theta \right) (2\iota+1) r^{2\iota}. \end{aligned}$$

Similarly, $\frac{\partial}{\partial r} (*)$ is equal to

$$3A_1^2 \cos \theta \sin \theta r^2 - \frac{\partial}{\partial r} [2A_1 q_1(r, \theta) \cos \theta \sin \theta r^3] + \frac{\partial}{\partial r} \left[\left(q_1(r, \theta) \right)^2 \cos \theta \sin \theta r^3 \right],$$

where

$$(29) \quad q_1(r, \theta) = \sum_{\iota=1}^{k_1-1} \left(\sum_{i=0}^{\iota} a_{2i, 2\iota-1} \cos^{2\iota-1-2i} \theta \sin^{2i} \theta \right) r^{2\iota-1}$$

and

$$\left(q_1(r, \theta) \right)^2 = \sum_{\iota=1}^{\diamond} \sum_{i=0}^{\iota} \sum_{\tilde{\iota}=1}^{k_1-1} \sum_{\tilde{i}=0}^{\tilde{\iota}} a_{2i, 2\iota-1} a_{2\tilde{i}, 2\tilde{\iota}-1} \cos^{2\iota+2\tilde{\iota}-2i-2\tilde{i}-2} \theta \sin^{2i+2\tilde{i}} \theta r^{2\iota+2\tilde{\iota}-2}.$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial r} (*) &= 3A_1^2 \cos \theta \sin \theta r^2 - 2A_1 \sum_{\iota=1}^{k_1-1} \sum_{i=0}^{\iota} a_{2i, 2\iota-1} \cos^{2\iota-2i} \theta \sin^{2i+1} \theta (2\iota+2) r^{2\iota+1} \\ &\quad + \sum_{\iota=1}^{\diamond} a_{2i, 2\iota-1} a_{2\tilde{i}, 2\tilde{\iota}-1} \cos^{2\iota+2\tilde{\iota}-2i-2\tilde{i}-1} \theta \sin^{2i+2\tilde{i}+1} \theta (2\tilde{\iota}+2\iota+1) r^{2\tilde{\iota}+2\iota}. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial r} F_2(r, \theta) &= \sum_{\iota=1}^{\lfloor \frac{n_2-2}{2} \rfloor} \sum_{i=1}^{\iota} b_{2i-1, 2\iota} \cos^{2\iota-2i+1} \theta \sin^{2i} \theta (2\iota+2) r^{2\iota+1} - 2A_2 r \sin \theta \\ (30) \quad &+ \sum_{\iota=1}^{k_2-1} \sum_{i=0}^{\iota-1} b_{2i, 2\iota-1} \cos^{2\iota-2i-1} \theta \sin^{2i+1} \theta (2\iota+1) r^{2\iota} \\ &+ \sum_{\iota=1}^{k_2-1} \sum_{i=0}^{\iota-1} b_{2i, 2\iota-1} \cos^{2\iota-2i-1} \theta \sin^{2i+1} \theta (2\iota+1) r^{2\iota} \\ &- 3A_1^2 \cos \theta \sin \theta r^2 + 2A_1 \sum_{\iota=1}^{k_1-1} \sum_{i=0}^{\iota} a_{2i, 2\iota-1} \cos^{2\iota-2i} \theta \sin^{2i+1} \theta (2\iota+2) r^{2\iota+1} \\ &- \sum_{\iota=1}^{\diamond} a_{2i, 2\iota-1} a_{2\tilde{i}, 2\tilde{\iota}-1} \cos^{2\iota+2\tilde{\iota}-2i-2\tilde{i}-1} \theta \sin^{2i+2\tilde{i}+1} \theta (2\tilde{\iota}+2\iota+1) r^{2\tilde{\iota}+2\iota}. \end{aligned}$$

Therefore,

$$(31) \quad \mathcal{I}_2(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r} F_2(r, \theta) \cdot y_1(r, \theta) d\theta$$

satisfies

$$(32) \quad \mathcal{I}_2(r) = \frac{A_1}{2\pi} \sum_{\iota=1}^{\lfloor \frac{n_2-2}{2} \rfloor} \left(\sum_{i=1}^{\iota} b_{2i-1, 2\iota} (2\iota+2) \int_0^{2\pi} \cos^{2\iota-2i+2} \theta \sin^{2i} \theta d\theta \right) r^{2\iota+3}$$

as shown (6) and (26).

