



**CENTRE DE RECERCA MATEMÀTICA**

This is a preprint of: *Topology of the intersections of quadrics II*  
Journal Information: *CRM Preprints*,  
Author(s): V.G. Gutiérrez and S. López de Medrano.  
Volume, pages: 1-22, DOI:[--]



CENTRE DE RECERCA MATEMÀTICA

Preprint núm. 1175

October 2013

## Topology of the intersections of quadrics II

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# TOPOLOGY OF THE INTERSECTIONS OF QUADRICS II

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*To Fico, on his 70th birthday.*

## INTRODUCTION.

We study the topology of the affine real variety  $V = F^{-1}(0)$  in  $\mathbf{R}^n$ , where  $F: \mathbf{R}^n \rightarrow \mathbf{R}^2$  is given by two quadratic forms. Due to homogeneity, it is enough to study the intersection of  $V$  with the unit sphere, so we will be dealing with the variety  $Y$  given by equations

$$\sum a_{ij}x_i x_j = 0$$

$$\sum b_{ij}x_i x_j = 0$$

$$\sum_{i=1}^n x_i^2 = 1$$

where  $a_{ij}, b_{ij}$  are the entries of real symmetric  $n \times n$  matrices  $A, B$ .

We will give a complete topological description of this variety in the generic case, that is, when the above system of equations is of rank 3 in  $Y$  and therefore  $Y$  is smooth. We show that a smooth  $Y$ , when non-empty, is always diffeomorphic to either:

- a) The unit tangent bundle of a sphere, or
- b) The product of two spheres, or
- c) The product of three spheres, or
- d) The connected sum of an odd number of manifolds, each of them a product of two spheres.

The case to which a specific  $Y$  corresponds, as well as the dimensions of the spheres and, in case d), the number of summands, can be read from a normal form to which the couple of quadratic maps can be deformed.

This study was started by C.T.C. Wall in [6] where he gave the normal form in general and, in the *diagonal* case (that is, when the two quadratic forms are simultaneously diagonalizable) the homology groups of  $Y$  and the diffeomorphism type in case c).

In January 1984 the second-named author of this paper began to look at this problem, starting from a question by Alberto Verjovsky. Without knowing the work of Wall he rediscovered the normal form and his results in the diagonal case and went further to describe the diffeomorphism type of  $Y$  in d) in most diagonal

cases. This result was obtained along that year and on the next one the details of the proof were completed. The result was announced in various meetings along 1986 and finally published in what it should now be called *Topology of the Intersection of Quadrics I*: [4].

The topological description was still lacking for:

- a) the unsettled diagonal cases,
- b) all the non-diagonal ones and
- c) the intersections of three or more quadrics.

Though much work around these varieties was done in other directions, nothing new was obtained about those open questions until in [1] several new roads were opened towards the understanding of c). In [2], these roads were developed to obtain several new results, thanks to the combination of two different approaches to the study of the same type of objects that had coexisted for a long time without any contact between them.

Following the impulse and experience of these developments, new results have been obtained, including two old dreams of ours: the topological description of cases a) and b) above (which we give in the present article) and the answer to new questions by Alberto. A survey of results recently obtained can be found in [5]<sup>1</sup> while new lines of research are being explored.

In Section 1 we recall briefly the normal form for the equations of  $Y$  and state the Main Theorem of this article, of which we give a brief proof in the simple cases.

In Section 2 we recall basic facts about group actions and polytopes associated to the diagonal cases.

In Section 3 we describe the geometry of three operations producing new varieties from old. At the end of each description we give examples which are, in fact, proofs of different cases of the Main Theorem.

In Section 4 we give the homology groups in all cases, recalling it in the diagonal cases and deducing the general case through computations of the homology of the branched coverings that appear in the second operation.

In Section 5 we show that the first two operations preserve connected sums of sphere products. Together with the homology computations and the examples proved in the first three section this gives the complete proof of the Main Theorem.

Results, ideas, suggestions and comments by Francisco González Acuña, *Fico* form a dense set throughout the geometric proofs in this article and forthcoming ones, to which it is impossible to give credit at every point. Instead, the whole article is dedicated to him.

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<sup>1</sup>As well as a list of many fields where these varieties appear as interesting examples.

This work includes part of the Ph.D. Thesis of the first-named author. Some of the results were proved while the second-named author occupied the *Lluís Santaló* temporal position at the Centre de Recerca Matemàtica in Bellaterra, Barcelona and enjoyed its warm and rich academic atmosphere.

## 1. THE NORMAL FORM AND THE MAIN THEOREM.

Assume now that the above system is regular, so that  $Y$  is smooth. If we deform the coefficients of the equation without breaking the regularity condition then the diffeomorphism type of  $Y$  does not change. The description of the differentiable type of the manifold  $Y$  passes through the process of making such a deformation until we reach the simplest form:

By a small perturbation one can assume that the matrices  $A, B$  are invertible and that all the eigenvalues of  $C := A^{-1}B$  are different. Then  $C$  can be diagonalized over the complex numbers, with respect to a basis of eigenvectors that are orthogonal with respect to both quadratic forms  $A$  and  $B$ . Over the real numbers the two forms will split into common orthogonal subspaces of dimension 1 and 2 corresponding respectively to the real and complex eigenvalues. The 2-dimensional ones, depending essentially on a pair of conjugate complex numbers, can be deformed always into the unique form  $(u^2 - v^2, 2uv)$  without changing the regularity of the system nor the differentiable type of  $Y$ .

