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# SHARP PITT INEQUALITY AND LOGARITHMIC UNCERTAINTY PRINCIPLE FOR DUNKL TRANSFORM IN $L^2$

D. V. GORBACHEV, V. I. IVANOV, AND S. YU. TIKHONOV

ABSTRACT. We prove sharp Pitt's inequality for the Dunkl transform in  $L^2(\mathbb{R}^d)$  with the corresponding weights. As an application, we obtain the logarithmic uncertainty principle for the Dunkl transform.

## 1. INTRODUCTION

Let  $\Gamma(t)$  be the gamma function,  $\mathbb{R}^d$  be the real space of  $d$  dimensions, equipped with a scalar product  $\langle x, y \rangle$  and a norm  $|x| = \sqrt{\langle x, x \rangle}$ . Denote by  $\mathcal{S}(\mathbb{R}^d)$  the Schwartz space on  $\mathbb{R}^d$  and by  $L^2(\mathbb{R}^d)$  the Hilbert space of complex-valued functions endowed with a norm  $\|f\|_2 = \left( \int_{\mathbb{R}^d} |f(x)|^2 dx \right)^{1/2}$ . The Fourier transform is defined by

$$\widehat{f}(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, y \rangle} dx.$$

W. Beckner [2] proved the Pitt inequality for the Fourier transform

$$(1.1) \quad \||y|^{-\beta} \widehat{f}(y)\|_2 \leq C(\beta) \||x|^\beta f(x)\|_2, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad 0 < \beta < d/2,$$

with sharp constant

$$C(\beta) = 2^{-\beta} \frac{\Gamma(\frac{1}{2}(\frac{d}{2} - \beta))}{\Gamma(\frac{1}{2}(\frac{d}{2} + \beta))}.$$

Noting that  $\|(-\Delta)^{\beta/2} f(x)\|_2 = \||x|^\beta \widehat{f}(x)\|_2$ , Pitt's inequality can be viewed as a Hardy–Rellich inequality  $\||x|^{-\beta} f(x)\|_2 \leq C(\beta) \|(-\Delta)^{\beta/2} f(x)\|_2$ ; see the papers by D. Yafaev [19] and S. Eilertsen [7] for alternative proofs and extensions of (1.1).

For  $\beta = 0$ , (1.1) is the Plancherel theorem. If  $\beta > 0$  there is no extremiser in inequality (1.1) and its sharpness can be obtained on the set of radial functions.

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The proof of (1.1) in [2] is based on an equivalent integral realization as a Stein–Weiss fractional integral on  $\mathbb{R}^d$ . D. Yafaev in [19] used the following decomposition (see [17, Chapt. IV])

$$(1.2) \quad L^2(\mathbb{R}^d) = \sum_{n=0}^{\infty} \oplus \mathcal{R}_n^d,$$

where  $\mathcal{R}_0^d$  is the space of radial function, and  $\mathcal{R}_n^d = \mathcal{R}_0^d \otimes \mathcal{H}_n^d$  is the space of functions in  $\mathbb{R}^d$  that are products of radial functions and spherical harmonics of degree  $n$ . Thanks to this decomposition it is enough to study inequality (1.1) on the subsets of  $\mathcal{R}_n^d$  which are invariant under the Fourier transform.

In this paper, following [19] and using similar decomposition of the space  $L^2(\mathbb{R}^d)$  with the Dunkl weight, we prove sharp Pitt’s inequality for the Dunkl transform.

Let  $R \subset \mathbb{R}^d$  be a root system,  $R_+$  be the positive subsystem of  $R$ , and  $k: R \rightarrow \mathbb{R}_+$  be a multiplicity function with the property that  $k$  is  $G$ -invariant. Here  $G(R) \subset O(d)$  is a finite reflection group generated by reflections  $\{\sigma_a: a \in R\}$ , where  $\sigma_a$  is a reflection with respect to a hyperplane  $\langle a, x \rangle = 0$ .