To compute

$$y_2(r, \theta) = \int_0^\theta \overbrace{\left[\frac{\partial}{\partial r} F_1(r, \phi) \cdot y_1(r, \phi) + F_2(r, \phi) \right]}^{f(\phi)} d\phi$$

we apply $\sin^{2\tilde{i}} \phi = \sum_{\tilde{a}=0}^{\tilde{i}} \binom{\tilde{i}}{\tilde{a}} (-1)^{\tilde{a}} \cos^{2\tilde{a}} \phi$ and rewrite (27) as

$$\begin{aligned} \frac{\partial}{\partial r} F_1(r, \phi) &= -2A_1 r \sin \phi \\ &+ \sum_{\tilde{i}=1}^{k_1-1} \left(\sum_{\tilde{i}=0}^{\tilde{i}-1} a_{2\tilde{i}, 2\tilde{i}-1} \sum_{\tilde{a}=0}^{\tilde{i}} \binom{\tilde{i}}{\tilde{a}} (-1)^{\tilde{a}} \cos^{2\tilde{a}+2\tilde{i}-2\tilde{i}-1} \phi \sin \phi \right) (2\tilde{i}+1)r^{2\tilde{i}}. \end{aligned}$$

Thus, (26) gives the existence of $b_j \in \mathbb{R}$ and $\tilde{m} \in \mathbb{N}$ such that

$$\int_0^\theta f(\phi) d\phi = \sum_{j=0}^{\tilde{m}} b_j \cos^{\tilde{m}_j} \theta, \quad \tilde{m}_j \in \mathbb{N} \cup \{0\},$$

and so (6) shows that

$$\int_0^{2\pi} \left[\frac{\partial}{\partial r} F_1(r, \theta) \int_0^\theta f(\phi) d\phi \right] d\theta = 0.$$

Therefore, since $y_2(r, \theta) = \int_0^\theta f(\phi) d\phi + \int_0^\theta F_2(r, \phi) d\phi$, the integral

$$(33) \quad \mathcal{I}_1(r) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2} \frac{\partial}{\partial r} F_1(r, \theta) \cdot y_2(r, \theta) \right] d\theta$$

reduces to

$$(34) \quad \mathcal{I}_1(r) = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2} \frac{\partial}{\partial r} F_1(r, \theta) \cdot \int_0^\theta F_2(r, \phi) d\phi \right] d\theta.$$

On the other hand, we consider

$$F_2(r, \phi) = \overbrace{(q_2(r, \phi) - A_2)r^2 \sin \phi}^{h(\phi)} - \overbrace{(A_1 - q_1(r, \phi))^2 r^3 \cos \phi \sin \phi}^{g(\phi)}$$

and rewrite (29) as

$$q_1(r, \phi) = \sum_{\iota=1}^{k_1-1} \left(\sum_{i=0}^{\iota} a_{2i, 2\iota-1} \cos^{2\iota-1-2i} (1 - \cos^2 \phi)^i \right) r^{2\iota-1},$$

thus it is not difficult to prove the existence of $\hat{b}_j \in \mathbb{R}$ and $\hat{m} \in \mathbb{N}$ such that

$$\int_0^\theta g(\phi) d\phi = \sum_{j=0}^{\hat{m}} \hat{b}_j \cos^{\hat{m}_j} \theta, \quad \hat{m}_j \in \mathbb{N} \cup \{0\}.$$

In a similar way, as (25) gives $D_{q_2} \equiv 0$ (see (7)), there are $\bar{b}_j \in \mathbb{R}$ and $\bar{m}, \bar{m}_j \in \mathbb{N}$ such that

$$\int_0^\theta h(\phi) d\phi = \sum_{j=0}^{\bar{m}} \bar{b}_j \cos^{\bar{m}_j} \theta + \int_0^\theta A_{q_2}(r, \phi) r^2 \sin \phi d\phi,$$