With respect to the new coordinates, the quadratic forms will be

$$\begin{aligned} \Sigma_{i=1}^r a_i x_i^2 + \Sigma_{j=1}^s (u_j^2 - v_j^2) \\ \Sigma_{i=1}^r b_i x_i^2 + \Sigma_{j=1}^s 2u_j v_j \end{aligned}$$

The fact that we have a regular system is expressed by the condition that  $0 \in \mathbf{R}^2$  should not be a convex combination of any two of the  $\lambda_i = (a_i, b_i)$ , a property known as *normal hyperbolicity*, and the diffeomorphism type of  $Y$  is not changed if we move the points  $\lambda_i$  around  $\mathbf{R}^2$  without breaking this condition. With this type of deformation one can join together as many of them as possible in single points with multiplicity, then push them radially until they are in the unit circle and finally distribute them evenly along the circle. Then the  $\lambda_i$  can be assumed to be the  $(2\ell+1)$ -th roots of unity  $\rho^j$ , where  $\rho$  has argument  $2\pi/(2\ell+1)$  and  $\rho^j$  appears as coefficient with multiplicity  $n_j$ ,  $j = 1, \dots, 2\ell+1$ . The number of terms is always odd and the type of  $Y$  does not change if one permutes cyclically the partition (For details, see [4]). This puts the diagonal part of our pair of quadratic forms in a normal form.

So the normal form is completely described by the non-negative integers  $r$  and  $s$  and, if  $r > 0$ , the cyclic partition  $r = n_1 + n_2 + \dots + n_{2\ell+1}$ .

The differentiable type of  $Y$  will depend only on the numbers

$$d_i = n_i + \dots + n_{i+\ell-1}, \quad i = 1, \dots, 2\ell+1$$

where the sub-indices of the  $n_j$  are always reduced modulo  $2\ell+1$ . Now we can state our

**Main Theorem.** *Let  $Y$  be the variety corresponding to the non-negative integers  $r, s$  and, when  $r > 0$ , the cyclic partition  $r = n_1 + n_2 + \cdots + n_{2\ell+1}$ . Then*

- (i) *if  $r = 0, s \leq 1$ , or if  $r > 0, \ell = 0, s = 0$ ,  $Y$  is empty,*
- (ii) *if  $r = 0, s > 1$ ,  $Y$  is diffeomorphic to the unit tangent bundle of  $S^{s-1}$ ,*
- (iii) *if  $r > 0, \ell = 0, s > 0$ ,  $Y$  is diffeomorphic to the product  $S^{s-1} \times S^{r+s-2}$ ,*
- (iv) *if  $r > 0, \ell = 1, s = 0$ ,  $Y$  is diffeomorphic to the product*  

$$S^{n_1-1} \times S^{n_2-1} \times S^{n_3-1}$$
- (v) *and if  $r > 0$  and  $\ell + s > 1$ ,  $Y$  is diffeomorphic to the connected sum*  

$$\#_{i=1}^{2\ell+1} (S^{d_i+s-1} \times S^{n-d_i+s-2}).$$

Case (i) is immediate while in case (ii) the normal form coincides with the well-known complex equations of the unit tangent bundle of  $S^{s-1}$ . It is easy to see that in case (iii) one gets the sphere bundle of the stabilized tangent bundle of the sphere, which is trivial. Case (iv) follows easily: a direct manipulation transforms the equations in the normal form into three equations of spheres with separate variables<sup>2</sup>. The proof of case (v) will occupy the rest of this article.

## 2. ACTIONS AND POLYTOPES.

When  $s = 0$ , the pair of quadratic forms are simultaneously diagonalizable, so the variety  $Y$  is given by

$$(1) \quad \sum_{i=1}^r \lambda_i x_i^2 = 0$$

$$(2) \quad \sum_{i=1}^r x_i^2 = 1.$$

$$\lambda_i \in \mathbf{R}^2, \quad i = 1, \dots, r$$

The manifold  $Y$  admits a  $\mathbf{Z}_2^r$  action by changing the signs of the coordinates. The quotient can be identified with the intersection of  $Y$  with the first orthant of  $\mathbf{R}^r$  and  $Y$  can be reconstructed from this intersection by reflecting it iteratively on all the coordinate hyperplanes.

By a simple change of coordinates  $r_i = x_i^2$ , this quotient-intersection can be identified with the  $d$ -dimensional convex polytope  $P$  given by

$$\sum_{i=1}^r \lambda_i r_i = 0$$

$$\sum_{i=1}^r r_i = 1$$

$$r_i \geq 0.$$

<sup>2</sup>Statements (ii) and (iii), as well as a different proof of (iv), appear in [6]. Case (v) for  $s = 0$  with some restrictions was proved in [4].

The weak hyperbolicity condition is equivalent to the fact that  $P$  is a *simple polytope*, meaning that each vertex is exactly in  $d$  facets of  $P$ .

As a first example, consider the case  $s = 0$ ,  $5 = 1 + 1 + 1 + 1 + 1$ . By symmetry,  $P$  touches every coordinate hyperplane  $r_i = 0$ , so  $Y$  is connected and  $P$  is a pentagon. Then the Euler characteristic of  $Y$  can be computed from its cell decomposition given by  $P$ , all its faces and all their reflections on the coordinate hyperplanes. As a result  $Y$  is the surface of genus 5.

### 3. GEOMETRIC OPERATIONS.

We will be using three geometric operations that associate to one of our varieties a different one<sup>3</sup>:

- a) For  $r > 0$ , the operation  $Y \mapsto Y'$  which increases by 1 one of the  $n_i$  and therefore also the dimension of the variety. This operation depends on the choice of  $i$ , but we will not need to specify this in the notation.
- b) The operation  $Y \mapsto \tilde{Y}$  which increases  $s$  by 1, thus increasing the dimension of the variety by 2.
- c) For  $s = 0$ , the operation of truncating the quotient polytope  $Y/\mathbf{Z}_2^n$  and constructing from the result a new variety  $Y^\wedge$  of the same dimension. This operation adds one variable and one equation, so applied to  $Y$  it would get us out of our field of study. Instead, we will apply it to a variety  $Z$  given by just one quadratic form, to obtain one of our varieties  $Y$ .

We now proceed to describe more precisely each of these operations and give their geometric description.

**3.1. Operation  $Y \mapsto Y'$ .** It adds a new variable  $x_0$  to which we assign an old pair of coefficients, which we can assume to be the first one  $a_1, b_1$ , thus increasing  $n_1, r$  and the dimension of the variety by 1 each.

