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PAPER

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Smoothness of functions and Fourier coefficients

M. I. Dyachenko, A. B. Mukanov and S. Yu. Tikhonov

Abstract. We consider functions represented as trigonometric series with general monotone Fourier coefficients. The main result of the paper is the equivalence of the L_p modulus of smoothness, $1 < p < \infty$, of such functions to certain sums of their Fourier coefficients. As applications, for such functions we give a description of the norm in the Besov space and sharp direct and inverse theorems in approximation theory.

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§ 1. Introduction

In this paper we are concerned with investigating interrelations between the smoothness of functions in the space L_p and the growth of their Fourier coefficients.

1.1. General conditions on the Fourier coefficients of functions in L_p .

Let f be a 2π -periodic integrable function with the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (1.1)$$

Let

$$\omega_l(f, \delta)_p := \sup_{|h| \leq \delta} \|\Delta_h^l f(\cdot)\|_p$$

denote the modulus of smoothness of order $l \geq 1$ of the function f in L_p , where

$$\Delta_h^l f(x) := \Delta_h(\Delta_h^{l-1} f(x)) \quad \text{and} \quad \Delta_h f(x) := f(x+h) - f(x).$$

First we give the simplest estimates for the modulus of smoothness of $f \in L_p$, $1 \leq p \leq \infty$, in terms of its Fourier coefficients:

$$|a_n| + |b_n| \lesssim \omega_l\left(f, \frac{1}{n}\right)_p \lesssim \frac{1}{n^l} \sum_{k=1}^n k^l (|a_k| + |b_k|) + \sum_{k=n+1}^{\infty} (|a_k| + |b_k|).$$

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The left-hand inequality is the well-known Lebesgue estimate for $p = 1$ (see [1]) and the right-hand inequality follows from the Fourier representation for $\Delta_h^l f$ in the case $p = \infty$ (see [2], (2.1), for example).

Throughout this paper, $T \lesssim S$ means that there exists $c > 0$ such that $T \leq cS$. Moreover, $T \asymp S$ means $T \lesssim S \lesssim T$.

$\text{Lip}(\alpha, p)$ will denote the Lipschitz class:

$$\text{Lip}(\alpha, p) := \{f \in L_p([0, 2\pi]) : \omega(f, \delta)_p = O(\delta^\alpha)\},$$

where $\omega(f, \delta)_p := \omega_1(f, \delta)_p$ is the L_p modulus of continuity of f . In the middle of the last century, necessary and sufficient conditions for a function f to belong to the Lipschitz class were found. In particular, Lorentz [3] showed that for $2 \leq p \leq \infty$ and $0 < \alpha < 1$ the condition

$$\sum_{k=n}^{\infty} (|a_k|^{p'} + |b_k|^{p'}) = O\left(\frac{1}{n^{\alpha p'}}\right), \quad (1.2)$$

implies that $f \in \text{Lip}(\alpha, p)$. Throughout this paper p' denotes the conjugate index of p : $p' = p/(p-1)$. Note that for any positive α condition (1.2) is equivalent to

$$\sum_{k=n}^{2n} (|a_k|^{p'} + |b_k|^{p'}) = O\left(\frac{1}{n^{\alpha p'}}\right).$$

For $1 < p \leq 2$, condition (1.2) is necessary for f to belong to $\text{Lip}(\alpha, p)$; see [4].

A more detailed picture of the relationship between the modulus of smoothness of $f \in L_p$ and its Fourier coefficients is given in the recent paper [5], Theorem 2.1.

Theorem 1. *Let (1.1) be the Fourier series of a function $f \in L_p([0, 2\pi])$.*

(A) *Let $1 < p \leq 2$. Then, for $p \leq q \leq p'$,*

$$\begin{aligned} & \frac{1}{n^l} \left(\sum_{k=1}^n k^{(l+1-1/p-1/q)q} (|a_k|^q + |b_k|^q) \right)^{1/q} \\ & + \left(\sum_{k=n+1}^{\infty} k^{(1-1/p-1/q)q} (|a_k|^q + |b_k|^q) \right)^{1/q} \lesssim \omega_l\left(f, \frac{1}{n}\right)_p. \end{aligned} \quad (1.3)$$

(B) *Let $2 \leq p < \infty$ and let $(\sum_{n=1}^{\infty} n^{(1-1/p-1/q)q} (|a_n|^q + |b_n|^q))^{1/q} < \infty$, where $p' \leq q \leq p$. Then*

$$\begin{aligned} & \frac{1}{n^l} \left(\sum_{k=1}^n k^{(l+1-1/p-1/q)q} (|a_k|^q + |b_k|^q) \right)^{1/q} \\ & + \left(\sum_{k=n+1}^{\infty} k^{(1-1/p-1/q)q} (|a_k|^q + |b_k|^q) \right)^{1/q} \gtrsim \omega_l\left(f, \frac{1}{n}\right)_p. \end{aligned} \quad (1.4)$$

Note that this theorem is sharp with respect to the conditions on p and q . Moreover, for $0 < \alpha < l$ Theorem 1 implies the following fact. For the condition $\omega_l(f, \delta)_p = O(\delta^\alpha)$ to hold it is necessary (when $p \leq 2$) and sufficient (when $p \geq 2$)

that (1.2) holds. In other words, in this case the choice of the parameter $q = p'$ is the best possible. However, in some cases this is no longer true. For instance, if $|a_n| = |b_n| \asymp n^{-l-1/p'} \ln^{-1/p} n$ then for $q = p$ Theorem 1 gives

$$\omega_l\left(f, \frac{1}{n}\right)_p \lesssim \frac{(\ln \ln n)^{1/p}}{n^l}$$

in the case $p \geq 2$ and the reverse inequality in the case $p \leq 2$. For other values of q the estimates are weaker.

1.2. The behaviour of the Fourier coefficients of functions in L_p under additional conditions. Imposing some additional conditions (monotonicity, general monotonicity) on the coefficients of the series (1.1) it is possible to describe the interrelation between the smoothness of a function and the behaviour of its Fourier coefficients more fully. In particular, Konyushkov [6] showed that for functions with monotone Fourier coefficients condition (1.2) is equivalent to the condition $f \in \text{Lip}(\alpha, p)$.

Theorem 2. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Also let (1.1) be the Fourier series of a function $f \in L_p([0, 2\pi])$, and let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be nonincreasing sequences. Then the following conditions are equivalent:*

- (i) $f \in \text{Lip}(\alpha, p)$;
- (ii) $|a_n|, |b_n| = O(n^{1/p-\alpha-1})$;
- (iii) $\sum_{k=n}^\infty (|a_k|^{p'} + |b_k|^{p'}) = O(n^{-\alpha p'})$.

It was later shown that the behaviour of the modulus of smoothness can be characterized in terms of the Fourier coefficients of the function.

Theorem 3 (see [7] and [8]). *Under the conditions of Theorem 2 the equivalence*

$$\omega_l\left(f, \frac{1}{n}\right)_p \asymp \frac{1}{n^l} \left(\sum_{k=1}^n k^{(l+1)p-2} (a_k^p + b_k^p) \right)^{1/p} + \left(\sum_{k=n+1}^\infty k^{p-2} (a_k^p + b_k^p) \right)^{1/p}$$

holds.

Similar problems were considered in [9] and [10]. These results are very important in characterizing certain smooth function spaces (see, for example, [2], [5], [8] and [11]–[17]). In particular, Askey [10] proved the following result.

Theorem 4. *Let $0 < \alpha < 2$, $1 < p < \infty$ and $1 \leq q \leq \infty$. Also let $\sum_{n=1}^\infty a_n \cos nx$ be the Fourier series of a function $f \in L_p([0, 2\pi])$ and $\{a_n\}_{n=1}^\infty$ a nonincreasing sequence. Then*

$$\left(\int_0^\pi \left[\int_0^\pi \left| \frac{f(x+t) - 2f(x) + f(x-t)}{t^\alpha} \right|^p dx \right]^{q/p} \frac{dt}{t} \right)^{1/q} < \infty \quad (1.5)$$

if and only if

$$\left(\sum_{n=1}^\infty a_n^q n^{q(\alpha+1-1/p)-1} \right)^{1/q} < \infty.$$

Note that (1.5) is equivalent to the condition $f \in B_{p,q}^\alpha$, where the Besov space $B_{p,q}^\alpha$ is defined as follows.

Definition 1. Let $1 \leq p \leq \infty$ and $\tau, r > 0$. The Besov space $B_{p,\tau}^r([0, 2\pi])$ is the set of functions $f \in L_p([0, 2\pi])$ such that

$$\|f\|_{B_{p,\tau}^r} := \|f\|_{L_p} + \left(\int_0^1 \left(\frac{\omega_l(f, t)_p}{t^r} \right)^\tau \frac{dt}{t} \right)^{1/\tau} < \infty,$$

where $l > r$.