where $A_{q_2}(r, \phi) = A_{q_2}(r \cos \phi, r \sin \phi)$ satisfies

$$\begin{aligned} \int_0^\theta A_{q_2}(r, \phi) r^2 \sin \phi d\phi &= \sum_{\tilde{i}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{i}} b_{2\tilde{i}-1, 2\tilde{i}} \int_0^\theta \cos^{2\tilde{i}-2\tilde{i}+1} \phi \sin^{2\tilde{i}} \phi d\phi \right) r^{2\tilde{i}+2} \\ &= \sum_{\tilde{i}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{i}} b_{2\tilde{i}-1, 2\tilde{i}} \sum_{\tilde{a}=0}^{\tilde{i}-\tilde{i}} \binom{\tilde{i}-\tilde{i}}{\tilde{a}} \frac{\sin^{2\tilde{a}+2\tilde{i}+1} \theta}{2\tilde{a}+2\tilde{i}+1} (-1)^{\tilde{a}} \right) r^{2\tilde{i}+2} \end{aligned}$$

Therefore, as $\int_0^\theta F_2(r, \phi) d\phi = \sum_{j=0}^{\bar{m}} \bar{b}_j \cos^{\bar{m}_j} \theta + \int_0^\theta A_{q_2}(r, \phi) r^2 \sin \phi d\phi - \sum_{j=0}^{\hat{m}} \hat{b}_j \cos^{\hat{m}_j} \theta$ a direct application of (6) and (34) give

$$\begin{aligned} \mathcal{I}_1(r) &= \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2} \frac{\partial}{\partial r} F_1(r, \theta) \cdot \int_0^\theta A_{q_2}(r, \phi) r^2 \sin \phi d\phi \right] d\theta \\ (35) \quad &= \frac{A_1}{2\pi} \sum_{\tilde{i}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{i}} b_{2\tilde{i}-1, 2\tilde{i}} \sum_{\tilde{a}=0}^{\tilde{i}-\tilde{i}} \binom{\tilde{i}-\tilde{i}}{\tilde{a}} \frac{\int_0^{2\pi} \sin^{2\tilde{a}+2\tilde{i}+2} \theta d\theta}{2\tilde{a}+2\tilde{i}+1} (-1)^{\tilde{a}+1} \right) r^{2\tilde{i}+3}. \end{aligned}$$

By using (28), (33) and (31) we have $F_{3,0}(r)$ is equal to $\mathcal{I}_1(r) + \mathcal{I}_2(r) + \frac{1}{2\pi} \int_0^{2\pi} F_3(r, \theta) d\theta$. Therefore, (35) and (32) imply

$$\begin{aligned} F_{3,0}(r) &= \frac{A_1}{2\pi} \sum_{\tilde{i}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{i}} b_{2\tilde{i}-1, 2\tilde{i}} \sum_{\tilde{a}=0}^{\tilde{i}-\tilde{i}} \binom{\tilde{i}-\tilde{i}}{\tilde{a}} \frac{\int_0^{2\pi} \sin^{2\tilde{a}+2\tilde{i}+2} \theta d\theta}{2\tilde{a}+2\tilde{i}+1} (-1)^{\tilde{a}+1} \right) r^{2\tilde{i}+3} \\ (36) \quad &+ \frac{A_1}{2\pi} \sum_{\iota=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\iota=1}^{\iota} b_{2\iota-1, 2\iota} (2\iota+2) \int_0^{2\pi} \cos^{2\iota-2\iota+2} \theta \sin^{2\iota} \theta d\theta \right) r^{2\iota+3} \\ &+ \frac{1}{2\pi} \int_0^{2\pi} F_3(r, \theta) d\theta. \end{aligned}$$

To compute this integral we apply (29) and (24), thus (6) implies that

$$\frac{1}{2\pi} \int_0^{2\pi} \left[(q_1(r, \theta) - A_1)^3 r^2 \cos^2 \theta - A_3 \right] r^2 \sin \theta d\theta = 0$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} F_3(r, \theta) d\theta = \frac{r^2}{2\pi} \int_0^{2\pi} q_3(r, \theta) \sin \theta d\theta + \frac{A_1 r^3}{\pi} \int_0^{2\pi} q_2(r, \theta) \cos \theta \sin \theta d\theta$$

$$-\frac{r^3}{\pi} \int_0^{2\pi} \cos \theta q_1(r, \theta) q_2(r, \theta) \sin \theta d\theta.$$