Let  $Y$  be given, in short notation, by

$$a_1 x_1^2 + G_1(y) = 0$$

$$b_1 x_1^2 + G_2(y) = 0$$

$$x_1^2 + |y|^2 = 1$$

(where  $y \in \mathbf{R}^{n-1}$  includes all variables other than  $x_1$ ).

Then  $Y'$  is given by

$$a_1(x_0^2 + x_1^2) + G_1(y) = 0$$

$$b_1(x_0^2 + x_1^2) + G_2(y) = 0$$

$$x_0^2 + x_1^2 + |y|^2 = 1.$$

<sup>3</sup>These operations can be defined for intersections of any number  $k$  homogeneous quadrics and the unit sphere, but are only used in this article for  $k = 2$  (a and b) and  $k = 1$  (c).



From the form of these equations there is an action of  $S^1$  on  $Y'$  by rotation in the coordinate plane  $(x_0, x_1)$ , whose quotient can be identified with the submanifold with boundary

$$Y_+ = Y \cap \{x_1 \geq 0\}.$$

Let also:

$$Y_0 = Y \cap \{x_1 = 0\}$$

$$Y'_+ = Y \cap \{x_0 \geq 0\}$$

so we have a retraction:  $Y' \rightarrow Y_+$  given by  $(x_0, x_1, y) \mapsto (\sqrt{x_0^2 + x_1^2}, y)$  that restricts to retractions  $Y'_+ \rightarrow Y_+$  and  $Y \rightarrow Y_+$ .

From this, the following proposition follows easily:

**Proposition.**

- (i)  $Y'$  is an open book with binding  $Y_0$ , page  $Y_+$  and trivial monodromy.
- (ii)  $Y_+$  is a deformation retract of  $Y'_+$  and  $Y'_+$  is diffeomorphic to  $Y_+ \times D^1$ .
- (iii) The inclusion of  $Y$  in  $Y'_+$  as its boundary is homotopic to the retraction  $Y \rightarrow Y_+,.$

**Example.** In the case  $s = 0$ ,  $5 = 1 + 1 + 1 + 1 + 1$  we have seen that  $Y$  is the surface of genus 5 (end of Section 2) and that  $Y_0 = S^1 \times S^0 \times S^0$ , since its normal form is  $4 = 2 + 1 + 1$  (see the end of Section 1). It follows that  $Y_+$  (being a retraction of  $Y$ , having  $Y_0$  as boundary and  $Y$  as double) can only be a torus minus four disks. It is easy to see that  $Y'_+ = Y_+ \times D^1$  is the connected sum along the boundary of 5 copies of  $S^1 \times D^2$  and  $Y' = \#_5(S^1 \times S^2)$ . By induction, when  $s = 0, r = q + 1 + 1 + 1 + 1$  we have  $Y = \#_5(S^1 \times S^q)$ . This will also follow from Section 3.3.

**3.2. Operation  $Y \mapsto \tilde{Y}$ .** It adds two new variables  $(u, v)$  and thus increases  $s$  by 1 and the dimension of the variety by 2:

Let  $Y$  be given, in short notation, by equations in  $\mathbf{R}^n$

$$F_1(x) = 0$$

$$F_2(x) = 0$$

$$|x|^2 = 1$$

then  $\tilde{Y}$  is given by equations

$$F_1(x) + u^2 - v^2 = 0$$

$$F_2(x) + 2uv = 0$$

$$|x|^2 + u^2 + v^2 = 1.$$

**Proposition.**

- (i)  $\tilde{Y}$  is a double cover of  $S^{n-1}$ , ramified over  $Y$ .

- (ii) *There is an embedding  $Y' \subset \tilde{Y}$  such that  $\Phi(Y')$  is a manifold with boundary diffeomorphic to  $Y'_+ \subset S^{n-1}$ .*
- (iii)  *$\tilde{Y}$  is diffeomorphic to the union of two copies of the exterior of  $Y'_+$  in  $S^{n-1}$  glued by a diffeomorphism of their boundaries.*

**Proof.** The mapping in (i) is given by

$$\Phi(x, u, v) = x/|x|$$

We can also write the equations of  $Y$  in complex notation

$$F(x) + w^2 = 0$$

$$|x|^2 + |w|^2 = 1$$

where  $F(x) = F_1(x) + iF_2(x)$  and  $w = u + iv$ .

Clearly,  $\Phi$  is well defined (since  $x$  cannot be null at  $\tilde{Y}$ ) and simple computations show that

- the inverse image of every point in the sphere consists of two points except for the points in  $Y$  where it is just one.
- Outside  $Y$ ,  $\Phi$  is a local diffeomorphism.
- $\Phi$  is equivalent to the mapping

$$\{(x, w) : F(x) + w^2 = 0, |x|^2 = 1\} \longrightarrow \{(x, z) : F(x) + z = 0, |x|^2 = 1\}$$

given by  $(x, w) \mapsto (x, w^2)$ , the equivalence given by diffeomorphisms  $\phi(x, w) = \frac{1}{|x|}(x, w)$  and  $\psi(x) = (x, -F(x))$ , which proves (i).

For (ii) let  $\tilde{Y}_0 = \tilde{Y} \cap \{v = 0\}$ . We can identify this manifold with  $Y'$  if we add the variable  $u$  to the collection of variables with coefficient  $(1, 0)$ . The image of  $\tilde{Y}_0$  under  $\Phi$  is the manifold with boundary given by

$$\{x \in S^{n-1} : F_1(x) \leq 0, \quad F_2(x) = 0\}$$

and can be identified with  $Y'_+$ .

As for (iii), observe that  $\tilde{Y}$  is the union of two manifolds with boundary

$$W := \tilde{Y} \cap \{v \geq 0\}$$

$$\tilde{Y} \cap \{v \leq 0\}$$

whose intersection is their boundary. They are both diffeomorphic to the exterior of  $\Phi(\tilde{Y}_0)$  and are interchanged by the diffeomorphism  $(x, w) \mapsto (x, -w)$ . If this diffeomorphism restricted to its boundary could be extended to one of  $W$  with itself one would conclude that  $\tilde{Y}$  would be the double of  $W$  and the proof of our main theorem would be enormously simplified. Since we have not been able to prove this in general, we will take a long detour to arrive at the theorem.