Note that Theorems 3 and 4 have been generalized in various papers (see [5], [8], [14], [15] and [18]–[23]) by weakening the monotonicity condition on the Fourier coefficients.

We consider the following class of general monotone sequences (see [14] and [15]).

Definition 2. We say that a sequence of real numbers $\{a_n\}_{n=1}^\infty$ is *general monotone* and write $\{a_n\}_{n=1}^\infty \in \text{GM}$ if there exist constants $C > 0$ and $\gamma > 1$ such that, for any natural n ,

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \leq C \sum_{k=n/\gamma}^{\gamma n} \frac{|a_k|}{k}.$$

It is easy to show that monotone (or more generally, quasimonotone) sequences belong to the class GM. More detail about the various subclasses of GM can be found in [15], [24] and [25].

The main aim of this paper is to prove a generalization of Theorem 3 to the case of sequences in GM. Note that the main difficulty in the proof is that the general monotone sequences under consideration are not necessarily positive. This allows us to extend the class of functions under consideration significantly.

Without loss of generality in the definition of the class GM we can assume that $\lambda = 2^\nu$, where ν is a natural number.

The main result in this paper is the following theorem.

Theorem 5. Let $f(x) \in L_p([0, 2\pi])$, $1 < p < \infty$, and let

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \text{GM}$. Then, for $l \in \mathbb{N}$,

$$\omega_l\left(f, \frac{1}{n}\right)_p \asymp \frac{1}{n^l} \left(\sum_{k=1}^n k^{lp+p-2} (|a_k|^p + |b_k|^p) \right)^{1/p} + \left(\sum_{k=n}^{\infty} k^{p-2} (|a_k|^p + |b_k|^p) \right)^{1/p}. \quad (1.6)$$

Remark 1. It is sufficient to prove relation (1.6) for a cosine series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx.$$

Moreover, all the auxiliary lemmas will also be proved for cosine Fourier series. The key estimates used in proving Theorem 5 are stated below in Lemmas 5 and 6. The proof of Theorem 5 in the case of nonnegative general monotone sequences follows from [5] and [14].

Remark 2. All the constants that appear in the proof of Theorem 5 depend only on p , l , C and ν .

The question of extending the conditions guaranteeing the validity of Theorem 5 arises in a natural way. Note that in the case when $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ belong to the wider class of weak monotone sequences the statement of Theorem 5 is not true. We recall the definition of the class WM (see [26], and also [27] and [28]).

Definition 3. We say that a sequence of real numbers $\{a_n\}_{n=1}^\infty$ is *weak monotone* (and write $\{a_n\}_{n=1}^\infty \in \text{WM}$) if there exist $C > 0$ and $\gamma > 1$ such that, for any natural number $n \in \mathbb{N}$,

$$|a_n| \leq C \sum_{k=n/\gamma}^{\infty} \frac{|a_k|}{k}.$$

As we have already noted, $\text{GM} \subsetneq \text{WM}$. The following result demonstrates, in particular, that (1.6) does not hold for weak monotone sequences, and Theorem 1 provides the best estimates in this case.

Theorem 6. Let $l \in \mathbb{N}$.

(A) If $p > 2$, then there exists a continuous function

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \text{WM},$$

such that (1.3) does not hold for any $q > 0$.

(B) If $1 < p < 2$, then there exists a continuous function

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad \{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \text{WM},$$

such that inequality (1.4) does not hold for any $q > 0$.

The paper is organised as follows. Section 2 provides us with some useful properties of sequences in the class GM. In §§3 and 4 we give upper estimates for the sums $\frac{1}{n^l} \sum_{k=1}^n |a_k|^p k^{(l+1)p-2}$ and $\sum_{k=n}^{\infty} |a_k|^p k^{(l+1)p-2}$, respectively, which we use in the proof of Theorem 5. In §5 we prove Theorems 5 and 6. In §6 we give some corollaries of Theorem 5 and, in particular, some generalizations of Theorems 2 and 4.

Note that the results of this paper are valid for moduli of smoothness of fractional order ($l > 0$).

§ 2. Auxiliary results for general monotone sequences

Recall that $\nu \in \mathbb{N}$ is such that $\gamma = 2^\nu$, where γ is the number in the definition of the class GM. Given a sequence $\{a_n\}_{n=1}^\infty$ we set

$$\begin{aligned} A_n &:= \max_{2^n \leq k \leq 2^{n+1}} |a_k|, & B_n &:= \max_{2^{n-2\nu} \leq k \leq 2^{n+2\nu}} |a_k|, \\ M_n &:= \left\{ k \in [2^{n-\nu}, 2^{n+\nu}] : |a_k| > \frac{A_n}{8C2^{2\nu}} \right\}, \\ M_n^+ &:= \{k \in M_n : a_k > 0\} \quad \text{and} \quad M_n^- := M_n \setminus M_n^+. \end{aligned}$$

The following concept was introduced in [26] in the case $l = 1$.

Definition 4. Let $\{a_n\}_{n=1}^\infty \in \text{GM}$. We say that a nonnegative integer n is *good* if either $n \leq 2\nu$ or

$$B_n \leq 2^{4\nu} A_n.$$

The other nonnegative integers are called *bad*.

The following result is proved similarly to Lemma 2.2 in [26].

Lemma 1 (see [26], Lemma 2.2). *Let the infinitesimal sequence $\{a_n\}_{n=1}^\infty$ belong to the class GM. Let $N_0 := [\log_2(C^3 2^{10\nu+8})] + 1$. Then for any good $n \geq N_0$ there exists a segment of positive integers $[l_n, m_n] \subseteq [2^{n-\nu}, 2^{n+\nu}]$ such that one of the following two conditions holds:*

(1) *for any $k \in [l_n, m_n]$, $a_k \geq 0$ and*

$$|M_n^+ \cap [l_n, m_n]| \geq \frac{2^n}{C^3 2^{(12l+6)\nu+8}};$$

(2) *for any $k \in [l_n, m_n]$, $a_k \leq 0$ and*

$$|M_n^- \cap [l_n, m_n]| \geq \frac{2^n}{C^3 2^{(12l+6)\nu+8}}.$$

Set

$$I_s(x) := \sum_{k=2^s}^{2^{s+1}-1} a_k \cos kx.$$

Then

$$I_s^{(l)}(x) = \sum_{k=2^s}^{2^{s+1}-1} k^l a_k \cos\left(kx + \frac{l\pi}{2}\right).$$

Lemma 2. *Let $\{a_n\}_{n=1}^\infty \in \text{GM}$ and let $n \geq N_0$ be a good number, where N_0 is defined in Lemma 1. Then*

$$\left\| \sum_{s=n-\nu}^{n+\nu} I_s(\cdot) \right\|_p^p \gtrsim \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{p-2} |a_k|^p.$$

Proof. Let $n \geq N_0$ be a good number. Without loss of generality we can assume that condition (1) in Lemma 1 holds, and we consider the sum

$$Q_n(x) := \sum_{k=l_n}^{m_n} a_k \cos kx.$$

Note that for any $0 \leq x \leq 1/2^{n+\nu}$ all the terms of $Q_n(x)$ are nonnegative. Using the inequality $\cos t \geq 3t/(2\pi)$ for $t \in [0, \pi/3]$ together with Lemma 1, for any

$$0 \leq x \leq \frac{\pi}{3} \frac{1}{2^{n+\nu}}$$

we obtain

$$\begin{aligned} Q_n(x) &= \sum_{k=l_n}^{m_n} a_k \cos kx \geq \frac{3}{2\pi} x \sum_{k=l_n}^{m_n} a_k k \\ &\geq \frac{3}{2\pi} x \sum_{k \in [l_n, m_n] \cap M_n^+} a_k k \geq \frac{3}{2\pi} x 2^{(n-\nu)} \frac{A_n}{8C2^{2\nu}} \frac{2^n}{C^3 2^{(12l+6)\nu+8}} \gtrsim 2^{2n} A_n x. \end{aligned}$$

Using the last inequality and the fact that $\|S_M(f, \cdot) - S_N(f, \cdot)\|_p \lesssim \|f\|_p$, we derive

$$\begin{aligned} \left\| \sum_{s=n-\nu}^{n+\nu} I_s(\cdot) \right\|_p^p &\gtrsim \|Q_n(\cdot)\|_p^p \geq \int_0^{\frac{\pi}{3} \frac{1}{2^{n+\nu}}} Q_n^p(x) dx \\ &\gtrsim 2^{2np} A_n^p \int_0^{\frac{\pi}{3} \frac{1}{2^{n+\nu}}} x^p dx \gtrsim 2^{(p-1)n} A_n^p \gtrsim \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{p-2} |a_k|^p. \end{aligned}$$

Similarly to Lemma 2 we can obtain the following result.