Similarly, (11) gives

$$\frac{r^2}{2\pi} \int_0^{2\pi} q_3(r, \theta) \sin \theta d\theta = \frac{r^3}{2\pi} \sum_{\iota=1}^{k_3-1} \left(\sum_{i=1}^{\iota} \tilde{a}_{2i-1, 2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota-2},$$

the form of $A_{q_2}(x, y)$ as in (7) implies that

$$\int_0^{2\pi} q_2(r, \theta) \cos \theta \sin \theta d\theta = \sum_{\tilde{\iota}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{\iota}} b_{2\tilde{i}-1, 2\tilde{\iota}} \int_0^{2\pi} \cos^{2\tilde{\iota}-2\tilde{i}+2} \theta \sin^{2\tilde{i}} \theta d\theta \right) r^{2\tilde{\iota}}$$

and it is not difficult to see that

$$\int_0^{2\pi} \cos \theta q_1(r, \theta) q_2(r, \theta) \sin \theta d\theta = 0.$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} F_3(r, \theta) d\theta &= \frac{r^3}{2\pi} \sum_{\iota=1}^{k_3-1} \left(\sum_{i=1}^{\iota} \tilde{a}_{2i-1, 2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota-2} \\ &\quad + \frac{A_1 r^3}{\pi} \sum_{\tilde{\iota}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{\iota}} b_{2\tilde{i}-1, 2\tilde{\iota}} \int_0^{2\pi} \cos^{2\tilde{\iota}-2\tilde{i}+2} \theta \sin^{2\tilde{i}} \theta d\theta \right) r^{2\tilde{\iota}}, \end{aligned}$$

and (36) shows that

$$\begin{aligned} (37) \quad F_{3,0}(r) &= \frac{A_1}{2\pi} \sum_{\tilde{\iota}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{\iota}} b_{2\tilde{i}-1, 2\tilde{\iota}} \sum_{\tilde{a}=0}^{\tilde{\iota}-\tilde{i}} \binom{\tilde{\iota}-\tilde{i}}{\tilde{a}} \frac{\int_0^{2\pi} \sin^{2\tilde{a}+2\tilde{i}+2} \theta d\theta}{2\tilde{a}+2\tilde{i}+1} (-1)^{\tilde{a}+1} \right) r^{2\tilde{\iota}+3} \\ &\quad + \frac{A_1}{2\pi} \sum_{\iota=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{i=1}^{\iota} b_{2i-1, 2\iota} (2\iota+2) \int_0^{2\pi} \cos^{2\iota-2i+2} \theta \sin^{2i} \theta d\theta \right) r^{2\iota+3} \\ &\quad + \frac{r^3}{2\pi} \sum_{\iota=1}^{k_3-1} \left(\sum_{i=1}^{\iota} \tilde{a}_{2i-1, 2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota-2} \\ &\quad + \frac{A_1 r^3}{\pi} \sum_{\tilde{\iota}=1}^{\left[\frac{n_2-2}{2}\right]} \left(\sum_{\tilde{i}=1}^{\tilde{\iota}} b_{2\tilde{i}-1, 2\tilde{\iota}} \int_0^{2\pi} \cos^{2\tilde{\iota}-2\tilde{i}+2} \theta \sin^{2\tilde{i}} \theta d\theta \right) r^{2\tilde{\iota}}. \end{aligned}$$

We can choose the coefficients in (37) in such a way that $F_{3,0}(r)$ has exactly $\max\{k_3-2; \left[\frac{n_2-2}{2}\right]-1\}$ real positive roots. Thus, Theorem 1.1 holds.

4. PROOF OF THEOREM 1.2

Theorem 1.2 is obtained by using the results of Section 2 and the structure is similar to Section 3. The next polynomial will be needed

$$(38) \quad P_\ell(r, \theta) = \frac{E^2}{A_\ell^2} \left(1 + A_\ell r \cos \theta\right) \left(1 + A_\ell + A_\ell r \cos \theta\right).$$

4.1. Proof of statement (a) of Theorem 1.2. As before we apply (8), and rewrite (4) with $\ell = 1$ as

$$\begin{aligned} \dot{r} &= \varepsilon \left(q_1(r, \theta) - A_1 \right) \left(r^2 - P_1(r, \theta) \right) \sin \theta, \\ \dot{\theta} &= 1 + \frac{\varepsilon}{r} \left(q_1(r, \theta) - A_1 \right) \left(r^2 - P_1(r, \theta) \right) \cos \theta. \end{aligned}$$