Throughout this paper we let

$$v_k(x) = \prod_{a \in R_+} |\langle a, x \rangle|^{2k(a)}$$

denote the Dunkl weight. Moreover,  $d\mu_k(x) = c_k v_k(x) dx$ , where

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2} v_k(x) dx$$

is the Macdonald–Mehta–Selberg integral. Let  $L^2(\mathbb{R}^d, d\mu_k)$  be the Hilbert space of complex-valued functions endowed with a norm  $\|f\|_{2, d\mu_k} = \left( \int_{\mathbb{R}^d} |f(x)|^2 d\mu_k(x) \right)^{1/2}$ .

Introduced by C. F. Dunkl, a family of differential-difference operators (Dunkl’s operators) associated with  $G$  and  $k$  are given by

$$D_j f(x) = \frac{\partial f(x)}{\partial x_j} + \sum_{a \in R_+} k(a) \langle a, e_j \rangle \frac{f(x) - f(\sigma_a x)}{\langle a, x \rangle}, \quad j = 1, \dots, d.$$

The Dunkl kernel  $e_k(x, y) = E_k(x, iy)$  is the unique solution of the joint eigenvalue problem for the corresponding Dunkl operators:

$$D_j f(x) = i y_j f(x), \quad j = 1, \dots, d, \quad f(0) = 1.$$

Let us define the Dunkl transforms as follows

$$F_k(f)(y) = \int_{\mathbb{R}^d} f(x) \overline{e_k(x, y)} d\mu_k(x), \quad F_k^{-1}(f)(x) = F_k(f)(-x),$$

where  $F_k(f)$  and  $F_k^{-1}(f)$  are the direct and inverse transforms correspondingly (see, e.g., [12]). For  $k \equiv 0$  we have  $F_0(f) = \widehat{f}$ .

Our goal is to study Pitt's inequality for the Dunkl transform

$$(1.3) \quad \left\| |y|^{-\beta} F_k(f)(y) \right\|_{2, d\mu_k} \leq C(\beta, k) \left\| |x|^\beta f(x) \right\|_{2, d\mu_k}, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

with the sharp constant  $C(\beta, k)$ .

Let us first recall some known results on Pitt's inequality for the Hankel transform. Let  $\lambda \geq -1/2$ . Denote by  $J_\lambda(t)$  the Bessel function of degree  $\lambda$  and by  $j_\lambda(t) = 2^\lambda \Gamma(\lambda + 1) t^{-\lambda} J_\lambda(t)$  the normalized Bessel function. Setting

$$b_\lambda = \left( \int_0^\infty e^{-t^2/2} t^{2\lambda+1} dt \right)^{-1} = \frac{1}{2^\lambda \Gamma(\lambda + 1)}$$

and  $d\nu_\lambda(r) = b_\lambda r^{2\lambda+1} dr$ , we define  $\|f\|_{2, d\nu_\lambda} = \left( \int_{\mathbb{R}_+} |f(r)|^2 d\nu_\lambda(r) \right)^{1/2}$ .

The Hankel transform is defined by

$$H_\lambda(f)(\rho) = \int_{\mathbb{R}_+} f(r) j_\lambda(\rho r) d\nu_\lambda(r).$$

Note that  $H_\lambda^{-1} = H_\lambda$  (see [3, 5, 8]). Pitt's inequality for the Hankel transform is written as

$$(1.4) \quad \left\| \rho^{-\beta} H_\lambda(f)(\rho) \right\|_{2, d\nu_\lambda} \leq c(\beta, \lambda) \left\| r^\beta f(r) \right\|_{2, d\nu_\lambda}, \quad f \in \mathcal{S}(\mathbb{R}_+),$$

where  $c(\beta, \lambda)$  is the sharp constant in (1.4) and  $\mathcal{S}(\mathbb{R}_+)$  is the Schwartz space on  $\mathbb{R}_+$ . Note that if  $f \in \mathcal{R}_0^d$ , a study of the Hankel transform is of special interest since the Fourier transform of a radial function can be written as the Hankel transform.