Lemma 3. Let $n \geq N_0$ be a good number, where N_0 is defined in Lemma 1. Then

$$\left\| \sum_{s=n-\nu}^{n+\nu} I_s^{(l)}(\cdot) \right\|_p^p \gtrsim \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{(l+1)p-2} |a_k|^p.$$

Remark 3. Let the infinitesimal sequence $\{a_n\}_{n=1}^\infty$ belong to the class GM. Then for any bad number $r \in \mathbb{N}$ either there exists a set of integers

$$r = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_s =: \xi_{r,s} \quad (2.1)$$

or

$$r = \xi_0 > \xi_1 > \xi_2 > \dots > \xi_s =: \xi_{r,s} \quad (2.2)$$

such that $\xi_1, \xi_2, \dots, \xi_{s-1}$ are bad, $\xi_{r,s}$ is good and

$$\begin{aligned} A_r &< 2^{-4l\nu} A_{\xi_1} < 2^{-8l\nu} A_{\xi_2} < \dots < 2^{-4ls\nu} A_{\xi_{r,s}}, \\ |\xi_i - \xi_{i+1}| &\leq 2\nu, \quad i = 0, \dots, s-1. \end{aligned}$$

The sets (2.1) and (2.2) are uniquely constructed for any bad number r . In the sequel, we say that the sets (2.1) and (2.2) are *increasing* and *decreasing chains* of the number r , respectively, and we call s the *length of the bad number* r . Moreover, in this case we say that the bad number r *transforms* into the good number $\xi_{r,s}$.

In fact, let r be a bad number. Then $A_r < 2^{-4l\nu} B_r$. Note that there exists an integer ξ such that $B_r = A_\xi$ and $-2\nu \leq \xi - r \leq 2\nu - 1$. Set

$$\xi_1 := \min\{\xi : -2\nu \leq \xi - r \leq 2\nu - 1, B_r = A_\xi\}.$$

Assume first that $\xi_1 < r$. Then either ξ_1 is a good number or there exists an integer ξ such that $-2\nu \leq \xi - \xi_1 < 2\nu - 1$ and $A_{\xi_1} < 2^{-4l\nu} B_{\xi_1} = 2^{-4l\nu} A_\xi$. Set

$$\xi_2 := \min\{\xi : -2\nu \leq \xi - \xi_1 < 2\nu - 1, B_{\xi_1} = A_\xi\}.$$

Since $\xi_1 < r$, it follows that

$$[2^{\xi_1}, 2^{\xi_1+2\nu}] \subset [2^{r-2\nu}, 2^{r+2\nu}].$$

Therefore, for any $k \in [2^{\xi_1}, 2^{\xi_1+2\nu}] \cap \mathbb{Z}$ we have $|a_k| \leq A_{\xi_1} = B_r$. Hence ξ_2 cannot be greater¹ than ξ_1 , that is, $\xi_2 < \xi_1$.

Continuing this procedure, we arrive at a finite sequence (since $\{\xi_j\}$ is a decreasing sequence)

$$r = \xi_0 > \xi_1 > \cdots > \xi_{s-1} > \xi_s,$$

where the numbers $\xi_0, \xi_1, \dots, \xi_{s-1}$ are bad, and ξ_s is good. Moreover, $\xi_j - \xi_{j+1} \leq 2\nu$ and $A_{\xi_j} < 2^{-4l\nu} A_{\xi_{j+1}}$ for any $0 \leq j \leq s-1$.

Now let $\xi_1 > r$. Then, either ξ_1 is a good number or there exists $\xi_2 > \xi_1$ such that $\xi_2 - \xi_1 \leq 2\nu - 1$ and $A_{\xi_1} < 2^{-4l\nu} A_{\xi_2}$. Continuing this procedure and taking the fact that the sequence $\{a_n\}_{n=1}^\infty$ converges to 0 into account, we obtain a finite sequence of numbers

$$r = \xi_0 < \xi_1 < \cdots < \xi_{s-1} < \xi_s,$$

where $\xi_0, \xi_1, \dots, \xi_{s-1}$ are bad, and ξ_s is good. Moreover, for any $0 \leq j \leq s-1$ the inequalities $\xi_{j+1} - \xi_j \leq 2\nu$ and $A_{\xi_j} < 2^{-4l\nu} A_{\xi_{j+1}}$ hold.

Thus we have proved the statement in Remark 3.

We set

$$\begin{aligned} P_n &:= \sum_{k=2^n}^{2^{n+1}-1} k^{p-2} |a_k|^p, & P_{n,\nu} &:= \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{p-2} |a_k|^p, \\ \tilde{P}_n &:= \sum_{k=2^n}^{2^{n+1}-1} k^{(l+1)p-2} |a_k|^p, & \tilde{P}_{n,\nu} &:= \sum_{k=2^{n-\nu}}^{2^{n+\nu}} k^{(l+1)p-2} |a_k|^p. \end{aligned}$$

Lemma 4. *Let $n \geq N_0$ be a good number and let R_n be the set of bad numbers transforming into n . Then*

$$\sum_{r \in R_n} P_r \lesssim P_{n,\nu}. \quad (2.3)$$

Moreover, if A is a subset of the set of good numbers such that $\min_{n \in A} n \geq N_0$ and B is the set of bad numbers transforming only into good numbers in A , then

$$\sum_{r \in B} P_r \lesssim \sum_{m \in A} P_{m,\nu}. \quad (2.4)$$

¹If $\xi_2 > \xi_1$, then $[2^{\xi_2}, 2^{\xi_2+1}] \cap \mathbb{Z} \subset [2^{\xi_1}, 2^{\xi_1+2\nu}] \cap \mathbb{Z}$. On the other hand, we have $A_{\xi_2} > A_{\xi_1} = B_r$, which leads to a contradiction.

The expressions P_r and $P_{m,\nu}$ in the estimates (2.3) and (2.4) can be replaced by \tilde{P}_r and $\tilde{P}_{m,\nu}$, respectively.

Proof. First we prove (2.3). We divide the set R_n into two disjoint subsets:

$$R_n = Q_n^1 \sqcup Q_n^2,$$

where Q_n^1 is the set of bad numbers which transform into n via a decreasing chain, and Q_n^2 is the set of bad numbers which transform into n via an increasing chain.

Consider the set Q_n^1 . Let $r \in Q_n^1$ be a bad number of length s . Then, by Remark 3, $r \leq n + 2s\nu$ and $A_r \leq 2^{-4ls\nu} A_n$. Therefore,

$$\begin{aligned} P_r &= \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \\ &\leq 2^{-4ls\nu p} A_n^p 2^{(n+2s\nu)(p-1)} \lesssim 2^{-2ps\nu} A_n^p 2^{n(p-1)}. \end{aligned} \quad (2.5)$$

On the other hand, by Lemma 1,

$$\begin{aligned} P_{n,\nu} &= \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p k^{p-2} \geq \sum_{k=l_n}^{m_n} |a_k|^p k^{p-2} \geq \sum_{k \in [l_n, m_n] \cap M_n} |a_k|^p k^{p-2} \\ &\gtrsim A_n^p \sum_{k \in [l_n, m_n] \cap M_n} k^{p-2} \gtrsim A_n^p 2^{n(p-2)} |[l_n, m_n] \cap M_n| \gtrsim A_n^p 2^{n(p-1)}. \end{aligned} \quad (2.6)$$

Combining (2.5) and (2.6), for the bad number $r \in Q_n^1$ of length $s = s_r$ we obtain

$$P_r \lesssim 2^{-2ps_r\nu} P_{n,\nu}.$$

Note that the set Q_n^1 consists of at most 2ν bad numbers of length 1, at most $(2\nu)^2$ bad numbers of length 2, and so on. Hence

$$\sum_{r \in Q_n^1} P_r \lesssim P_{n,\nu} \sum_{r \in Q_n^1} 2^{-2ps_r\nu} \leq P_{n,\nu} \sum_{s=1}^{\infty} (2\nu)^s 2^{-2ps\nu} \lesssim P_{n,\nu}, \quad (2.7)$$

since $\sum_{s=1}^{\infty} (2\nu)^s 2^{-2ps\nu} < \infty$.