Consequently,

$$\frac{dr}{d\theta} = \varepsilon \left(q_1(r, \theta) - A_1 \right) \left(r^2 - P_1(r, \theta) \right) \sin \theta + \mathcal{O}(\varepsilon^2).$$

The definition of $F_{1,0}(r)$, in Section 2 joint to (6) imply that

$$F_{1,0}(r) = \frac{r^2}{2\pi} \int_0^{2\pi} q_1(r, \theta) \sin \theta d\theta - \frac{1}{2\pi} \int_0^{2\pi} q_1(r, \theta) P_1(r, \theta) \sin \theta d\theta.$$

Moreover, since

$$P_1(r, \theta) = \left(\frac{E^2}{A_1^2} + \frac{E^2}{A_1} \right) + r \left(E^2 + \frac{2E^2}{A_1} \right) \cos \theta + r^2 E^2 \cos^2 \theta,$$

the equations (11) and (6) imply that

$$\begin{aligned} (39) \quad F_{1,0}(r) &= \frac{r^3}{2\pi} \sum_{\iota=1}^{k_1-1} \left(\sum_{i=1}^{\iota} a_{2i-1, 2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota-2} \\ &+ \frac{r}{2\pi} \left(\frac{E^2}{A_1^2} + \frac{E^2}{A_1} \right) \sum_{\iota=1}^{k_1-1} \left(\sum_{i=1}^{\iota} a_{2i-1, 2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i} \theta \sin^{2i} \theta d\theta \right) r^{2\iota-2} \\ &+ \frac{r}{2\pi} \left(E^2 + \frac{2E^2}{A_1} \right) \sum_{\iota=1}^{\left[\frac{n_1-2}{2} \right]} \left(\sum_{i=1}^{\iota} a_{2i-1, 2\iota} \int_0^{2\pi} \cos^{2\iota-2i+2} \theta \sin^{2i} \theta d\theta \right) r^{2\iota} \\ &+ \frac{r^2 E^2}{2\pi} \sum_{\iota=1}^{k_1-1} \left(\sum_{i=1}^{\iota} a_{2i-1, 2\iota-1} \int_0^{2\pi} \cos^{2\iota-2i+2} \theta \sin^{2i} \theta d\theta \right) r^{2\iota-1}. \end{aligned}$$

Therefore, there is a polynomial in (39) with exactly $\max \{k_1 - 2; \left[\frac{n_1-2}{2} \right] - 1\}$ real positive roots. Thus, statement (a) of Theorem 1.2 holds.

Remark 4.1. If we consider (4) with $\ell = 1$ and $E = \varepsilon > 0$, its system in polar coordinates induces the equation

$$\frac{dr}{d\theta} = \varepsilon r^2 (q_1(r, \theta) - A_1) \sin \theta + \mathcal{O}(\varepsilon^2),$$

and the respective $F_{1,0}(r)$ satisfies (10) and (11). Therefore, some $F_{1,0}(r)$ has exactly $k_1 - 2$ simple positive roots. Therefore, if $E = \varepsilon > 0$ then we obtain the same conclusions of Statement (a) in Theorem 1.1.

4.2. Proof of statement (b) of Theorem 1.2. We apply (8) joint to (38), and rewrite (4) with $\ell = 1, 2$

$$\begin{aligned} \dot{r} &= \varepsilon \left((q_1(r, \theta) - A_1)(r^2 - P_1(r, \theta)) + \varepsilon (q_2(r, \theta) - A_2)(r^2 - P_2(r, \theta)) \right) \sin \theta, \\ \dot{\theta} &= 1 + \frac{\varepsilon}{r} \left((q_1(r, \theta) - A_1)(r^2 - P_1(r, \theta)) + \varepsilon (q_2(r, \theta) - A_2)(r^2 - P_2(r, \theta)) \right) \cos \theta. \end{aligned}$$

Thus system (4) induces the equation

$$\frac{dr}{d\theta} = \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + \mathcal{O}(\varepsilon^3),$$

where

$$(40) \quad \begin{cases} F_1(r, \theta) = (q_1(r, \theta) - A_1)(r^2 - P_1(r, \theta)) \sin \theta, \\ F_2(r, \theta) = \left[(q_2(r, \theta) - A_2)(r^2 - P_2(r, \theta)) - \frac{1}{r} (q_1(r, \theta) - A_1)^2 (r^2 - P_1(r, \theta))^2 \cos \theta \right] \sin \theta. \end{cases}$$