L. De Carli [3] proved that  $c(\beta, \lambda)$  is finite only if  $0 \leq \beta < \lambda + 1$ . In [3, 4] it was also studied the sharp constant in  $(L^p, L^q)$  Pitt's inequality for the Hankel transform of type (1.4) in the case of  $1 \leq p \leq q \leq \infty$ .

For  $\lambda = d/2 - 1$ ,  $d \in \mathbb{N}$ , the constant  $c(\beta, \lambda)$  was calculated by D. Yafaev [19], and in the general case by S. Omri [11]. The proof of Pitt's inequality in [11] is rather technical and uses the Stein–Weiss type estimate for the so-called B-Riesz potential operator. Following [19], we give a direct and simple proof of inequality (1.4) in Section 2.

Let  $|k| = \sum_{a \in R_+} k(a)$  and  $\lambda_k = d/2 - 1 + |k|$ . For a radial function  $f(r)$ ,  $r = |x|$ , Pitt's inequality for the Dunkl transform (1.3) corresponds to Pitt's inequality for the Hankel transform (1.4) with  $\lambda = \lambda_k$ . Therefore the condition

$$(1.5) \quad 0 \leq \beta < \lambda_k + 1$$

is necessary for  $C(\beta, k) < \infty$ . Our goal is to show that in fact  $C(\beta, k) = c(\beta, \lambda_k)$  if condition (1.5) holds. This will be proved in Section 3. Moreover, in Section 4, we use Pitt's inequality for the Dunkl transform to obtain the logarithmic uncertainty principle (see [2, 11, 15]). It is worth mentioning that different uncertainty principles for the Dunkl transform have been recently studied in, e.g., [9, 10, 13, 14].

Note that for the one-dimensional Dunkl weight

$$v_\lambda(t) = |t|^{2\lambda+1}, \quad d\mu_\lambda(t) = \frac{v_\lambda(t) dt}{2^{\lambda+1}\Gamma(\lambda+1)}, \quad \lambda \geq -1/2,$$

and the corresponding Dunkl transform

$$F_\lambda(f)(s) = \int_{\mathbb{R}} f(t) \overline{e_\lambda(st)} |t|^{2\lambda+1} d\mu_\lambda(t), \quad e_\lambda(t) = j_\lambda(t) - i j'_\lambda(t),$$

F. Soltani [15] proved Pitt's inequality that can be equivalently written as

$$(1.6) \quad \| |s|^{-\beta} F_\lambda(f)(s) \|_{2, d\mu_\lambda} \leq \max \{c(\beta, \lambda), c(\beta, \lambda+1)\} \| |t|^\beta f(t) \|_{2, d\mu_\lambda}$$

for  $f \in \mathcal{S}(\mathbb{R})$  and  $0 \leq \beta < \lambda+1$ . Since  $c(\beta, \lambda) \geq c(\beta, \lambda+1)$  (see [19]), then in fact (1.6) holds with the constant  $c(\beta, \lambda)$  and therefore, we have in this case  $C(\beta, k) = c(\beta, \lambda_k)$ .

Finally, we remark that Pitt's inequality in  $L^2$  for the multi-dimensional Dunkl transform has been recently established in [16] in the case of  $\lambda_k - 1/2 < \beta < \lambda_k + 1$ . The obtained constant is not sharp.

## 2. PITT'S INEQUALITY FOR HANKEL TRANSFORM

**Theorem 2.1.** *Let  $\lambda \geq -1/2$  and  $0 \leq \beta < \lambda+1$ , then for  $f \in \mathcal{S}(\mathbb{R}_+)$  inequality (1.4) holds with the sharp constant*

$$c(\beta, \lambda) = 2^{-\beta} \frac{\Gamma(\frac{1}{2}(\lambda+1-\beta))}{\Gamma(\frac{1}{2}(\lambda+1+\beta))}.$$