Similarly, for Q_n^2 we have

$$\sum_{r \in Q_n^2} P_r \lesssim P_{n,\nu}. \quad (2.8)$$

From (2.7) and (2.8) we obtain

$$\sum_{r \in R_n} P_r = \sum_{r \in Q_n^1} P_r + \sum_{r \in Q_n^2} P_r \lesssim P_{n,\nu}.$$

Second, we establish (2.4). Enumerating the elements of the set A we have $A = \{m_1, m_2, \dots, m_{|A|}\}$, where $|A|$ denotes the cardinality² of the set A . We divide the set B into the following disjoint sets:

$$B = R_{m_1} \sqcup R_{m_2} \sqcup \dots \sqcup R_{m_{|A|}},$$

²We allow cases when $|A| = \infty$.

where R_{m_i} is the subset of bad numbers in B which transform into m_i , $i = 1, 2, \dots, |A|$. Then it follows from (2.3) that

$$\sum_{r \in B} P_r = \sum_{i=1}^{|A|} \sum_{r \in R_{m_i}} P_r \lesssim \sum_{i=1}^{|A|} P_{m_i, \nu} = \sum_{m \in A} P_{m, \nu}.$$

The lemma is proved.

§ 3. An upper estimate for the sum $n^{-lp} \sum_{k=1}^n |a_k|^p k^{(l+1)p-2}$

Lemma 5. Let $p > 2$, let $f(x) \in L_p([0, 2\pi])$, where

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx$$

and let $\{a_n\}_{n=1}^{\infty} \in \text{GM}$. Then

$$\frac{1}{n^{lp}} \sum_{k=1}^n |a_k|^p k^{(l+1)p-2} \lesssim \frac{1}{n^{lp}} \|S_{2^{\nu+2}n}^{(l)}\|_p^p + \|f - S_{[2^{-\nu-2}n]}\|_p^p. \quad (3.1)$$

Proof. Choose N such that $2^{N-1} \leq n < 2^N$. Then

$$\begin{aligned} \frac{1}{n^{lp}} \sum_{k=1}^n |a_k|^p k^{(l+1)p-2} &\leq \frac{1}{2^{(N-1)lp}} \sum_{k=1}^{2^N-1} |a_k|^p k^{(l+1)p-2} \\ &\lesssim \frac{1}{2^{Nlp}} \sum_{r=0}^N \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{(l+1)p-2} = \frac{1}{2^{Nlp}} \sum_{r=0}^N \tilde{P}_r. \end{aligned} \quad (3.2)$$

A. First let $N > N_0$. We subdivide the interval $[0, N] \cap \mathbb{Z}$ into six parts:

$$[0, N] \cap \mathbb{Z} = ([0, N_0] \cap \mathbb{Z}) \sqcup G_N \sqcup B_N^1 \sqcup B_N^2 \sqcup B_N^3 \sqcup B_N^4,$$

where

- 1) $G_N := G \cap [N_0, N]$ is the set of good numbers $r \in [N_0, N]$;
- 2) B_N^1 is the set of bad numbers $r \in [N_0, N]$ with increasing chain such that $\xi_{r,s} \leq N$ (see the definition of $\xi_{r,s}$ in Remark 3), so that the following inequality holds:

$$N_0 \leq r < \xi_{r,s} \leq N;$$

- 3) B_N^2 is the set of bad numbers $r \in [N_0, N]$ with increasing chain such that $\xi_{r,s} > N$, so that the following inequality holds:

$$N_0 \leq r \leq N < \xi_{r,s};$$

- 4) B_N^3 is the set of bad numbers $r \in [N_0, N]$ with decreasing chain such that $\xi_{r,s} \geq N_0$, so that the following inequality holds:

$$N_0 \leq \xi_{r,s} < r \leq N;$$

- 5) B_N^4 is the set of bad numbers $r \in [N_0, N]$ with decreasing chain such that $\xi_{r,s} < N_0$, so that the following inequality holds:

$$\xi_{r,s} < N_0 \leq r \leq N.$$

Therefore,

$$\begin{aligned} \frac{1}{2^{Nlp}} \sum_{r=0}^N \tilde{P}_r &= \frac{1}{2^{Nlp}} \sum_{r \in [0, N_0-1] \cup B_N^4} \tilde{P}_r + \frac{1}{2^{Nlp}} \sum_{r \in G_N \cup B_N^1 \cup B_N^3} \tilde{P}_r \\ &\quad + \frac{1}{2^{Nlp}} \sum_{r \in B_N^2} \tilde{P}_r =: J_1 + J_2 + J_3. \end{aligned} \quad (3.3)$$

Step 1. An estimate for J_1 . Let $r \in B_N^4$; then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}}$$

and

$$\xi_{r,s} \leq N_0 \leq r \leq \xi_{r,s} + 2s\nu.$$

Then

$$\begin{aligned} \tilde{P}_r &= \sum_{k=2^r}^{2^{r+1}-1} k^{(l+1)p-2} |a_k|^p \lesssim A_r^p 2^{r((l+1)p-1)} \\ &\leq 2^{-4ls\nu p} A_{\xi_{r,s}}^p 2^{(\xi_{r,s}+2s\nu)((l+1)p-1)} \leq 2^{-2s\nu} A_{\xi_{r,s}}^p 2^{\xi_{r,s}((l+1)p-1)} \\ &< 2^{-2s\nu} A_{\xi_{r,s}}^p 2^{N_0((l+1)p-1)} \lesssim 2^{-2s\nu} A_{\xi_{r,s}}^p. \end{aligned}$$

Repeating the arguments in the proof of Lemma 4, since

$$\sum_{s=1}^{\infty} (2\nu)^s 2^{-2s\nu} < \infty,$$

we obtain

$$\sum_{r \in B_N^4} \tilde{P}_r \lesssim \sum_{\substack{\xi \in G \\ \xi \leq N_0}} A_{\xi}^p \leq N_0 \max_{1 \leq k \leq 2^{N_0+1}} |a_k|^p \leq N_0 \max_{1 \leq k \leq 2^{N_0+1}} |k^l a_k|^p. \quad (3.4)$$

Now consider $r \in [0, N_0 - 1]$. It is easy to see that

$$\tilde{P}_r = \sum_{k=2^r}^{2^{r+1}-1} k^{(l+1)p-2} |a_k|^p \leq \max_{1 \leq k \leq 2^{N_0+1}} |k^l a_k|^p \sum_{k=1}^{2^{N_0}-1} k^{p-2} \lesssim \max_{1 \leq k \leq 2^{N_0+1}} |k^l a_k|^p.$$

Therefore,

$$\sum_{r=0}^{N_0-1} \tilde{P}_r \lesssim \max_{1 \leq k \leq 2^{N_0+1}} |k^l a_k|^p. \quad (3.5)$$

Note that for any $k \leq 2^{N_0+1}$ the expression $|k^l a_k|$ is the absolute value of the k th Fourier coefficient of the function $S_{2^{\nu+2n}}^{(l)}(x)$. Using (3.4), (3.5) and Hölder's inequality we obtain

$$J_1 = \frac{1}{2^{Nlp}} \sum_{r \in [0, N_0-1] \sqcup B_N^4} \tilde{P}_r \lesssim \frac{1}{2^{Nlp}} \max_{1 \leq k \leq 2^{N_0+1}} |k^l a_k|^p \lesssim \frac{1}{n^{lp}} \|S_{2^{\nu+2n}}^{(l)}\|_p^p. \quad (3.6)$$

Step 2. An estimate for J_2 . Since all the bad numbers $r \in B_N^1 \sqcup B_N^3$ only transform into good numbers $m \in [N_0, N]$, from Lemma 4 we obtain

$$J_2 = \frac{1}{2^{Nlp}} \sum_{r \in G_N} \tilde{P}_r + \frac{1}{2^{Nlp}} \sum_{r \in B_N^1 \sqcup B_N^3} \tilde{P}_r \lesssim \frac{1}{2^{Nlp}} \sum_{m \in G_N} \tilde{P}_{m,\nu}.$$

Using Lemma 3 we derive

$$J_2 \lesssim \frac{1}{2^{Nlp}} \sum_{m \in G_N} \tilde{P}_{m,\nu} \lesssim \frac{1}{2^{Nlp}} \sum_{m \in G_N} \left\| \sum_{s=m-\nu}^{m+\nu} I_s^{(l)}(\cdot) \right\|_p^p \leq \frac{1}{2^{Nlp}} \sum_{m=\nu}^N \left\| \sum_{s=m-\nu}^{m+\nu} I_s^{(l)}(\cdot) \right\|_p^p. \quad (3.7)$$