To compute $F_{2,0}(r)$ we assume $F_{1,0}(r) \equiv 0$, thus from (39) we assume

$$(41) \quad \begin{aligned} a_{2i-1, 2\iota-1} &= 0, & 1 \leq i, \iota \leq k_1 - 1; \\ a_{2i-1, 2\iota} &= 0, & 1 \leq i, \iota \leq \left\lfloor \frac{n_1-2}{2} \right\rfloor. \end{aligned}$$

From (41) and (7) we have that $q_1 = B_{q_1} + C_{q_1}$, so by using that $\sin^2 \theta = 1 - \cos^2 \theta$ it is easy to obtain the existence of $n, n_j \in \mathbb{N} \cup \{0\}$ and $c_j(r) \in \mathbb{R}$ such that

$$(42) \quad (q_1(r, \theta) - A_1)(r^2 - P_1(r, \theta)) = \sum_{j=0}^n c_j(r) \cos^{n_j} \theta.$$

Similarly, (40) and $y_1(r, \theta) = \int_0^\theta F_1(r, \phi) d\phi$ show that

$$(43) \quad y_1(r, \theta) = \sum_{j=0}^{\tilde{n}} \tilde{c}_j(r) \cos^{\tilde{n}_j} \theta \quad \text{and} \quad \frac{\partial}{\partial r} F_1(r, \theta) = \left(\sum_{j=0}^{\tilde{n}} \frac{\partial}{\partial r} \tilde{c}_j(r) \cos^{\tilde{n}_j} \theta \right) \sin \theta,$$

where $\tilde{n}, \tilde{n}_j \in \mathbb{N} \cup \{0\}$ and $\tilde{c}_j(r) \in \mathbb{R}$. Therefore, (6) implies that

$$\int_0^{2\pi} \frac{\partial}{\partial r} F_1(r, \theta) \cdot y_1(r, \theta) d\theta = 0$$

and

$$(44) \quad F_{2,0}(r) = \frac{1}{2\pi} \int_0^{2\pi} F_2(r, \theta) d\theta$$

In a similar way $\left(q_1(r, \theta) - A_1\right) \left(r^2 - P_1(r, \theta)\right) = \sum_{j=0}^n c_j(r) \cos^{n_j} \theta$ and (6) implies that

$$\int_0^{2\pi} \left[\frac{1}{r} \left(q_1(r, \theta) - A_1\right)^2 \left(r^2 - P_1(r, \theta)\right)^2 \cos \theta \right] \sin \theta d\theta = 0$$

Thus, by using (44) and (40) we conclude that

$$F_{2,0}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(q_2(r, \theta) - A_2\right) \left(r^2 - P_2(r, \theta)\right) \sin \theta d\theta.$$

satisfies

$$(45) \quad F_{2,0}(r) = \frac{r^2}{2\pi} \int_0^{2\pi} q_2(r, \theta) \sin \theta d\theta - \frac{1}{2\pi} \int_0^{2\pi} q_2(r, \theta) P_2(r, \theta) \sin \theta d\theta,$$

where

$$P_2(r, \theta) = \left(\frac{E^2}{A_2^2} + \frac{E^2}{A_2}\right) + r \left(E^2 + \frac{2E^2}{A_2}\right) \cos \theta + r^2 E^2 \cos^2 \theta.$$

Therefore, $F_{2,0}(r)$ has the fom (39) with $q_2(x, y) = \sum_{d=1}^{n_2-2} \left(\sum_{j=0}^d b_{j,d} x^{d-j} y^j\right)$ instead of q_1 , and we conclude that some $F_{2,0}(r)$ has exactly $\max\{k_2 - 2; \lceil \frac{n_2-2}{2} \rceil - 1\}$ real positive roots. Thus, statement (b) of Theorem 1.2 holds.