**Examples.** In simple cases this is enough to determine  $\tilde{Y}$ :

- Consider the case  $s = 0, r = 1 + p + q$  and assume  $p \leq q$ . Then  $Y = S^0 \times S^{p-1} \times S^{q-1}$ . One can take  $Y'_+ = S^0 \times S^{p-1} \times D^q$  which inside the  $(p+q)$ -sphere is standard. So one can assume that  $Y'_+$  is contained in an equator and the reflection on this equator extends the glueing diffeomorphism for  $\tilde{Y}$  to the exterior  $W$ . Therefore  $\tilde{Y}$  is the double of  $W$  and, the latter being  $S^{p+q}$  minus two copies of  $S^{p-1} \times D^{q+1}$ ,  $\tilde{Y}$  is  $(S^1 \times S^{p+q-1}) \# 2(S^p \times S^q)$ .
- Cases where  $Y$  has dimension 2 not covered so far are  $s = 0, r = 1 + 1 + 1 + 1 + 1$  and  $s = 1, r = 1 + 1 + 1$  for which we know  $Y$  is the surface of genus 5 and 3 respectively.  $Y_+$  will be in both cases a connected surface with non-empty boundary and therefore  $Y'_+ = Y_+ \times D^1$  is a connected sum along the boundary of copies of  $S^1 \times D^2$ . By a theorem of Hosokawa and Kawauchi, [3], this means that the surfaces are unknotted and  $\tilde{Y}$  is the double cover of  $S^4$  over a standard surface. The symmetry argument in the previous paragraph shows that  $\tilde{Y}$  is the connected sum of copies of  $S^2 \times S^2$  (5 and 3, respectively).

These results cover all the varieties  $Y$  of dimension not bigger than 4, with the exception of the case  $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$ , which will be addressed by the third operation.

**3.3. Operation  $Z \mapsto Z^\wedge$ .** Let  $Z$  be given by two equations in  $\mathbf{R}^{p+q+2}$ :

$$\begin{aligned} \sum_{i=0}^p x_i^2 - \sum_{j=0}^q y_j^2 &= 0 \\ \sum_{i=0}^p x_i^2 + \sum_{j=0}^q y_j^2 &= 1. \end{aligned}$$

By elementary manipulations of the equations it is evident that  $Z$  is diffeomorphic to the product  $S^p \times S^q$  and that the corresponding polytope  $P = Z/\mathbf{Z}_2^{p+q+2}$  is the product of two simplices  $\Delta^p \times \Delta^q$  given by the corresponding equations in  $\mathbf{R}_+^{p+q+2}$ :

$$\begin{aligned} \sum_{i=0}^p t_i - \sum_{j=0}^q s_j &= 0 \\ \sum_{i=0}^p t_i + \sum_{j=0}^q s_j &= 1 \end{aligned}$$

Now we consider the operation of cutting  $P$  in two pieces  $U$  and  $V$  by an hyperplane  $H$  which we assume does not contain any of the vertices of  $P$ , so that both pieces are also simple polytopes. We will construct varieties  $Y(U), Y(V)$  which

have them as polytopes<sup>4</sup> so that  $Z = Y_+(U) \cup Y_+(V)$ . The general idea<sup>5</sup> is that if  $Y_+(U)$  is simple enough we should be able to describe its complement  $Y_+(V)$  in  $Z$  and therefore the double of  $Y_+(V)$  which is  $Y(V)$ . Details in our cases follow:

**A) Cutting faces.** For  $0 \leq b < q, \epsilon > 0$  let  $H \subset \mathbf{R}^{p+q}$  be the hyperplane given by the equation

$$\sum_{i=1}^p t_i + \sum_{j=b+1}^q s_j = \epsilon$$

Let  $U$  be the intersection of  $P$  with the half-space

$$\sum_{i=1}^p t_i + \sum_{j=b+1}^q s_j \leq \epsilon$$

and  $V$  the intersection of  $P$  with the half-space

$$\sum_{i=1}^p t_i + \sum_{j=b+1}^q s_j \geq \epsilon$$

If  $\epsilon$  is sufficiently small,  $U$  will be a neighborhood of the face  $(1/2, 0, \dots, 0) \times \Delta^b \subset \Delta^p \times \Delta^q$ . So  $U$  is equivalent to  $\Delta^0 \times \Delta^b \times \Delta^{p+q-b}$ . This product has  $p + q + 2$  facets: the new facet contained in  $H$  and the other are subsets of the facets of  $P$ .

We can represent  $U$  and  $V$  as the polytopes of intersections of quadrics as follows:

For  $U$  introduce a new variable  $u$  and an equation joining it with the previous one:

$$-u = \sum_{i=1}^p t_i + \sum_{b+1}^q s_j - \epsilon$$

to obtain a system that clearly represents  $U$  inside  $\mathbf{R}_+ \times \mathbf{R}_+^{p+1} \times \mathbf{R}^{q+1}$ :

$$\begin{aligned} u + \sum_{i=1}^p t_i + \sum_{j=b+1}^q s_j &= \epsilon \\ \sum_{i=0}^p t_i - \sum_{j=0}^q s_j &= 0 \end{aligned}$$

<sup>4</sup>It can be shown that any simple polytope can be realized as the associated polytope of an intersection of diagonal quadrics, in general more than two, but here the construction will be explicite.

<sup>5</sup>This idea was used in [2] to find the topology of some intersections of more than 2 quadrics.

$$\sum_{i=0}^p t_i + \sum_{j=0}^q s_j = 1$$

To put this system in the standard form we just have to combine adequately these equations and then substitute  $u = x_0^2, t_i = x_i^2, s_j = y_j^2$  and we obtain the equations of an intersection of quadrics  $Y(U) \subset \mathbf{R}^{p+q+3}$  whose normal form can be easily identified (see below for the case of  $V$ ). But we can also argue as follows:  $Y(U)$  is obtained from  $U = \Delta^b \times \Delta^{p+q-b}$  by reflecting it in all the coordinate hyperplanes. If we reflect it in all the coordinate hyperplanes that touch it (i.e., in all its facets) we would obtain  $S^b \times S^{p+q-b}$ , but we have still one hyperplane on which to reflect and it does not intersect  $U$ . Therefore  $Y(U)$  consists of two copies of  $S^b \times S^{p+q-b}$ . If we consider only the part  $Y_+(U)$  with  $x_0 \geq 0$  (that is if we reflect on all hyperplanes except  $u = 0$  we clearly obtain two copies of  $S^b \times D^{p+q-b}$ ).