*Proof.* For  $\beta = 0$  we have  $c(\beta, \lambda) = 1$  and (1.4) becomes Plancherel's identity

$$\|H_\lambda(f)(\rho)\|_{2, d\nu_\lambda} = \|f(r)\|_{2, d\nu_\lambda}.$$

Let  $\beta > 0$ . Setting  $g(r) = f(r)r^{\beta+\lambda+1/2}$  in (1.4), we arrive at

$$\left( \int_0^\infty \left| \int_0^\infty g(r) J_\lambda(\rho r) (\rho r)^{1/2-\beta} dr \right|^2 d\rho \right)^{1/2} \leq c(\beta, \lambda) \left( \int_0^\infty |g(r)|^2 dr \right)^{1/2}.$$

Hence,  $c(\beta, \lambda)$  is the norm of the integral operator

$$A_\lambda g(\rho) = \int_0^\infty a_\lambda(\rho r) g(r) dr: L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+),$$

where  $a_\lambda(t) = J_\lambda(t)t^{1/2-\beta}$ .

By [1, Sect. 7.7], the Mellin transform of the function  $a_\lambda(\cdot)$  for  $0 < \beta < \lambda+1$  and  $\eta \in \mathbb{R}$  is given by

$$\begin{aligned} M a_\lambda(\eta) &= \int_0^\infty a(t) t^{-1/2-i\eta} dt = \int_0^\infty J_\lambda(t) t^{-\beta-i\eta} dt \\ &= 2^{-\beta-i\eta} \frac{\Gamma(\frac{1}{2}(\lambda+1-\beta-i\eta))}{\Gamma(\frac{1}{2}(\lambda+1+\beta+i\eta))}. \end{aligned}$$

Basic properties of the gamma function imply that  $Ma_\lambda(\eta)$  is continuous and

$$Ma_\lambda(0) = 2^{-\beta} \frac{\Gamma(\frac{1}{2}(\lambda + 1 - \beta))}{\Gamma(\frac{1}{2}(\lambda + 1 + \beta))}, \quad |Ma_\lambda(\eta)| \sim 2^{-\beta} |\eta|^{-\beta} \rightarrow 0, \quad \eta \rightarrow \infty.$$

In [19, 20] it was proved that

$$\|A_\lambda\| = \max_{\mathbb{R}} |Ma_\lambda(\eta)| = Ma_\lambda(0)$$

and that the norm is not attained; that is to say, there is no function  $g$  such that  $\|A_\lambda\| = \|A_\lambda g\|_{L^2(\mathbb{R}_+)}$  with  $\|g\|_{L^2(\mathbb{R}_+)} = 1$ .  $\square$

**Corollary 2.2.** *If  $\lambda_k = d/2 - 1 + |k|$  and  $0 \leq \beta < \lambda_k + 1$ , then for  $f \in \mathcal{S}(\mathbb{R}^d) \cap \mathcal{R}_0^d$  Pitt's inequality for the Dunkl transform (1.3) holds with sharp constant  $c(\beta, \lambda_k)$ .*

### 3. PITT'S INEQUALITY FOR DUNKL TRANSFORM

Let  $\mathbb{S}^{d-1}$  be the unit sphere in  $\mathbb{R}^d$ ,  $x' \in \mathbb{S}^{d-1}$ , and  $dx'$  be the Lebesgue measure on the sphere. Set  $a_k^{-1} = \int_{\mathbb{S}^{d-1}} v_k(x') dx'$ ,  $d\omega_k(x') = a_k v_k(x') dx'$ , and  $\|f\|_{2, d\omega_k} = (\int_{\mathbb{S}^{d-1}} |f(x')|^2 d\omega_k(x'))^{1/2}$ . We have

$$(3.1) \quad c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2} v_k(x) dx = \int_0^\infty e^{-r^2/2} r^{d-1+2|k|} dr \int_{\mathbb{S}^{d-1}} v_k(x') dx' = b_{\lambda_k}^{-1} a_k^{-1}.$$