4.3. Proof of statement (c) of Theorem 1.2. We develop the same idea of applying (8) to the system (4) with $\ell = 1, 2, 3$. This induces the equation

$$\frac{dr}{d\theta} = \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + \varepsilon^3 F_3(r, \theta) + \mathcal{O}(\varepsilon^4)$$

where $F_1(r, \theta)$ and $F_2(r, \theta)$ satisfy (24) but

$$(46) \quad \begin{aligned} F_3(r, \theta) = & \left(q_3(r, \theta) - A_3\right) \left(r^2 - P_3(r, \theta)\right) \sin \theta \\ & - \frac{\left(q_2(r, \theta) - A_2\right) \left(r^2 - P_2(r, \theta)\right) \cos \theta}{r} F_1(r, \theta) \\ & - \frac{\left(q_1(r, \theta) - A_1\right) \left(r^2 - P_1(r, \theta)\right) \cos \theta}{r} F_2(r, \theta). \end{aligned}$$

As before, we assume $F_{1,0} \equiv F_{2,0} \equiv 0$, that is

$$(47) \quad \begin{aligned} a_{2i-1,2\iota-1} &= 0, & 1 \leq i, \iota \leq k_1 - 1; \\ a_{2i-1,2\iota} &= 0, & 1 \leq i, \iota \leq \left\lfloor \frac{n_1-2}{2} \right\rfloor; \\ b_{2i-1,2\iota-1} &= 0, & 1 \leq i, \iota \leq k_2 - 1; \\ b_{2i-1,2\iota} &= 0, & 1 \leq i, \iota \leq \left\lfloor \frac{n_2-2}{2} \right\rfloor. \end{aligned}$$

Thus (43) and (6) imply

$$\int_0^{2\pi} \frac{1}{2} \left(y_1(r, \theta) \right)^2 \frac{\partial^2}{\partial r^2} F_1(r, \theta) \cdot y_1(r, \theta) d\theta = 0.$$

Similarly, (47) shows that $\frac{\partial}{\partial r} F_2(r, \theta)$ has the form of (43), and consequently

$$\int_0^{2\pi} \frac{\partial}{\partial r} F_2(r, \theta) \cdot y_1(r, \theta) d\theta = 0.$$

Moreover, by using this idea, it is not difficult to prove that $y_2(r, \theta)$ has the form of (42), and so

$$\int_0^{2\pi} \frac{1}{2} \frac{\partial}{\partial r} F_1(r, \theta) \cdot y_2(r, \theta) d\theta = 0.$$

Therefore, $F_{3,0}(r) = \frac{1}{2\pi} \int_0^{2\pi} F_3(r, \theta) d\theta$, satisfies

$$F_{3,0}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(q_3(r, \theta) - A_3 \right) \left(r^2 - P_3(r, \theta) \right) \sin \theta d\theta,$$

or equivalently

$$(48) \quad F_{3,0}(r) = \frac{r^2}{2\pi} \int_0^{2\pi} q_3(r, \theta) \sin \theta d\theta - \frac{1}{2\pi} \int_0^{2\pi} q_3(r, \theta) P_3(r, \theta) \sin \theta d\theta.$$

This equation (48) is similar to (45), so we conclude that that some $F_{3,0}(r)$ has exactly $\max \{k_3 - 2, \left\lfloor \frac{n_3-2}{2} \right\rfloor - 1\}$ real positive roots. Thus, Theorem 1.2 holds.

REFERENCES

1. A. A. Andronov, E. A. Leontovich E A, I. I. Gordon and A. G. Maier, Qualitative theory of second-order dynamic systems. Halsted Press, New York–Toronto, 1973.
2. A. Buică and J. Llibre. Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* **128** (2004), 7–22.
3. A. Buica; J. P. Francoise; J. Llibre. Periodic solutions of nonlinear periodic differential systems with a small parameter. *Commun. Pure Appl. Anal.* **6** (2007), 103–111.
4. C. J. Christopher; N. G. Lloyd, On the paper of X. Jin and D. Wang concerning the conditions for a centre in certain cubic systems *Bull. London Math. Soc.* **22** (1990), 5–12.
5. J. Chavarriga; I. A. García; E. Sáez; I. Szántó. Limit cycles in Kukles systems of arbitrary degree with invariant ellipse. *Nonlinear Anal.* **67** (2007), 1005–1014.
6. J. Chavarriga; E. Sáez; I. Szántó; M. Grau. Coexistence of limit cycles and invariant algebraic curves for a Kukles system. *Nonlinear Anal.* **59** (2004), 673–693.
7. H. Giacomini; J. Llibre; M. Viano. On the shape of limit cycles that bifurcate from Hamiltonian centers. *Nonlinear Anal.* **41** (2000), 523–537.