Now we look at  $V$ . We introduce again a new variable  $u$  and an equation joining it with the previous one:

$$u = \sum_{i=1}^p t_i + \sum_{b+1}^q s_j - \epsilon$$

So we can describe  $V$  by the following equations in  $\mathbf{R}_+ \times \mathbf{R}_+^{p+1} \times \mathbf{R}_+^{q+1}$ :

$$\begin{aligned} u - \sum_{i=1}^p t_i - \sum_{j=b+1}^q s_j &= -\epsilon \\ \sum_{i=0}^p t_i - \sum_{j=0}^q s_j &= 0 \\ \sum_{i=0}^p t_i + \sum_{j=0}^q s_j &= 1 \end{aligned}$$

If we add to the first equation the third one multiplied by  $\epsilon$  we get:

$$u + \epsilon t_0 + \sum_{i=1}^p (-1 + \epsilon) t_i + \sum_{j=0}^b \epsilon s_j + \sum_{j=b+1}^q (-1 + \epsilon) s_j = 0$$

If to the third one we add a small multiple of the latter we obtain an equation with all the positive coefficients. We can substitute this one with the equation of the unit simplex in  $\mathbf{R}_+ \times \mathbf{R}_+^{p+1} \times \mathbf{R}_+^{q+1}$  without changing the combinatorial type of the polytope. So we can write  $V$  by the system:

$$u + \epsilon t_0 + \sum_{i=1}^p (-1 + \epsilon) t_i + \sum_{j=0}^b \epsilon s_j + \sum_{j=b+1}^q (-1 + \epsilon) s_j = 0$$

$$\sum_{i=0}^p t_i - \sum_{j=0}^q s_j = 0$$

$$u + \sum_{i=0}^p t_i + \sum_{j=0}^q s_j = 1$$

The coefficients of this system are the following, where we have marked in Figure 1 their multiplicities:

- (1)  $(1, 0)$  with multiplicity 1 (corresponds to the variable  $u$ )
- (2)  $(\epsilon, 1)$  with multiplicity 1 (variable  $t_0$ )
- (3)  $(-1 + \epsilon, 1)$  with multiplicity  $p$  (variables  $t_1, \dots, t_p$ )
- (4)  $(-1 + \epsilon, -1)$  with multiplicity  $q - b$  (variables  $s_{b+1}, \dots, s_q$ )
- (5)  $(\epsilon, -1)$  with multiplicity  $b + 1$  (variables  $s_0, \dots, s_b$ )

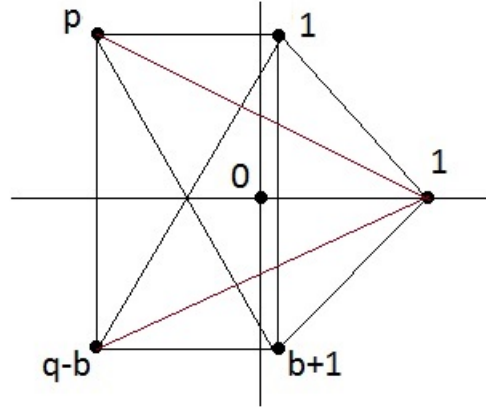


FIGURE 1. The configuration corresponding to  $Z(V)$ .

Therefore we have a configuration of coefficients that can clearly be put in the standard form  $1 + 1 + p + (q - b) + (b + 1)$ .

But, on the other hand we have that  $Y(V)$  is the double of  $Y_+(V)$  and  $Y_+(V) \cup Y_+(U) = Z = S^p \times S^q$ . So we can describe  $Y(V)$  as follows: consider  $S^p \times S^q$  and remove from it two copies of the interior of  $S^b \times D^{p+q-b}$ . Now take the double of this manifold with boundary.

In some cases (and probably in all) we can describe completely this manifold: recall that  $b < q$  and assume now that  $b < p$ . Then the core spheres  $S^b \times \{0\}$  are contractible in  $S^p \times S^q$  so it can be assumed that they lie inside a ball. Furthermore, they can be assumed to be unknotted and unlinked in that ball, so  $Y_+(V)$  can be thought as the connected sum of  $S^p \times S^q$  with the sphere  $S^{p+q}$  minus the tubular neighborhood of two disjoint standard spheres. Then it is easy to see that its double,  $Y(V)$ , is diffeomorphic to the connected sum

$$Y(V) = 2(S^p \times S^q) \# 2(S^{p+q-b-1} \times S^{b+1}) \# (S^{p+q-1} \times S^1).$$

**Example.** Consider the case  $s = 0$ ,  $r = 1 + 1 + n_3 + n_4 + n_5$  where we can assume  $n_3 \geq n_5$  (inverting the order of the coordinates in the normal form turns the coefficients into their conjugates). Then take  $p = n_3$ ,  $b = n_5 - 1$  and  $q = n_4 + n_5 - 1$  and all the conditions of the above result are satisfied so this case is included.

This proves all non-simply-connected cases with  $\ell = 2$  of the Main Theorem.