Step 3. An estimate for J_3 . Let $r \in B_N^2$; then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}} \quad (3.8)$$

and

$$r < \xi_{r,s} \leq r + 2s\nu \leq N + 2s\nu. \quad (3.9)$$

Then (3.8) and $r < \xi_{r,s}$ imply that

$$\begin{aligned} \frac{1}{2^{Nlp}} \tilde{P}_r &= \frac{1}{2^{Nlp}} \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{(l+1)p-2} \lesssim \frac{1}{2^{Nlp}} A_r^p 2^{r((l+1)p-1)} \\ &\leq \frac{1}{2^{Nlp}} 2^{-4ls\nu p} A_{\xi_{r,s}}^p 2^{\xi_{r,s}((l+1)p-1)}. \end{aligned} \quad (3.10)$$

To be definite, assume that condition (1) in Lemma 1 is satisfied. Using Lemma 1, from inequality (3.10) we obtain

$$\begin{aligned} \frac{1}{2^{Nlp}} \tilde{P}_r &\leq \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}((l+1)p-1)} A_{\xi_{r,s}}^p \\ &\leq \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}((l+1)p-1)} \frac{(8C2^{2\nu})^p}{|[l_{\xi_{r,s}}, m_{\xi_{r,s}}] \cap M_{\xi_{r,s}}^+|} \sum_{k \in [l_{\xi_{r,s}}, m_{\xi_{r,s}}] \cap M_{\xi_{r,s}}^+} |a_k|^p \\ &\lesssim \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}((l+1)p-1)} \frac{1}{2^{\xi_{r,s}}} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p \\ &= \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}lp} 2^{\xi_{r,s}(p-2)} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p \\ &\lesssim \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}lp} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2}. \end{aligned} \quad (3.11)$$

Combining inequality (3.11) with the inequality $\xi_{r,s} \leq N + 2s\nu$ we derive that

$$\begin{aligned} \frac{1}{2^{Nlp}} \tilde{P}_r &\lesssim \frac{1}{2^{Nlp}} 2^{-4ls\nu p} 2^{\xi_{r,s}lp} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2} \\ &= 2^{(\xi_{r,s}-N-2s\nu)lp} 2^{-2ls\nu p} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2} \\ &\leq 2^{-2ls\nu p} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{p-2} = 2^{-2ls\nu p} P_{\xi_{r,s},\nu}. \end{aligned}$$

Thus, for any bad number $r \in B_N^2$ we have

$$\frac{1}{2^{Nlp}} \tilde{P}_r \lesssim 2^{-2ls\nu p} P_{\xi_{r,s},\nu}.$$

Since $\sum_{s=1}^{\infty} (2\nu)^s 2^{-2ls\nu p} < \infty$, using similar arguments to those in the proof of Lemma 4 we obtain

$$\sum_{r \in B_N^2} \frac{1}{2^{Nlp}} \tilde{P}_r \lesssim \sum_{\substack{\xi \in G \\ \xi > N}} P_{\xi,\nu}.$$

Hence, by Lemma 2,

$$J_3 = \sum_{r \in B_N^2} \frac{1}{2^{Nlp}} \tilde{P}_r \lesssim \sum_{\substack{\xi \in G \\ \xi > N}} P_{\xi,\nu} \lesssim \sum_{\substack{\xi \in G \\ \xi > N}} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k(\cdot) \right\|_p^p \leq \sum_{\xi=N+1}^{\infty} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k(\cdot) \right\|_p^p. \quad (3.12)$$

Now we prove an estimate for the sum $\frac{1}{2^{Nlp}} \sum_{r=0}^N \tilde{P}_r$. Applying (3.6), (3.7) and (3.12) we write

$$\begin{aligned} \frac{1}{2^{Nlp}} \sum_{r=0}^N \tilde{P}_r &= J_1 + J_2 + J_3 \\ &\lesssim \frac{1}{2^{Nlp}} \sum_{m=\nu}^N \left\| \sum_{k=m-\nu}^{m+\nu} I_k^{(l)}(\cdot) \right\|_p^p + \sum_{\xi=N+1}^{\infty} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k(\cdot) \right\|_p^p + \frac{1}{n^{lp}} \|S_{2^\nu n}^{(l)}\|_p^p. \end{aligned} \quad (3.13)$$

Using Jensen's inequality and the Littlewood-Paley theorem, we obtain

$$\begin{aligned} \sum_{m=\nu}^N \left\| \sum_{k=m-\nu}^{m+\nu} I_k^{(l)}(\cdot) \right\|_p^p &= \int_0^{2\pi} \sum_{m=\nu}^N \left| \sum_{k=m-\nu}^{m+\nu} I_k^{(l)}(x) \right|^p dx \\ &\leq \int_0^{2\pi} \left(\sum_{m=\nu}^N \left| \sum_{k=m-\nu}^{m+\nu} I_k^{(l)}(x) \right|^2 \right)^{p/2} dx \leq \int_0^{2\pi} \left((2\nu+1)^2 \sum_{k=0}^{N+\nu} |I_k^{(l)}(x)|^2 \right)^{p/2} dx \\ &\lesssim \left\| \left(\sum_{k=0}^{N+\nu} |I_k^{(l)}(\cdot)|^2 \right)^{1/2} \right\|_p^p \lesssim \|S_{2^{\nu+2}n}^{(l)}\|_p^p. \end{aligned} \quad (3.14)$$

In a similar way,

$$\begin{aligned}
 \sum_{\xi=N+1}^{\infty} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k(\cdot) \right\|_p^p &= \int_0^{2\pi} \sum_{\xi=N+1}^{\infty} \left| \sum_{k=\xi-\nu}^{\xi+\nu} I_k(x) \right|^p dx \\
 &\leq \int_0^{2\pi} \left(\sum_{\xi=N+1}^{\infty} \left| \sum_{k=\xi-\nu}^{\xi+\nu} I_k(x) \right|^2 \right)^{p/2} dx \\
 &\lesssim \left\| \left(\sum_{k=N+1-\nu}^{\infty} |I_k(\cdot)|^2 \right)^{1/2} \right\|_p^p \lesssim \|f - S_{[2^{-\nu-2}n]}\|_p^p. \quad (3.15)
 \end{aligned}$$

Combining (3.13)–(3.15) we complete the proof of Lemma 5 in the case when $N > N_0$.

B. Suppose now that $N \leq N_0$. Then

$$\begin{aligned}
 \frac{1}{2^{Nlp}} \sum_{r=0}^N \tilde{P}_r &= \frac{1}{2^{Nlp}} \sum_{r=0}^N \sum_{k=2^r}^{2^{r+1}-1} k^{(l+1)p-2} |a_k|^p \\
 &\lesssim \frac{1}{2^{Nlp}} \max_{1 \leq k \leq 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{n^{lp}} \|S_{2^{\nu+2}n}^{(l)}\|_p^p.
 \end{aligned}$$

Lemma 5 is proved.

§ 4. An upper estimate for the sum $\sum_{k=n}^{\infty} |a_k|^p k^{p-2}$

Lemma 6. Let $p > 2$, let $f(x) \in L_p([0, 2\pi])$, where

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx,$$

and let $\{a_n\}_{n=1}^{\infty} \in \text{GM}$. Then

$$\sum_{k=n}^{\infty} |a_k|^p k^{p-2} \lesssim \frac{1}{n^{lp}} \|S_{2^{\nu+2}n}^{(l)}\|_p^p + \|f - S_{[2^{-\nu-2}n]}\|_p^p. \quad (4.1)$$

Proof. Let $n \in \mathbb{N}$. Choose N such that $2^N \leq n < 2^{N+1}$. Then

$$\sum_{k=n}^{\infty} |a_k|^p k^{p-2} \leq \sum_{k=2^N}^{\infty} |a_k|^p k^{p-2} = \sum_{r=N}^{\infty} \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} = \sum_{r=N}^{\infty} P_r. \quad (4.2)$$

A. Suppose that $N < N_0$. We divide the set $[N, \infty) \cap \mathbb{Z}$ into six subsets as follows:

$$[N, \infty) \cap \mathbb{Z} = ([N, N_0) \cap \mathbb{Z}) \sqcup T_N \sqcup K_N^1 \sqcup K_N^2 \sqcup K_N^3 \sqcup K_N^4,$$

where

- 1) $T_N := G \cap [N_0, \infty)$ is the set of good numbers $r \in [N_0, \infty)$;
- 2) K_N^1 is the set of bad numbers $r \in [N_0, \infty)$ with increasing chain, and hence the following inequality holds:

$$N < N_0 \leq r < \xi_{r,s};$$

- 3) K_N^2 is the set of bad numbers $r \in [N_0, \infty)$ with decreasing chain such that $\xi_{r,s} \geq N_0$, and hence the following inequality holds:

$$N < N_0 \leq \xi_{r,s} < r;$$

- 4) K_N^3 is the set of bad numbers $r \in [N_0, \infty)$ with decreasing chain such that $N < \xi_{r,s} < N_0$, and hence the following inequality holds:

$$N < \xi_{r,s} < N_0 \leq r;$$

- 5) K_N^4 is the set of bad numbers $r \in [N_0, \infty)$ with decreasing chain such that $\xi_{r,s} \leq N$, and hence the following inequality holds:

$$\xi_{r,s} \leq N < N_0 \leq r.$$

Therefore,

$$\sum_{r=N}^{\infty} P_r = \sum_{r \in [N, N_0-1] \sqcup K_N^3} P_r + \sum_{r \in T_N \sqcup K_N^1 \sqcup K_N^2} P_r + \sum_{r \in K_N^4} P_r =: \Theta_1 + \Theta_2 + \Theta_3.$$