Let us denote by  $\mathcal{H}_n^d(v_k)$  the subspace of  $k$ -spherical harmonics of degree  $n \in \mathbb{Z}_+$  in  $L^2(\mathbb{S}^{d-1}, d\omega_k)$  (see [6, Chap. 5]). Let  $\Delta_k = \sum_{j=1}^d D_j^2$  be the Dunkl Laplacian and  $\mathcal{P}_n^d$  be the space of homogeneous polynomials of degree  $n$  in  $\mathbb{R}^d$ . Then  $\mathcal{H}_n^d(v_k)$  is the restriction of  $\ker \Delta_k \cap \mathcal{P}_n^d$  to the sphere  $\mathbb{S}^{d-1}$ .

If  $l_n$  is the dimension of  $\mathcal{H}_n^d(v_k)$ , we denote by  $\{Y_n^j: j = 1, \dots, l_n\}$  the real-valued orthonormal basis  $\mathcal{H}_n^d(v_k)$  in  $L^2(\mathbb{S}^{d-1}, d\omega_k)$ . A union of these bases forms an orthonormal basis in  $L^2(\mathbb{S}^{d-1}, d\omega_k)$  consisting of  $k$ -spherical harmonics, i.e., we have

$$(3.2) \quad L^2(\mathbb{S}^{d-1}, d\omega_k) = \sum_{n=0}^\infty \oplus \mathcal{H}_n^d(v_k).$$

Using (3.2) and the following Funk-Hecke formula for  $k$ -spherical harmonic  $Y \in \mathcal{H}_n^d(v_k)$  (see [18])

$$(3.3) \quad \int_{\mathbb{S}^{d-1}} Y(y') \overline{e_k(x, y')} d\omega_k(y') = \frac{(-i)^n \Gamma(\lambda_k + 1)}{2^n \Gamma(n + \lambda_k + 1)} Y(x') r^n j_{n+\lambda_k}(r), \quad x = rx' \in \mathbb{R}^d,$$

similarly to (1.2) we have the direct sum decomposition of  $L^2(\mathbb{R}^d, d\mu_k)$ :

$$(3.4) \quad L^2(\mathbb{R}^d, d\mu_k) = \sum_{n=0}^\infty \oplus \mathcal{R}_n^d(v_k), \quad \mathcal{R}_n^d(v_k) = \mathcal{R}_0^d \otimes \mathcal{H}_n^d(v_k),$$



and that the space  $\mathcal{R}_n^d(v_k)$  is invariant under the Dunkl transform. An example of the orthogonal basis in  $L^2(\mathbb{R}^d, d\mu_k)$  consisting of eigenfunctions of the Dunkl transform was constructed in [6].

The next result provides a sharp constant in the Pitt inequality for the Dunkl transform (1.3).

**Theorem 3.1.** *Let  $\lambda_k = d/2 - 1 + |k|$  and  $0 \leq \beta < \lambda_k + 1$ , then for  $f \in \mathcal{S}(\mathbb{R}^d)$  we have*

$$C(\beta, k) = 2^{-\beta} \frac{\Gamma(\frac{1}{2}(\lambda_k + 1 - \beta))}{\Gamma(\frac{1}{2}(\lambda_k + 1 + \beta))}.$$

*Sharpness of this inequality can be seen by considering radial functions.*

*Proof.* For  $\beta = 0$  we have  $C(\beta, k) = 1$  and Pitt's inequality (1.3) becomes Plancherel's identity

$$\|F_k(f)(y)\|_{2, d\mu_k} = \|f(x)\|_{2, d\mu_k}.$$

Let  $0 < \beta < \lambda_k + 1$ . If  $f \in \mathcal{S}(\mathbb{R}^d)$ , then

$$\begin{aligned} f_{nj}(r) &= \int_{\mathbb{S}^{d-1}} f(rx') Y_n^j(x') d\omega_k(x') \in \mathcal{S}(\mathbb{R}_+), \\ f(rx') &= \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} f_{nj}(r) Y_n^j(x'), \\ \int_{\mathbb{S}^{d-1}} |f(rx')|^2 d\omega_k(x') &= \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} |f_{nj}(r)|^2. \end{aligned}$$

