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ON THE COEFFICIENTS OF THE PERMANENT AND THE DETERMINANT OF A CIRCULANT MATRIX. APPLICATIONS

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ABSTRACT. Let $d(N)$ (resp. $p(N)$) be the number of summands in the determinant (resp. permanent) of an $N \times N$ circulant matrix $A = (a_{ij})$ given by $a_{ij} = X_{i+j}$ where $i+j$ should be considered mod N . This short note is devoted to prove that $d(N) = p(N)$ if and only if N is a prime power. We then give an application to homogeneous monomial ideals failing the Weak Lefschetz property.

1. INTRODUCTION

We say that a square $N \times N$ matrix A is a *circulant* matrix if each row is obtained applying the cycle $(\alpha_N, \alpha_0, \dots, \alpha_{N-1})$ to the preceding row. Circulant matrices have been studied for a long time and from many different perspectives, for instance in [3], [11], [12], [13], [14], [15] and [18]. In this note, we will focus our attention on the generic circulant matrix $Circ(x_0, x_1, \dots, x_{N-1})$ given by the first row $(x_0 \ x_1 \ \dots \ x_{N-1})$ (where x_0, \dots, x_{N-1} are indeterminates) and we will address the problem of determining the number of terms of its determinant $\det(Circ(x_0, \dots, x_{N-1}))$ (resp. permanent $\text{per}(Circ(x_0, \dots, x_{N-1}))$). More precisely, $\det(Circ(x_0, \dots, x_{N-1}))$ and $\text{per}(Circ(x_0, \dots, x_{N-1}))$ are homogeneous polynomials of degree N in $k[x_0, x_1, \dots, x_{N-1}]$ and we would like to determine the number $d(N)$ (resp. $p(N)$) of monomials appearing in the expansion of $\det(Circ(x_0, \dots, x_{N-1}))$ (resp. $\text{per}(Circ(x_0, \dots, x_{N-1}))$).

In [3], it was shown that the monomials $x_0^{a_0} x_1^{a_1} \dots x_{N-1}^{a_{N-1}}$ appearing in $\text{per}(Circ(x_0, \dots, x_{N-1}))$ are precisely those given by the solutions of the system

$$\begin{aligned} a_0 + 2a_1 + \dots + Na_{N-1} &\equiv 0 \pmod{N} \\ a_0 + \dots + a_{N-1} &= N. \end{aligned}$$

Unfortunately an analogous result for the monomials $x_0^{a_0} x_1^{a_1} \dots x_{N-1}^{a_{N-1}}$ appearing in the expansion of $\det(Circ(x_0, \dots, x_{N-1}))$ is not known. We easily check that $d(N) \leq p(N)$. In addition, in [14], Thomas proved that $d(N) = p(N)$ for N a power of a prime and asked

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whether the converse is true. This short note is devoted to prove this result $d(N) = p(N)$ if and only if N is a power of a prime.

In the second part of this note, we generalize the results in [7] and we prove that the artinian ideal $I_{0,1,\dots,N-1}^N \subset k[x_0, x_1, \dots, x_{N-1}]$ generated by the monomials appearing in the expansion of $\text{per}(\text{Circ}(x_0, \dots, x_{N-1}))$ is a *Togliatti system*, i.e. it fails the *Weak Lefschetz property* in degree $N - 1$ (see Definition 4.1); even more, it is a *GT-system* (see section 4) and we analyze whether it is minimal. In [7], the minimality problem of GT-systems in $k[x, y, z]$ was related to the vanishing of the coefficients in the determinant of certain circulant matrices. As an application of our results about the number of monomials appearing in the expansion of the determinant and the permanent of a circulant matrix we are able to conclude that $I_{0,1,\dots,N-1}^N$ is a *minimal* Togliatti system if and only if N is a power of a prime (see Theorem 4.8).

Next we outline the structure of this note. In Section 2, we collect the definition and basic results on the determinant (resp. permanent) of a circulant matrix and we collect examples to illustrate that the number of terms $d(N)$ in the determinant of an $N \times N$ circulant matrix A could be strictly less than the number of terms in the permanent of A . Section 3 contains the main result of this paper, namely, $d(N) = p(N)$ if and only if N is a power of a prime (see Theorem 3.5). In the last section, we apply this result to study whether a GT-system is minimal. We prove that, given an integer $N \geq 3$, the GT-system $I_{0,1,\dots,N-1}^N \subset K[x_0, x_1, \dots, x_{N-1}]$ is minimal if and only if N is power of a prime integer (see Theorem 4.8) and we finish our paper with a Conjecture based on our previous results and on many examples computed with Macaulay2.