**B) A deeper cut.** Now we want to build up the variety  $Y$  corresponding to the primitive case  $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$  given by equations in normal form with coefficients  $(a_i, b_i)$  which are the components of  $\rho^i$ ,  $\rho^7 = 1$ .

$$\begin{aligned}\Sigma_1^7 a_i x_i^2 &= 0 \\ \Sigma_1^7 b_i x_i^2 &= 0 \\ \Sigma_1^7 x_i^2 &= 1\end{aligned}$$

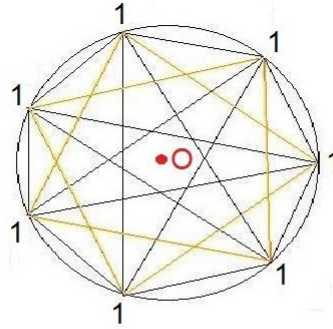
$Z$  will now be the variety given by the two last equations restricted to  $x_7 = 0$ , and the corresponding polytope is  $P = \Delta^2 \times \Delta^2$ :

$$\begin{aligned}\Sigma_1^6 b_i t_i &= 0 \\ \Sigma_1^6 t_i &= 1.\end{aligned}$$

Let now  $H$  be given by

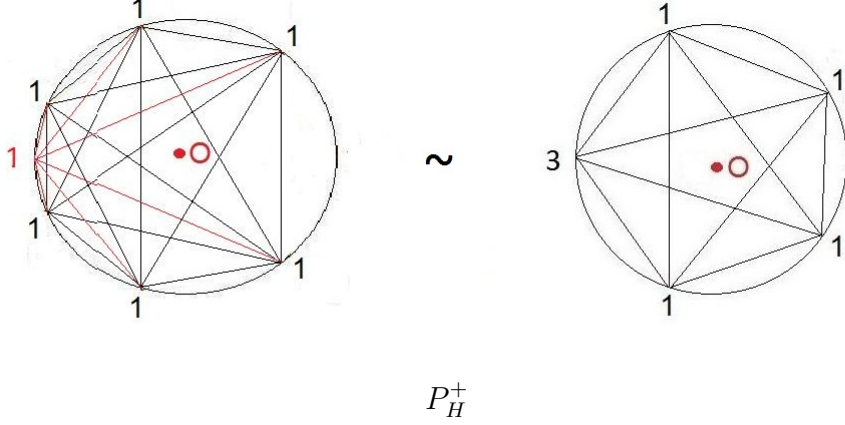
$$\Sigma_1^6 a_i t_i = 0$$

so we are in the situation above with  $p = q = 2$  and  $P_H^-$  is the polytope corresponding to  $Y$ .



$P_H^-$

While  $P_H^+$  has coefficients  $(a_i, b_i)$ ,  $i = 1 \dots 6$  and  $(-1, 0)$ , corresponding to the case  $7 = 3 + 1 + 1 + 1 + 1 + 1$ :



Therefore

$$Y(P_H^+) = 5(S^1 \times S^3)$$

and  $Y_+(P_H^+)$  is the connected sum along the boundary of 5 copies of  $S^1 \times D^3$

Since  $Z(P) = S^2 \times S^2$  is 4 dimensional and simply connected,  $Y_+(P_H^+)$  can be deformed into an open disk  $D^4 \subset S^2 \times S^2$  and assumed to be standard there. So

$$Y_+(P_H^-) = (S^2 \times S^2) \setminus Y_+(P_H^+) = (S^2 \times S^2) \# (S^4 \setminus \amalg 5(S^1 \times D^3)).$$

The second summand is then the connected sum along the boundary of 5 copies of  $S^2 \times D^2$  and therefore

$$Y_+(P_H^-) = (S^2 \times S^2) \# \amalg_5 (S^2 \times D^2).$$

Its double is clearly

$$Y = \#_7 (S^2 \times S^2).$$

With this we have proved all 4-dimensional cases of our Main Theorem.

#### 4. HOMOLOGY.

**Theorem.** *Let  $Y$  correspond to the data  $0 < r = n_1 + n_2 + \dots + n_{2\ell+1}$  and  $s$ . Then the homology of  $Z$  coincides with that of the connected sum<sup>6</sup>*

$$\#_{i=1}^{2\ell+1} (S^{d_i+s-1} \times S^{n-d_i+s-2}).$$

Observe that the number of summands in the connected sum only depends on  $\ell$ , and so, even if the Betti numbers of  $Y$  depend on the numbers  $n_i$ , their sum

<sup>6</sup>See Section 1 for the definition of  $d_i$ . Observe that when  $s = 0, \ell = 1$ ,  $Y$  is not even homotopy equivalent to the connected sum in the theorem, but it does have the same homology groups.



does not. We will call the sum of the Betti numbers of a space  $X$  its *total amount of homology*<sup>7</sup> and denote it by  $\beta(X)$ .

The diagonal case  $s = 0$  was proved in [6] and [4] with totally different arguments. So we consider this case proved.

For the general case it is natural to try to start with cases  $s = 0$  and go up inductively using the geometry of the  $\tilde{Y}$  construction and the Mayer-Vietoris exact sequence of the triple  $(\tilde{Y}; \tilde{Y}^+, \tilde{Y}^-)$ . But this requires the knowledge of some homology properties of the glueing diffeomorphism, a fact that we do not know for the moment how to prove directly. Instead, we will take a detour which at the end of the proof will imply that fact. An important step will be to prove first that the total amount of homology is preserved by the operations  $Y \mapsto Y'$  and  $Y \mapsto \tilde{Y}$ :

**Proposition (Preservation of the total amount of homology).** *The operations  $Y \mapsto Y'$  and  $Y \mapsto \tilde{Y}$  preserve the total amount of homology.*

**Proof.** This is clear in the known case  $s = 0$  and is easy to prove in general for the operation  $Y'$ :

From the retraction (recall Section 3.1)  $Y \rightarrow Y_+$  it follows that we have an isomorphism:

$$H_i(Y) \cong H_i(Y_+) \oplus H_i(Y, Y_+).$$

Since

$$H_i(Y, Y_+) \cong H_i(Y_-, Y_0) \cong H^{n-i}(Y_-)$$

(by the excision isomorphism and Lefschetz Duality, where  $n$  is the dimension of  $Y$ ), we have

$$H_i(Y) \cong H_i(Y_+) \oplus H^{n-i}(Y_+).$$

The same applies to  $Y'$ :

$$H_i(Y') \cong H_i(Y'_+) \oplus H^{n+1-i}(Y'_+).$$

Since we know that  $Y_+$  and  $Y'_+$  are homotopy equivalent, the total amount of homology of  $Y$  and of  $Y'$  is the same.