8. J. Giné; M. Grau; J. Llibre. Averaging theory at any order for computing periodic orbits. *Phys. D*, **250** (2013), 58–65.
9. J. Guckenheimer; P. Holmes. Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Revised and corrected reprint of the 1983 original. Appl. Math. Sci. **42**. Springer-Verlag, New York, 1990. xvi+459 pp.
10. I. S. Kukles. Sur quelques cas de distinction entre un foyer et un centre. (French) *C. R. (Doklady) Acad. Sci. URSS (N. S.)* **42** (1944), 208–211.
11. D. Hilbert Mathematical problems. Reprinted from Bull. Amer. Math. Soc. 8 (1902), 437–479. Bull. Amer. Math. Soc. (N.S.) **37** (2000), 407–436.
12. J. Llibre. Averaging theory and limit cycles for quadratic systems. *Rad. Mat.* **11** (2002/03), 215–228.
13. J. Llibre and A. C. Mereu. Limit cycles for generalized Kukles polynomial differential systems. *Nonlinear Anal.* **74** (2011), 1261–1271.
14. J. Llibre; A. C. Mereu; M. A. Teixeira.) Limit cycles of the generalized polynomial Linard differential equations. *Math. Proc. Cambridge Philos. Soc.* **148** (2010), 363–383
15. J. Llibre; R. Rabanal, Planar real polynomial differential systems of degree $n \geq 3$ having a weak focus of high order. *Rocky Mountain J. Math.* **42** (2012), 657–693.
16. J. llibre; C. Valls. Limit cycles for a generalization of polynomial Liénard differential systems *Chaos Solitons Fractals* **46** (2013), 65–74
17. J. Pearson; N. G. Lloyd. Kukles revisited: advances in computing techniques. *Comput. Math. Appl.* **60** (2010), 2797–2805.
18. H. Poincaré. Memoire sur les Courbes Dfinies par Une quation Diffrentielle, Edit. Jacques Gabay, Paris, 1993. Reprinted from the original papers published in the Journal de Mathematiques **7** (1881), 375–422, **8** (1882), 251–296, **1** (1885), 167–244, and **2** (1886), 151–217.
19. V. G. Romanovski; D. S. Shafer, The center and cyclicity problems: a computational algebra approach. Birkhuser Boston, Inc., Boston, MA, 2009. xvi+330 pp.
20. C. Rousseau; B. Toni Local bifurcations of critical periods in the reduced Kukles system. *Canad. J. Math.* **49** (1997) 338–358.
21. C. Rousseau; D. Schlomiuk; P. Thibaut The centres in the reduced Kukles system. *Nonlinearity* **8** (1995), 541–569.
22. A. P. Sadovskii. Cubic systems of nonlinear oscillations with seven limit cycles. *Differ. Equ.* **39** (2003), 505–516
23. E. Sáez; I. Szántó. Bifurcations of limit cycles in Kukles systems of arbitrary degree with invariant ellipse. *Appl. Math. Lett.* **25** (2012), 1695–1700.
24. E. Sáez; I. Szántó. Coexistence of algebraic and nonalgebraic limit cycles in Kukles systems. *Period. Math. Hungar.* **56** (2008), 137–142
25. S. Smale. Mathematical problems for the next century. Mathematics: frontiers and perspectives, 271–294, Amer. Math. Soc., Providence, RI, 2000.
26. F. Verhulst. Nonlinear differential equations and dynamical systems. Second edition. Universitext. Springer-Verlag, Berlin, 1996. x+303 pp.
27. H. Zang; T. Zhang; Y. C. Tian; M. Tadé. Limit cycles for the Kukles system. *J. Dyn. Control Syst.* **14** (2008), 283–298.
28. L. Zhenhai; E. Sáez; I. Szántó. Limit cycles and invariant parabola in a Kukles system of degree three. *Acta Math. Sci. Ser. B Engl. Ed.* **28** (2008), 865–869.

DEPARTAMENTO DE CIENCIAS
 PONTIFICIA UNIVERSIDAD CATÓLICA DEL PERÚ
 AV. UNIVERSITARIA 1801, LIMA 32; PERÚ
E-mail address: rrabanal@pucp.edu.pe