2. PRELIMINARIES

Throughout this paper, for any $N \times N$ matrix $A = (a_{i,j})$, we denote by $\det(A)$ its determinant and by $\text{per}(A)$ its permanent defined, as usual, by

$$\det(A) = \sum_{\sigma \in \Sigma_N} (-1)^{\epsilon(\sigma)} \prod a_{i\sigma(i)} \quad \text{and}$$

$$\text{per}(A) = \sum_{\sigma \in \Sigma_N} \prod a_{i\sigma(i)}$$

where the sum extends over all elements σ of the symmetric group Σ_N and $\epsilon(\sigma)$ denotes the signature of the permutation σ . In this paper, we are interested in computing the non-zero terms of the determinant and of the permanent of a circulant matrix. So, let us start this section by recalling the definition and the basic properties on circulant matrices needed in the sequel. The reader should read [14], [15] and [13] for more details.

Definition 2.1. An $N \times N$ *circulant matrix* is a matrix of the form

$$\text{Circ}(v_0, \dots, v_{N-1}) := \begin{pmatrix} v_0 & v_1 & \cdots & v_{N-2} & v_{N-1} \\ v_{N-1} & v_0 & \cdots & v_{N-3} & v_{N-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ v_1 & v_2 & \cdots & v_{N-1} & v_0 \end{pmatrix}$$

where successive rows are circular permutations of the first row v_0, \dots, v_{N-1} .

A circulant matrix $\text{Circ}(v_0, \dots, v_{N-1})$ is a particular form of a Toeplitz matrix, i.e., a matrix whose elements are constant along the diagonals. A circulant matrix $\text{Circ}(v_0, \dots, v_{N-1})$ has N eigenvalues, namely, $v_0 + e^p v_1 + e^{2p} v_2 + \cdots + e^{p(N-1)} v_{N-1}$, $0 \leq p \leq N-1$, where e is a primitive N -th root of unity. Therefore, it holds:

$$\begin{aligned} (1) \quad \det(\text{Circ}(v_0, \dots, v_{N-1})) &= \begin{vmatrix} v_0 & v_1 & \cdots & v_{N-2} & v_{N-1} \\ v_{N-1} & v_0 & \cdots & v_{N-3} & v_{N-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ v_1 & v_2 & \cdots & v_{N-1} & v_0 \end{vmatrix} \\ &= \prod_{j=0}^{N-1} (v_0 + e^j v_1 + e^{2j} v_2 + \cdots + e^{j(N-1)} v_{N-1}). \end{aligned}$$

The computation of the permanent is harder and, in spite of several combinatorial interpretations, very little is known so far. The computation of the permanent of a matrix is a challenging problem; it is computationally very hard, even for $(0, 1)$ -matrices. In fact, Valiant proved that computing the permanent of a $(0, 1)$ -matrix is $\#P$ -complete (see [18]). In general, we have:

$$(2) \quad \text{per}(\text{Circ}(v_0, \dots, v_{N-1})) = \sum_{\sigma \in \Sigma_N} v_{\sigma(0)} v_{\sigma(1)} \cdots v_{\sigma(N-1)}.$$

The product on the right hand side in the equations (1) and (2), when expanded out, contains $\binom{2N-1}{N}$ terms and it is still an open problem to find an efficient formula for the coefficients and decide whether they are zero or not. Some examples of these determinants (resp. permanents) of generic circulant matrices for small values of N are:

$$\begin{aligned}\det(\text{Circ}(x, y, z)) &= x^3 + y^3 + z^3 - 3xyz \\ \text{per}(\text{Circ}(x, y, z)) &= x^3 + y^3 + z^3 + 3xyz;\end{aligned}$$

$$\begin{aligned}\det(\text{Circ}(x, y, z, t)) &= x^4 - y^4 + z^4 - t^4 - 2x^2z^2 + 2y^2t^2 - 4x^2yt + 4xy^2z - 4yz^2t + 4xzt^2; \\ \text{per}(\text{Circ}(x, y, z, t)) &= x^4 + y^4 + z^4 + t^4 + 2x^2z^2 + 2y^2t^2 + 4x^2yt + 4xy^2z + 4yz^2t + 4xzt^2;\end{aligned}$$

$$\begin{aligned}\det(\text{Circ}(x, y, z, t, u)) &= x^5 + y^5 + z^5 + t^5 + u^5 - 5x^3yu - 5x^3zt - 5xy^3z - 5y^3tu - 5xz^3u \\ &\quad - 5yz^3t - 5xyt^3 - 5zy^3u - 5xtu^3 - 5yzu^3 + 5x^2y^2t + 5x^2yz^2 + 5x^2zu^2 \\ &\quad + 5x^2t^2u + 5xy^2u^2 + 5xz^2t^2 + 5y^2z^2u + 5y^2tu^2 + 5yt^2u^2 + 5z^2tu^2 - 5xyztu; \\ \text{per}(\text{Circ}(x, y, z, t, u)) &= x^5 + y^5 + z^5 + t^5 + u^5 + 5tu^3x + 5t^2ux^2 + 5t^2u^2y + 5t^3xy + 5ux^3y \\ &\quad + 5u^2xy^2 + 5tx^2y^2 + 5tuy^3 + 5t^3uz + 5u^2x^2z + 5tx^3z + 5u^3yz + 15tuxyz \\ &\quad + 5t^2y^2z + 5xy^3z + 5tu^2z^2 + 5t^2xz^2 + 5x^2yz^2 + 5uy^2z^2 + 5uxz^3 + 5tyz^3;\end{aligned}$$