For the operation  $Y \mapsto \tilde{Y}$  again it is not easy to prove it directly from a Mayer-Vietoris argument, except in the case where  $Y$  is highly connected:

**Lemma.** *Assume  $Y$  has dimension  $2p$  for  $p \geq 0$ ,  $H_i(Y) = 0$  for  $i = 1 \dots p-1$  and  $H_p(Y)$  is free of even rank  $2k$ . Then  $H_i(\tilde{Y}) = 0$  for  $i = 1 \dots p$  and  $H_{p+1}(\tilde{Y})$  is free of rank  $2k$ .*

**Proof.** If  $p = 0$  we are in the cases  $s = 0, 3 = 1 + 1 + 1$  or  $s = 1, r = 1$ , where  $Y = S^0 \times S^0 \times S^0$  ( $Y = S^0 \times S^0$ , respectively) and the double cover of  $S^2$  ramified over  $Y$  is well known to be the surface of genus 3 (genus 0, respectively), the Lemma is true for these cases. For  $p > 0$ , it follows from the previous formulas

<sup>7</sup>Sometimes this sum is called the *complexity* of the space.

that  $H_i(Y_+) = 0$  for  $i = 1 \dots p-1$  and  $H_p(Y_+)$  is free of rank  $k$  and that  $H_i(Y') = 0$  for  $i = 1 \dots p-1$  and  $H_p(Y')$  and  $H_{p+1}(Y')$  are both free of rank  $k$ .

Let  $W$  be the exterior of  $Y'_+$  in  $S^{2p+2}$ . By Alexander Duality,  $H_i(W) = 0$  for  $i \neq 0, p+1$  and  $H_{p+1}(W)$  is free of rank  $k$ .

The Mayer-Vietoris sequence for  $\tilde{Y} = W \cup_{Y'} W$  is interesting only in the range:

$$0 \rightarrow H_{p+2}(\tilde{Y}) \rightarrow H_{p+1}(Y') \rightarrow H_{p+1}(W) \oplus H_{p+1}(W) \rightarrow H_{p+1}(\tilde{Y}) \rightarrow H_p(Y') \rightarrow 0.$$

It follows that  $H_i(\tilde{Y}) = 0$  for  $i = 1 \dots p$  and, by Poincaré Duality, that  $H_{p+2}(\tilde{Y}) = 0$  and  $H_{p+1}(\tilde{Y})$  are free. It follows now from the exact sequence that it has rank  $2k$ .

Now we can finish the proof of the Proposition:  $Y$  arises from a primitive configuration (i.e., one with  $s = 0$  and  $n_i = 1$  for all  $i$ ) by applying the operations  $Y'$  and  $\tilde{Y}$  a number of times each. Since one can assume all the  $\tilde{Y}$  operations are applied first (because they commute), it follows inductively from the Lemma and the preservation of the total amount of homology by the operation  $Y'$ , that  $Y$  has the same amount of homology as that primitive configuration. Now,  $Y$  and  $\tilde{Y}$  arise from the same primitive configuration, so the proposition is proved.

With this information we can now dare into the Mayer-Vietoris sequence in general:

Assume that  $Y$  is connected and non-empty. Due to the retraction  $Y \rightarrow Y_+$  this implies that  $Y_+$  is connected and so are  $Y'_+$  and  $Y'$ .

Let

$$r_i = \beta_i(Y_+).$$

We have  $r_0 = 1, r_d = r_{d+1} = 0$ .

As shown before,

$$\begin{aligned} \beta_i(Y) &= r_i + r_{d-i} \\ \beta_i(Y') &= r_i + r_{d+1-i} \end{aligned}$$

and, therefore,

$$(3) \quad \beta_i(Y') = \beta_i(Y) + r_{d+1-i} - r_{d-i}$$

(while  $\beta_0(W) = 1$  and  $\beta_{d+1}(W) = 0$ ).

First we have the Mayer-Vietoris sequence of the triple  $(S^{d+2}; Y'_+ \times D^1, W)$ :

$$H_{i+1}(S^{d+2}) \rightarrow H_i(Y') \rightarrow H_i(Y'_+ \times D^1) \oplus H_i(W) \rightarrow H_i(S^{d+2}).$$

For  $1 \leq i \leq d$  this implies  $H_i(Y') \rightarrow H_i(W)$  is surjective (even for  $i = d+2$ ).

Second, consider the Mayer-Vietoris sequence of  $(\tilde{Y}; W, W)$

$$H_{i+1}(Y') \rightarrow H_{i+1}(W) \oplus H_{i+1}(W) \rightarrow H_{i+1}(\tilde{Y}) \rightarrow H_i(Y') \rightarrow H_i(W) \oplus H_i(W).$$

For  $i = 1$  we get  $H_1(\tilde{Y}) = 0$  and, by Poincaré Duality,  $H_{d+1}(\tilde{Y}) = 0$ .

Let  $S_i$  be the image of  $H_i(Y') \rightarrow H_i(W) \oplus H_i(W)$  and  $s_i$  its rank. Then, since the homomorphism into the first summand is surjective, we have  $s_i \geq \beta_i(W)$ . The sequence splits into shorter ones

$$0 \rightarrow (H_{i+1}(W) \oplus H_{i+1}(W))/S_{i+1} \rightarrow H_{i+1}(\tilde{Y}) \rightarrow H_i(Y') \rightarrow S_i \rightarrow 0.$$

So we have

$$(2\beta_{i+1}(W) - s_{i+1}) - \beta_{i+1}(\tilde{Y}) + \beta_i(Y') - s_i = 0.$$