Step 1_A. An estimate for Θ_1 . Let $r \in K_N^3$; then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}}$$

and

$$r \leq \xi_{r,s} + 2s\nu.$$

The last two inequalities yield

$$\begin{aligned} P_r &= \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \leq A_{\xi_{r,s}}^p 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \\ &\leq A_{\xi_{r,s}}^p 2^{-2ls\nu p} 2^{N_0(p-1)} \lesssim A_{\xi_{r,s}}^p 2^{-2ls\nu p}. \end{aligned}$$

Combining this with arguments from the proof of Lemma 4 we write

$$\sum_{r \in K_N^3} P_r \lesssim \sum_{\substack{\xi \in G \\ N \leq \xi < N_0}} A_{\xi}^p \leq N_0 \max_{2^N \leq k \leq 2^{N_0+1}} |a_k|^p \lesssim \max_{2^N \leq k \leq 2^{N_0+1}} |a_k|^p. \quad (4.3)$$

On the other hand it is easy to obtain

$$\sum_{r=N}^{N_0-1} P_r = \sum_{r=N}^{N_0-1} \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim \max_{2^N \leq k \leq 2^{N_0}} |a_k|^p. \quad (4.4)$$

From (4.3) and (4.4) we have

$$\Theta_1 = \sum_{r=N}^{N_0-1} P_r + \sum_{r \in K_N^3} P_r \lesssim \max_{2^N \leq k \leq 2^{N_0+1}} |a_k|^p \lesssim \|f - S_{[2^{-\nu-2}n]}\|_p^p. \quad (4.5)$$

Step 2_A. An estimate for Θ_2 . Since all the bad numbers $r \in K_N^1 \sqcup K_N^2$ transform only into good $m \in [N_0, \infty)$, according to Lemma 4 we have

$$\Theta_2 = \sum_{r \in T_N} P_r + \sum_{r \in K_N^1 \sqcup K_N^2} P_r \lesssim \sum_{m \in T_N} P_{m, \nu}.$$

Now, by Lemma 2,

$$\Theta_2 \lesssim \sum_{m \in T_N} P_{m, \nu} \lesssim \sum_{m \in T_N} \left\| \sum_{k=m-\nu}^{m+\nu} I_k(\cdot) \right\|_p^p \leq \sum_{m=N}^{\infty} \left\| \sum_{k=m-\nu}^{m+\nu} I_k(\cdot) \right\|_p^p. \quad (4.6)$$

Step 3_A. An estimate for Θ_3 . Let $r \in K_N^4$; then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4lsv} A_{\xi_{r,s}}$$

and

$$r \leq \xi_{r,s} + 2s\nu.$$

Then, using the last inequalities we obtain

$$\begin{aligned} P_r &= \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \leq A_{\xi_{r,s}}^p 2^{-4lsvp} 2^{(\xi_{r,s}+2s\nu)(p-1)} \\ &\leq A_{\xi_{r,s}}^p 2^{-2lsvp} 2^{N_0(p-1)} \lesssim A_{\xi_{r,s}}^p 2^{-2lsvp} \leq 2^{N_0lp} \frac{1}{2^{Nlp}} A_{\xi_{r,s}}^p 2^{-2lsvp} \\ &\lesssim \frac{1}{2^{Nlp}} A_{\xi_{r,s}}^p 2^{-2lsvp} \leq 2^{-2lsvp} \frac{1}{2^{Nlp}} \max_{2^{\xi_{r,s}} \leq k \leq 2^{\xi_{r,s}+1}} |k^l a_k|^p. \end{aligned}$$

Repeating the arguments in the proof of Lemma 4 we obtain

$$\begin{aligned} \sum_{r \in K_N^4} P_r &\lesssim \sum_{\substack{\xi \in G \\ \xi < N}} \frac{1}{2^{Nlp}} \max_{2^\xi \leq k \leq 2^{\xi+1}} |k^l a_k|^p \\ &\leq \frac{N_0}{2^{Nlp}} \max_{1 \leq k \leq 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{2^{Nlp}} \max_{1 \leq k \leq 2^{N+1}} |k^l a_k|^p. \end{aligned}$$

Therefore,

$$\Theta_3 = \sum_{r \in K_N^4} P_r \lesssim \frac{1}{2^{Nlp}} \max_{1 \leq k \leq 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{n^{lp}} \|S_{2^{\nu+2}n}^{(l)}\|_p^p. \quad (4.7)$$

Now we estimate the sum $\sum_{r=N}^{\infty} P_r$. From (4.5)–(4.7) we see that

$$\sum_{r=N}^{\infty} P_r = \Theta_1 + \Theta_2 + \Theta_3 \lesssim \sum_{m=N}^{\infty} \left\| \sum_{k=m-\nu}^{m+\nu} I_k(\cdot) \right\|_p^p + \|f - S_{[2^{-\nu-2}n]}\|_p^p + \frac{1}{n^{lp}} \|S_{2^{\nu+2}n}^{(l)}\|_p^p.$$

In the same way as in the proof of inequality (3.14) we derive

$$\sum_{m=N}^{\infty} \left\| \sum_{k=m-\nu}^{m+\nu} I_k(\cdot) \right\|_p^p \lesssim \|f - S_{[2^{-\nu-2}n]}\|_p^p.$$

This completes the proof of Lemma 6 in the case when $N < N_0$.

B. Suppose now that $N \geq N_0$. We divide the set $[N, \infty) \cap \mathbb{Z}$ into five subsets:

$$[N, \infty) \cap \mathbb{Z} = T_N \sqcup K_N^1 \sqcup K_N^2 \sqcup K_N^3 \sqcup K_N^4,$$

where

- 1) $T_N := G \cap [N, \infty)$ is the set of good numbers $r \in [N, \infty)$;
- 2) K_N^1 is the set of bad numbers $r \in [N, \infty)$ with increasing chain, so that the following inequality holds:

$$N_0 \leq N \leq r < \xi_{r,s};$$

- 3) K_N^2 is the set of bad numbers $r \in [N, \infty)$ with decreasing chain such that $\xi_{r,s} \geq N$, so that the following inequality holds:

$$N_0 \leq N \leq \xi_{r,s} < r;$$

- 4) K_N^3 is the set of bad numbers $r \in [N, \infty)$ with decreasing chain such that $N_0 < \xi_{r,s} < N$, so that the following inequality holds:

$$N_0 < \xi_{r,s} < N \leq r;$$

- 5) K_N^4 is the set of bad numbers $r \in [N, \infty)$ with decreasing chain such that $\xi_{r,s} \leq N_0$, so that the following inequality holds:

$$\xi_{r,s} \leq N_0 \leq N \leq r.$$

Therefore,

$$\sum_{r=N}^{\infty} P_r = \sum_{r \in T_N \sqcup K_N^1 \sqcup K_N^2} P_r + \sum_{r \in K_N^3} P_r + \sum_{r \in K_N^4} P_r =: L_1 + L_2 + L_3.$$

Step 1_B. An estimate for L_1 . Similarly to the above estimate of Θ_2 (see Step 2_A) we obtain

$$L_1 = \sum_{r \in T_N} P_r + \sum_{r \in K_N^1 \sqcup K_N^2} P_r \lesssim \sum_{r=N}^{\infty} \left\| \sum_{k=r-\nu}^{r+\nu} I_k(\cdot) \right\|_p^p. \quad (4.8)$$

Step 2_B. An estimate for L_2 . Let $r \in K_N^3$; then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}} \quad (4.9)$$

and

$$r \leq \xi_{r,s} + 2s\nu \leq N + 2s\nu. \quad (4.10)$$

Then (4.9) and the inequality $r \leq \xi_{r,s} + 2s\nu$ yield

$$P_r = \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \leq 2^{-4ls\nu p} A_{\xi_{r,s}}^p 2^{(\xi_{r,s}+2s\nu)(p-1)}. \quad (4.11)$$