Using spherical coordinates, decomposition of  $L^2(\mathbb{R}^d, d\mu_k)$  (3.4), formulas (3.1) and (3.3), and the property  $e_k(tx, y) = e_k(ty, x)$ , we get that

$$\begin{aligned} (3.5) \quad \int_{\mathbb{R}^d} |x|^{2\beta} |f(x)|^2 d\mu_k(x) &= b_{\lambda_k} \int_0^{\infty} r^{2\beta+d-1+2|k|} \int_{\mathbb{S}^{d-1}} |f(rx')|^2 d\omega_k(x') dr \\ &= b_{\lambda_k} \int_0^{\infty} r^{2\beta+d-1+2|k|} \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} |f_{nj}(r)|^2 dr \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} \int_0^{\infty} |f_{nj}(r)|^2 r^{2\beta} d\nu_{\lambda_k}(r), \end{aligned}$$

$$\begin{aligned} F_k(f)(y) &= \int_{\mathbb{R}^d} f(x) \overline{e_k(x, y)} d\mu_k(x) = b_{\lambda_k} \int_0^{\infty} r^{d-1+2|k|} \int_{\mathbb{S}^{d-1}} f(rx') d\omega_k(x') dr \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} b_{\lambda_k} \int_0^{\infty} f_{nj}(r) r^{2\lambda_k+1} \int_{\mathbb{S}^{d-1}} Y_n^j(x') \overline{e_k(rx', \rho y')} d\omega_k(x') dr \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} \frac{(-i)^n \Gamma(\lambda_k + 1)}{2^n \Gamma(n + \lambda_k + 1)} Y_n^j(y') \int_0^{\infty} f_{nj}(r) j_{n+\lambda_k}(\rho r) (\rho r)^n d\nu_{\lambda_k}(r), \end{aligned}$$

and

$$(3.6) \quad \int_{\mathbb{R}^d} |y|^{-2\beta} |F_k(f)(y)|^2 d\mu_k(y) = \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} \frac{\Gamma^2(\lambda_k + 1)}{2^{2n} \Gamma^2(n + \lambda_k + 1)} \\ \times \int_0^{\infty} \left| \int_0^{\infty} f_{nj}(r) j_{n+\lambda_k}(\rho r) (\rho r)^n d\nu_{\lambda_k}(r) \right|^2 \rho^{-2\beta} d\nu_{\lambda_k}(\rho).$$

Suppose that  $g \in \mathcal{S}(\mathbb{R}_+)$ ,  $n \in \mathbb{Z}_+$ , and  $0 < \beta < \lambda_k + 1 + n$ . Let us show that

$$(3.7) \quad \left\| \rho^{-\beta} \int_0^{\infty} g(r) j_{n+\lambda_k}(\rho r) (\rho r)^n d\nu_{\lambda_k}(r) \right\|_{2, d\nu_{\lambda_k}} \\ \leq \frac{2^n \Gamma(n + \lambda_k + 1) c(\beta, \lambda_k + n)}{\Gamma(\lambda_k + 1)} \|r^\beta g(r)\|_{2, d\nu_{\lambda_k}},$$

where  $c(\beta, \lambda_k + n)$  is given in Theorem 2.1 with  $\lambda = \lambda_k + n$ . Setting in (3.7)  $g(r) = u(r)r^n$ , we rewrite it as follows:

$$(3.8) \quad \left\| \rho^{-\beta} \int_0^{\infty} u(r) j_{n+\lambda_k}(\rho r) d\nu_{\lambda_k+n}(r) \right\|_{2, d\nu_{\lambda_k+n}} \leq c(\beta, \lambda_k + n) \|r^\beta u(r)\|_{2, d\nu_{\lambda_k+n}},$$

which is (1.4) with  $\lambda = \lambda_k + n$ .