Adding this equations for  $i = -1$  to  $d+1$  we obtain

$$(2\Sigma_0^{d+2}\beta_i(W) - \Sigma_0^{d+2}s_i) - \Sigma_0^{d+2}\beta_i(\tilde{Y}) + \Sigma_0^{d+1}\beta_i(Y') - \Sigma_0^{d+1}s_i = 0$$

or, since the total amount of homology of both  $\tilde{Y}$  and  $Y'$  is equal to that of  $Y$ :

$$\Sigma_0^{d+2}(\beta_i(W) - s_i) = 0.$$

Since each summand  $\beta_i(W) - s_i$  is non positive, we conclude that

$$\beta_i(W) = s_i.$$

Plugging this information into the previous formula we get

$$\beta_{i+1}(\tilde{Y}) = \beta_i(Y') + \beta_{i+1}(W) - \beta_i(W).$$

Using formula (1) we obtain

$$\beta_{i+1}(\tilde{Y}) = \beta_i(Y) + r_{d+1-i} - r_{d-i} + \beta_{i+1}(W) - \beta_i(W).$$

Now, if  $1 \leq i \leq d$ ,  $\beta_i(W) = r_{d+1-i}$  by Alexander Duality (while  $\beta_0(W) = 1$  and  $\beta_{d+1}(W) = 0$ ). So, for  $1 \leq i \leq d-1$  we have

$$\beta_{i+1}(\tilde{Y}) = \beta_i(Y).$$

And we already know that  $H_1(\tilde{Y}) = 0$  and  $H_{d+1}(\tilde{Y}) = 0$ .

So we have proved that the Betti numbers of  $\tilde{Y}$  are as expected and, by induction, that the Betti numbers of a connected  $Y$  are equal to those of the connected sum in the Theorem. It remains to prove that the homology groups are all free and to extend the result for the (few) non-connected  $Y$ . All this can be done following the above proof, but we will skip this since we will only need the connected case in the proof of our Main Theorem, which will in turn imply the full version of the Theorem in this Section.

## 5. PRESERVATION OF CONNECTED SUMS.

We will show in this section that the operations  $Y'$  and  $\tilde{Y}$  preserve connected sums of sphere products.

For  $Y'$  this was proved in [2] in the case  $s = 0$  when  $Y$  is simply-connected and of dimension at least 5. The proof applies without change in the general case, with the same hypotheses. Cases not satisfying those conditions have been proved in previous sections.

For  $\tilde{Y}$  we start with a geometric version of our previous homology result when  $Y$  is highly connected.

**Proposition.** I. Assume that  $Y$  has dimension  $2p$  with  $p \geq 0$  and is  $(p-1)$ -connected, with  $H_p(Y)$  free of rank  $2k$ . Then:

- (a)  $Y$  is diffeomorphic to the connected sum of  $k$  copies of  $S^p \times S^p$
- (b)  $Y'$  is diffeomorphic to the connected sum of  $k$  copies of  $S^p \times S^{p+1}$
- (c)  $Y'_+$  is diffeomorphic to the connected sum along the boundary of  $k$  copies of  $S^p \times D^{p+1}$
- (d)  $\tilde{Y}$  is  $p$ -connected with  $H_{p+1}(\tilde{Y})$  free of rank  $2k$

II. Assume that  $Y$  has dimension  $2p+1$  with  $p \geq 0$  and is  $(p-1)$ -connected, with  $H_p(Y)$  free of rank  $k$ .

Then:

- (a)  $Y$  is diffeomorphic to the connected sum of  $k$  copies of  $S^p \times S^{p+1}$
- (b)  $Y'$  is diffeomorphic to the connected sum of  $k_1$  copies of  $S^p \times S^{p+2}$  and  $k_2$  copies of  $S^{p+1} \times S^{p+1}$ , with  $k_1 + k_2 = k$ .
- (c)  $Y'_+$  is diffeomorphic to the connected sum along the boundary of  $k_1$  copies of  $S^p \times D^{p+2}$  and  $k_2$  copies of  $S^{p+1} \times D^{p+1}$ .
- (d)  $\tilde{Y}$  is  $p$ -connected with  $H_{p+1}(\tilde{Y})$  free of rank  $k$

**Proof.** The cases with  $p = 0, 1, 2$  of I and  $p = 0, 1$  of II have already been proved at the end of Section 3.2, except for one proved at the end of Section 3.3. The other cases, follow from Theorem A1 from [2]<sup>8</sup> that gives sufficient conditions for a manifold with boundary to be a connected sum along the boundary of several manifolds of the form  $S^i \times D^{i+1}$ . The hypotheses of this theorem are immediately satisfied, except for the existence of a basis of the homology of  $Y'_+$ . One can get disjoint embedded spheres with stably trivial normal bundle (by the Hurewicz Theorem and the Whitney embedding theorem). To see they are trivial only in part II we need a proof: If  $p+1$  is even, then the type of the bundle is determined by the intersection number of 0 section with itself, but in  $Y \times D^1$  any sphere can be separated from itself by moving it in the  $D^1$  direction. For  $p+1$  odd the existence of a non-trivial bundle would give a summand of  $Y$  with torsion in its homology.

In both parts (a) follows,  $Y$  being the boundary of  $Y'_+$  and (b) also, since  $Y'$  is the double of  $Y'_+$  while (d) was proved in the previous section.  $\square$

Now we can prove all cases of our Main Theorem, starting with the cases with  $r > 0$ ,  $\ell > 1$ , of dimension at least 5 and simply connected, which follow by induction, applying always first  $s$  times the operation  $\tilde{Y}$  and starting from:

<sup>8</sup>This theorem formalizes an idea already used in [4].

When  $\ell \geq 3$ , the primitive case with  $s = 0$ .

When  $\ell = 2$  and for no two contiguous  $i$  we have  $n_i = 1$ , the case  $8 = 1 + 2 + 1 + 2 + 2$ ,  $s = 0$ .

When  $\ell = 2$ ,  $s > 0$ ,  $n_1 = n_2 = 1$ , the case  $6 = 1 + 1 + 1 + 1 + 2$ ,  $s = 1$ .

When  $\ell = 1$ ,  $s > 0$ , the case  $6 = 2 + 2 + 2$  with  $s = 1$ .

The non-simply connected cases with  $\ell = 1, 2$  are included in the examples in Sections 3.2 and 3.3 and, finally, all cases of dimension at most four were covered in Sections 2, 3.2 and 3.3.

We have finished the proof of our Main Theorem.

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