To be definite, suppose that condition (1) in Lemma 1 holds. From Lemma 1 and inequality (4.11) we obtain

$$\begin{aligned}
P_r &\leq 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} A_{\xi_{r,s}}^p \\
&\leq 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \frac{(8C2^{2\nu})^p}{|[l_{\xi_{r,s}}, m_{\xi_{r,s}}] \cap M_{\xi_{r,s}}^+|} \sum_{k \in [l_{\xi_{r,s}}, m_{\xi_{r,s}}] \cap M_{\xi_{r,s}}^+} |a_k|^p \\
&\lesssim 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \frac{1}{2^{\xi_{r,s}}} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p \\
&\lesssim 2^{-2ls\nu p} 2^{-2s\nu} 2^{-\xi_{r,s}lp} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{(l+1)p-2}.
\end{aligned}$$

Since $N \leq \xi_{r,s} + 2s\nu$, it follows that

$$P_r \leq 2^{-2s\nu} \frac{1}{2^{Nlp}} \sum_{k=2^{\xi_{r,s}-\nu}}^{2^{\xi_{r,s}+\nu}} |a_k|^p k^{(l+1)p-2} = 2^{-2s\nu} \frac{1}{2^{Nlp}} \tilde{P}_{\xi_{r,s}, \nu}.$$

Therefore, for any bad number $r \in K_N^3$,

$$P_r \lesssim 2^{-2s\nu} \frac{1}{2^{Nlp}} \tilde{P}_{\xi_{r,s}, \nu}.$$

In a similar way to the proof of Lemma 4 we obtain

$$\sum_{r \in K_N^3} P_r \lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi \in G \\ \xi < N}} \tilde{P}_{\xi, \nu}.$$

Hence, by Lemma 2,

$$\begin{aligned}
L_3 &= \sum_{r \in K_N^3} P_r \lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi \in G \\ \xi < N}} \tilde{P}_{\xi, \nu} \\
&\lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi \in G \\ \xi < N}} \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k^{(l)}(\cdot) \right\|_p^p \leq \frac{1}{2^{Nlp}} \sum_{\xi=\nu}^N \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k^{(l)}(\cdot) \right\|_p^p. \tag{4.12}
\end{aligned}$$

Step 3_B. An estimate for L_3 . Let $r \in K_N^4$; then there exists a good number $\xi_{r,s}$ such that

$$A_r < 2^{-4ls\nu} A_{\xi_{r,s}}$$

and

$$N \leq r \leq \xi_{r,s} + 2s\nu.$$

The last two inequalities imply that

$$\begin{aligned}
 P_r &= \sum_{k=2^r}^{2^{r+1}-1} |a_k|^p k^{p-2} \lesssim A_r^p 2^{r(p-1)} \leq A_{\xi_{r,s}}^p 2^{-4ls\nu p} 2^{(\xi_{r,s}+2s\nu)(p-1)} \\
 &\leq A_{\xi_{r,s}}^p 2^{-2ls\nu p} 2^{-2s\nu} 2^{N_0(p-1)} \lesssim A_{\xi_{r,s}}^p 2^{-2ls\nu p} 2^{-2s\nu} \\
 &\lesssim 2^{-2ls\nu p} 2^{-2s\nu} \frac{1}{2^{\xi_{r,s}lp}} \max_{2^{\xi_{r,s}} \leq k \leq 2^{\xi_{r,s}+1}} |k^l a_k|^p \\
 &\leq 2^{-2s\nu} \frac{1}{2^{Nlp}} \max_{2^{\xi_{r,s}} \leq k \leq 2^{\xi_{r,s}+1}} |k^l a_k|^p.
 \end{aligned}$$

Using the arguments from the proof of Lemma 4 we arrive at

$$\begin{aligned}
 L_3 &= \sum_{r \in K_N^4} P_r \lesssim \frac{1}{2^{Nlp}} \sum_{\substack{\xi \in G \\ \xi < N_0}} \max_{2^\xi \leq k \leq 2^{\xi+1}} |k^l a_k|^p \\
 &\leq \frac{N_0}{2^{Nlp}} \max_{1 \leq k \leq 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{2^{Nlp}} \max_{1 \leq k \leq 2^{N+1}} |k^l a_k|^p \lesssim \frac{1}{n^{lp}} \|S_{2^\nu+2n}^{(l)}\|_p^p. \quad (4.13)
 \end{aligned}$$

From (4.8), (4.12) and (4.13) it follows that

$$\begin{aligned}
 \sum_{r=N}^{\infty} P_r &= L_1 + L_2 + L_3 \\
 &\lesssim \sum_{r=N}^{\infty} \left\| \sum_{k=r-\nu}^{r+\nu} I_k(\cdot) \right\|_p^p + \frac{1}{2^{Nlp}} \sum_{\xi=\nu}^N \left\| \sum_{k=\xi-\nu}^{\xi+\nu} I_k^{(l)}(\cdot) \right\|_p^p + \frac{1}{n^{lp}} \|S_{2^\nu+2n}^{(l)}\|_p^p.
 \end{aligned}$$

It remains to apply Jensen's inequality and the Littlewood-Paley theorem to the first two terms on the right-hand side of the last inequality.

Lemma 6 is proved.

§ 5. The proof of Theorems 5 and 6

Here we use the following result for the realization of the modulus of smoothness (see [29] and [13]):

$$\omega_l^p \left(f, \frac{1}{n} \right)_p \asymp \frac{1}{n^{lp}} \|S_n^{(l)}\|_p^p + \|f - S_n\|_p^p, \quad 1 < p < \infty.$$

Proof of Theorem 5. The upper estimate for $\omega_l(f, 1/n)_p$. The upper estimate follows from Theorem 6.1 in [5], where functions with Fourier coefficients from the wider class $\text{GM}_1 \supset \text{GM}$ were considered, where

$$\text{GM}_1 = \left\{ \{a_n\}_{n=1}^{\infty} : \sum_{k=n}^{\infty} |a_k - a_{k+1}| \leq C \sum_{k=n/\gamma}^{\infty} \frac{|a_k|}{k}, \quad C > 0, \quad \gamma > 1 \right\}.$$

In spite of the fact that Theorem 6.1 in [5] is formulated for nonnegative Fourier coefficients, the proof of this part of the theorem is also valid for sequences with nonconstant sign.

The lower estimate for $\omega_l(f, 1/n)$. Let $p \leq 2$; then from the realization theorem for the modulus of smoothness and the Hardy-Littlewood theorem on Fourier coefficients (see [30]) we obtain

$$\omega_l^p\left(f, \frac{1}{n}\right)_p \asymp \frac{1}{n^{lp}} \|S_n^{(l)}\|_p^p + \|f - S_n\|_p^p \gtrsim \frac{1}{n^{lp}} \sum_{k=1}^n k^{(l+1)p-2} |a_k|^p + \sum_{k=n}^{\infty} k^{p-2} |a_k|^p.$$

Note that here we can also use Theorem 1 in the case $q = p$.

Consider the case $p > 2$. By the properties of the modulus of smoothness and Lemmas 5 and 6 we have

$$\begin{aligned} \omega_l^p\left(f, \frac{1}{n}\right)_p &\asymp \omega_l^p\left(f, \frac{1}{2^{\nu+2}n}\right)_p + \omega_l^p\left(f, \frac{1}{[2^{-\nu-2}n]}\right)_p \\ &\gtrsim \frac{1}{n^{lp}} \|S_{2^{\nu+2}n}^{(l)}\|_p^p + \|f - S_{[2^{-\nu-2}n]}\|_p^p \\ &\gtrsim \frac{1}{n^{lp}} \sum_{k=1}^n k^{(l+1)p-2} |a_k|^p + \sum_{k=n}^{\infty} k^{p-2} |a_k|^p. \end{aligned}$$

Theorem 5 is proved.

Remark 4. In Theorem 5 we can assume that we are dealing with a trigonometric series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ such that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \text{GM}$, and

$$\left(\sum_{k=n}^{\infty} k^{p-2} (|a_k|^p + |b_k|^p) \right)^{1/p} < \infty.$$

In this case, the function f can be defined to be the L_p -limit of the corresponding trigonometric polynomials.

Proof of Theorem 6. (A) We follow the proof of Theorem 4.2 in [26]. Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be the Rudin-Shapiro sequence, for which

$$\left| \sum_{n=0}^N \varepsilon_n e^{int} \right| < 5\sqrt{N+1}$$

for any $t \in [0, 2\pi]$ and $N = 0, 1, \dots$.