Since  $c(\beta, \lambda_k + n)$  is decreasing with  $n$  (see [19]), then using (3.5), (3.6), and (3.7), we arrive at

$$\int_{\mathbb{R}^d} |y|^{-2\beta} |F_k(f)(y)|^2 d\mu_k(y) \leq \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} c^2(\beta, \lambda_k + n) \int_0^{\infty} |f_{nj}(r)|^2 r^{2\beta} d\nu_{\lambda_k}(r) \\ \leq c^2(\beta, \lambda_k) \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} \int_0^{\infty} |f_{nj}(r)|^2 r^{2\beta} d\nu_{\lambda_k}(r) \\ = c^2(\beta, \lambda_k) \int_{\mathbb{R}^d} |x|^{2\beta} |f(x)|^2 d\mu_k(x). \quad \square$$

In the proof of Theorem 3.1 we obtained the following result (see (3.8)).

**Corollary 3.2.** *Let  $n \in \mathbb{N}$ ,  $\lambda_k = d/2 - 1 + |k|$ , and  $0 \leq \beta < \lambda_k + 1 + n$ , then for  $f \in \mathcal{S}(\mathbb{R}^d) \cap \mathcal{R}_n^d(v_k)$  we have Pitt's inequality for the Dunkl transform (1.3) with sharp constant  $c(\beta, \lambda_k + n)$ .*

For the Fourier transform Corollary 3.2 was established in [19].

## 4. LOGARITHMIC UNCERTAINTY PRINCIPLE FOR DUNKL TRANSFORM

W. Beckner in [2] proved the logarithmic uncertainty principle for the Fourier transform using Pitt's inequality (1.1): if  $f \in \mathcal{S}(\mathbb{R}^d)$ , then

$$\int_{\mathbb{R}^d} \ln(|x|) |f(x)|^2 dx + \int_{\mathbb{R}^d} \ln(|y|) |\widehat{f}(y)|^2 dy \geq \left( \psi\left(\frac{d}{4}\right) + \ln 2 \right) \int_{\mathbb{R}^d} |f(x)|^2 dx,$$

where  $\psi(t) = d \ln \Gamma(t)/dt$  is the  $\psi$ -function.

For the Hankel transform the logarithmic uncertainty principle reads as follows (see [11]): if  $f \in \mathcal{S}(\mathbb{R}_+)$  and  $\lambda \geq -1/2$ , then

$$(4.1) \quad \int_{\mathbb{R}_+} \ln(t) |f(t)|^2 t^{2\lambda+1} dt + \int_{\mathbb{R}_+} \ln(s) |H_\lambda(f)(s)|^2 s^{2\lambda+1} ds \\ \geq \left( \psi\left(\frac{\lambda+1}{2}\right) + \ln 2 \right) \int_{\mathbb{R}_+} |f(t)|^2 t^{2\lambda+1} dt.$$

For the one-dimensional Dunkl transform of functions  $f \in \mathcal{S}(\mathbb{R})$ , F. Soltani [15] has recently proved that

$$\int_{\mathbb{R}} \ln(|t|) |f(t)|^2 |t|^{2\lambda+1} dt + \int_{\mathbb{R}} \ln(|s|) |F_\lambda(f)(s)|^2 |s|^{2\lambda+1} ds \\ \geq \left( \min \left\{ \psi\left(\frac{\lambda+1}{2}\right), \psi\left(\frac{\lambda+2}{2}\right) \right\} + \ln 2 \right) \int_{\mathbb{R}} |f(t)|^2 |t|^{2\lambda+1} dt.$$

Since  $\psi$  is increasing the latter can be written as

$$(4.2) \quad \int_{\mathbb{R}} \ln(|t|) |f(t)|^2 |t|^{2\lambda+1} dt + \int_{\mathbb{R}} \ln(|s|) |F_\lambda(f)(s)|^2 |s|^{2\lambda+1} ds \\ \geq \left( \psi\left(\frac{\lambda+1}{2}\right) + \ln 2 \right) \int_{\mathbb{R}} |f(t)|^2 |t|^{2\lambda+1} dt,$$

which is the logarithmic uncertainty principle for the one-dimensional Dunkl transform.

Using Pitt's inequality (1.3) we obtain the logarithmic uncertainty principle the multi-dimensional Dunkl transform.

**Theorem 4.1.** *Let  $\lambda_k = d/2 - 1 + |k|$  and  $f \in \mathcal{S}(\mathbb{R}^d)$ . We have*

$$(4.3) \quad \int_{\mathbb{R}^d} \ln(|x|) |f(x)|^2 d\mu_k(x) + \int_{\mathbb{R}^d} \ln(|y|) |F_k(f)(y)|^2 d\mu_k(y) \\ \geq \left( \psi\left(\frac{\lambda_k+1}{2}\right) + \ln 2 \right) \int_{\mathbb{R}^d} |f(x)|^2 d\mu_k(x).$$

*Proof.* We write the Pitt inequality (1.3) in the following form

$$\int_{\mathbb{R}^d} |y|^{-\beta} |F_k(f)(y)|^2 d\mu_k(y) \leq c^2(\beta/2, \lambda_k) \int_{\mathbb{R}^d} |x|^\beta |f(x)|^2 d\mu_k(x), \quad 0 \leq \beta < 2(\lambda_k+1).$$

For  $\beta \in (-2(\lambda_k + 1), 2(\lambda_k + 1))$  define the function

$$\varphi(\beta) = \int_{\mathbb{R}^d} |y|^{-\beta} |F_k(f)(y)|^2 d\mu_k(y) - c^2(\beta/2, \lambda_k) \int_{\mathbb{R}^d} |x|^\beta |f(x)|^2 d\mu_k(x).$$

Since  $|\beta| < 2(\lambda_k + 1)$  and  $f, F_k(f) \in \mathcal{S}(\mathbb{R}^d)$ , then

$$\int_{|x| \leq 1} |\ln(|x|)| |x|^\beta v_k(x) dx = \int_0^1 |\ln(r)| r^{\beta+2\lambda_k+1} dr \int_{\mathbb{S}^{d-1}} v_k(x') dx' < \infty,$$

which gives

$$|y|^{-\beta} \ln(|y|) |F_k(f)(y)|^2 v_k(y) \in L^1(\mathbb{R}^d) \quad \text{and} \quad \ln(|x|) |x|^\beta |f(x)|^2 v_k(x) \in L^1(\mathbb{R}^d).$$

Therefore,

$$\begin{aligned} (4.4) \quad \varphi'(\beta) &= - \int_{\mathbb{R}^d} |y|^{-\beta} \ln(|y|) |F_k(f)(y)|^2 d\mu_k(y) \\ &\quad - c^2(\beta/2, \lambda_k) \int_{\mathbb{R}^d} |x|^\beta \ln(|x|) |f(x)|^2 d\mu_k(x) \\ &\quad - \frac{dc^2(\beta/2, \lambda_k)}{d\beta} \int_{\mathbb{R}^d} |x|^\beta |f(x)|^2 d\mu_k(x). \end{aligned}$$

Pitt's inequality and Plancherel's theorem imply that  $\varphi(\beta) \leq 0$  for  $\beta > 0$  and  $\varphi(0) = 0$  correspondingly, hence

$$\varphi'(0_+) = \lim_{\beta \rightarrow 0_+} \frac{\varphi(\beta) - \varphi(0)}{\beta} \leq 0.$$

Noting that

$$(4.5) \quad - \left. \frac{dc^2(\beta/2, \lambda_k)}{d\beta} \right|_{\beta=0} = \psi\left(\frac{\lambda_k + 1}{2}\right) + \ln 2,$$

we conclude that proof of (4.3) combining (4.4) and (4.5).  $\square$

For the radial functions inequality (4.3) with  $\lambda_k = \lambda$  implies inequality (4.1).

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