It was proved in [26] (see the proof of Theorem 4.1) that if an increasing function φ on $(0, 1)$ satisfies the condition

$$\int_0^u \varphi(t) \frac{dt}{t} = O(\varphi(u)) \quad \text{as } u \rightarrow 0,$$

then the function

$$f_{\varphi}(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n \varphi(1/n)}{n^{1/2}} \sin nx$$

satisfies the condition

$$\omega_l(f_{\varphi}, \delta)_C \lesssim \varphi(\delta) + \delta^l \sum_{k=1}^{[1/\delta]} k^{l-1} \varphi\left(\frac{1}{k}\right).$$

Consider $\varphi_0(x) = x^l$. Then the corresponding function

$$f_{\varphi_0}(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^{l+1/2}} \sin nx$$

is continuous and satisfies

$$\omega_l\left(f_{\varphi_0}, \frac{1}{n}\right)_C \lesssim \frac{\ln n}{n^l}.$$

On the other hand, for $p > 2$ the Fourier coefficients of the function f_{φ_0} satisfy

$$\frac{1}{n^l} \left(\sum_{k=1}^n k^{(l+1-1/p-1/q)q} |b_k|^q \right)^{1/q} + \left(\sum_{k=n+1}^{\infty} k^{(1-1/p-1/q)q} |b_k|^q \right)^{1/q} \asymp \frac{n^{1/2-1/p}}{n^l}.$$

This shows that inequality (1.3) does not hold for any $q > 0$.

(B) We consider the continuous function

$$g(x) = \sum_{n=1}^{\infty} \frac{\eta_n}{n^{l+1/2}} \left(\cos\left(nx - \frac{\pi l}{2}\right) + \sin\left(nx - \frac{\pi l}{2}\right) \right),$$

where the sequence $\eta_n = \pm 1$ is such that the series

$$\sum_{n=1}^{\infty} \frac{\eta_n}{n^{1/2}} (\cos nx + \sin nx) \quad (5.1)$$

is not a Fourier series (see [30], Ch. V, (8.14)). Note that for $1 < p < 2$ and any $q > 0$ the Fourier coefficients of g satisfy

$$\begin{aligned} & \frac{1}{n^l} \left(\sum_{k=1}^n k^{(l+1-1/p-1/q)q} (|a_k| + |b_k|)^q \right)^{1/q} \\ & + \left(\sum_{k=n+1}^{\infty} k^{(1-1/p-1/q)q} (|a_k| + |b_k|)^q \right)^{1/q} \asymp \frac{1}{n^l}. \end{aligned}$$

Therefore, inequality (1.4) implies that $\omega_l(f, \delta)_p = O(\delta^l)$. This relation is equivalent to $f^{(l)} \in L_p$. This contradicts the fact that the series (5.1) is not a Fourier series. Hence (1.4) is not true for any $q > 0$.

Theorem 6 is proved.

§ 6. Applications

In approximation theory the following direct and inverse estimates are well known (see [31], p. 210):

$$\frac{1}{n^l} \left(\sum_{\nu=0}^n (\nu+1)^{\tau l-1} E_{\nu}^{\tau}(f)_p \right)^{1/\tau} \lesssim \omega_l\left(f, \frac{1}{n}\right)_p \lesssim \frac{1}{n^l} \left(\sum_{\nu=0}^n (\nu+1)^{q l-1} E_{\nu}^q(f)_p \right)^{1/q}, \quad (6.1)$$

where $f \in L_p([0, 2\pi])$, $1 < p < \infty$, $l, n \in \mathbb{N}$, $q = \min(2, p)$, $\tau = \max(2, p)$ and $E_n(f)_p$ is the best approximation to the function f in L_p by trigonometric polynomials of degree n . Note that inequalities (6.1) are equivalent to the relations

$$t^l \left(\int_t^1 u^{-\tau l - 1} \omega_{l+1}^\tau(f, u)_p du \right)^{1/\tau} \lesssim \omega_l(f, t)_p \lesssim t^l \left(\int_t^1 u^{-q l - 1} \omega_{l+1}^q(f, u)_p du \right)^{1/q}$$

(see [32]). The following theorem provides a sharp relation between the moduli of smoothness $\omega_l(f, t)_p$ and $\omega_{l+1}(f, t)_p$ and also a relation between the modulus of smoothness $\omega_l(f, t)_p$ and the best approximations $E_k(f)_p$ for functions with general monotone Fourier coefficients.

Theorem 7. *Let $f \in L_p([0, 2\pi])$, $1 < p < \infty$, where*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \text{GM}$. Then

$$\begin{aligned} \omega_l(f, t)_p &\asymp t^l \left(\int_t^1 u^{-lp} \omega_{l+1}^p(f, u)_p \frac{du}{u} \right)^{1/p} \\ &\asymp t^l \left(\sum_{k=0}^{[1/t]} (k+1)^{lp-1} E_k^p(f)_p \right)^{1/p}, \quad 0 < t < \frac{1}{2}. \end{aligned}$$

Remark 5. The proof of Theorem 7 is similar to the proofs of Theorems 7.1 and 7.2 in [5] and uses Theorem 5.

Using Theorem 5, we can also obtain the following description of the Lebesgue spaces $L_p([0, 2\pi])$ and the Besov spaces $B_{p,\tau}^r([0, 2\pi])$.

Theorem 8. *Let $f \in L_1([0, 2\pi])$, where*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \text{GM}$. Then

$$\|f\|_p \asymp \left(\sum_{n=1}^{\infty} n^{p-2} (|a_n|^p + |b_n|^p) \right)^{1/p}, \quad 1 < p < \infty.$$

Theorem 9. *Let $1 < \tau \leq \infty$ and $1 < p \leq \tau$. Also, let $f \in L_p([0, 2\pi])$,*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \in \text{GM}$. Then $f \in B_{p,\tau}^r([0, 2\pi])$ if and only if

$$\sum_{n=1}^{\infty} n^{r\tau + \tau - \tau/p - 1} (|a_n|^\tau + |b_n|^\tau) < \infty \quad \text{if } 1 < \tau < \infty$$

and

$$\sup_n n^{r+1-1/p} (|a_n| + |b_n|) < \infty \quad \text{if } \tau = \infty.$$

Theorem 9 can be proved similarly to Theorem 7.3 in [5]. The necessity follows from the estimate

$$n^{1-1/p}(|a_n| + |b_n|) \lesssim \omega_l\left(f, \frac{1}{n}\right)_p, \quad 1 < p < \infty, \quad (6.2)$$

which is of independent interest (for monotone sequences see [33]). Inequality (6.2) can be obtained as follows. Since $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \text{GM}$, it follows that

$$|a_n| + |b_n| \lesssim \sum_{k=n/\gamma}^{n\gamma} \frac{|a_k| + |b_k|}{k} \lesssim \left(\sum_{k=n/\gamma}^{n\gamma} \frac{|a_k|^p + |b_k|^p}{k} \right)^{1/p} \lesssim n^{1/p-1} \omega_l\left(f, \frac{1}{n}\right)_p,$$

where we have used Hölder's inequality and Theorem 5. The sufficiency follows from Theorem 5 with the use of Hardy's inequalities.

Remark 6. Theorem 8 (the Hardy-Littlewood theorem) also follows from similar results for Lorentz spaces and weighted Lebesgue spaces proved in [34].

Theorems 8 and 9 imply the following corollary.

Corollary. *Let $1 < p < q < \infty$. Also let $f \in L_p([0, 2\pi])$, where*

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \text{GM}$. Then for $r = 1/p - 1/q$ we have

$$f \in B_{p,q}^r([0, 2\pi]) \iff f \in L_q([0, 2\pi]).$$

Remark 7. Theorem 9 for $l = 2$, $0 < r < 2$ and $1 < \tau \leq \infty$ implies Theorem 4. In the case when $\tau = \infty$, $l = 1$, $0 < r < 1$ and $1 < p < \infty$, Theorem 9 implies Theorem 2.

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Mikhail I. Dyachenko

Faculty of Mechanics and Mathematics,
Lomonosov Moscow State University,
Moscow, Russia
E-mail: dyach@mail.ru

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Askhat B. Mukanov

Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
Bellaterra (Barcelona), Spain;
Centre de Recerca Matemàtica,
Bellaterra (Barcelona), Spain;
Kazakhstan Branch
of Lomonosov Moscow State University,
Astana, Kazakhstan
E-mail: mukanov.askhat@gmail.com

Sergei Yu. Tikhonov

Centre de Recerca Matemàtica,
Bellaterra (Barcelona), Spain;
Institució Catalana de Recerca
i Estudis Avançats,
Barcelona, Spain;
Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
Bellaterra (Barcelona), Spain
E-mail: stikhonov@crm.cat