$$\begin{aligned}\det(\text{Circ}(x, y, z, t, u, v)) &= x^6 - y^6 + z^6 - t^6 + u^6 - v^6 + 6t^4uz + 6t^4vy + 3t^4x^2 - 6t^3u^2y - 12t^3uvx - \\ &\quad - 2t^3v^3 - 6t^3vz^2 - 12t^3xyz - 2t^3y^3 + 6t^2u^3x + 9t^2u^2v^2 - 9t^2u^2z^2 + \\ &\quad + 18t^2uxy^2 + 18t^2v^2xz - 9t^2v^2y^2 - 3t^2x^4 + 6t^2xz^3 + \\ &\quad + 9t^2y^2z^2 - 6tu^4v + 12tu^3yz - 18tu^2x^2y - 12tuv^3z + \\ &\quad + 12tuvx^3 + 12tuvz^3 - 12tuy^3z + 6tv^4y - 6tv^3x^2 - \\ &\quad + 18tvx^2z^2 + 6tvty^4 - 6tx^2y^3 + 12tx^3yz - 6tyz^4 - \\ &\quad + 6u^4xz - 3u^4y^2 + 6u^3v^2z + 12u^3vxy + 2u^3x^3 + 2u^3z^3 - 6u^2v^3y - 9u^2v^2x^2 - \\ &\quad + 18u^2vyz^2 + 9u^2x^2z^2 + 3u^2y^4 + 6uv^4x + 18uv^2y^2z - \\ &\quad + 12uvxy^3 - 6ux^4z + 6ux^3y^2 - 6uxz^4 + 6uy^2z^3 + 3v^4z^2 - \\ &\quad + 12v^3xyz - 2v^3y^3 + 6v^2x^3z + 9v^2x^2y^2 - 3v^2z^4 - 6vx^4y + 12vxyz^3 - 6vy^3z^2 + \\ &\quad + 2x^3z^3 - 9x^2y^2z^2 + 6xy^4z \\ \text{per}(\text{Circ}(x, y, z, t, u, v)) &= x^6 + y^6 + z^6 + t^6 + u^6 + v^6 + 6t^4uz + 6t^4vy + 3t^4x^2 + 6t^3u^2y + 12t^3uvx + \\ &\quad + 2t^3v^3 + 6t^3vz^2 + 12t^3xyz + 2t^3y^3 + 6t^2u^3x + 9t^2u^2v^2 + 9t^2u^2z^2 + 24t^2uvyz + \\ &\quad + 12t^2ux^2z + 18t^2uxy^2 + 18t^2v^2xz + 9t^2v^2y^2 + 12t^2vx^2y + 3t^2x^4 + 6t^2xz^3 + \\ &\quad + 9t^2y^2z^2 + 6tu^4v + 12tu^3yz + 24tu^2vzx + 12tu^2vy^2 + 18tu^2x^2y + 12tuv^3z + \\ &\quad + 24tuv^2xy + 12tuvx^3 + 12tuvz^3 + 24tuxyz^2 + 12tuy^3z + 6tv^4y + 6tv^3x^2 + \\ &\quad + 12tv^2yz^2 + 18tvx^2z^2 + 24tvxy^2z + 6tvty^4 + 6tx^2y^3 + 12tx^3yz + 6tyz^4 + \\ &\quad + 6u^4xz + 3u^4y^2 + 6u^3v^2z + 12u^3vxy + 2u^3x^3 + 2u^3z^3 + 6u^2v^3y + 9u^2v^2x^2 + \\ &\quad + 18u^2vyz^2 + 9u^2x^2z^2 + 12u^2xy^2z + 3u^2y^4 + 6uv^4x + 12uv^2xz^2 + 18uv^2y^2z + \\ &\quad + 24uvx^2yz + 12uvxy^3 + 6ux^4z + 6ux^3y^2 + 6uxz^4 + 6uy^2z^3 + 3v^4z^2 + \\ &\quad + 12v^3xyz + 2v^3y^3 + 6v^2x^3z + 9v^2x^2y^2 + 3v^2z^4 + 6vx^4y + 12vxyz^3 + 6vy^3z^2 + \\ &\quad + 2x^3z^3 + 9x^2y^2z^2 + 6xy^4z\end{aligned}$$

As we have seen in these examples when we expand $\det(\text{Circ}(x_0, \dots, x_{N-1}))$ we obtain a polynomial in the x_i and we define $d(N)$ to be the number of degree N monomials in this polynomial after like terms have been combined. So, $d(3) = 4$, $d(4) = 10$, $d(5) = 26$, $d(6) = 68$, etc. Similarly, we define $p(N)$ to be the number of terms in the permanent,

$\text{per}(\text{Circ}(x_0, \dots, x_{N-1}))$, of $\text{Circ}(x_0, \dots, x_{N-1})$. So, $p(3) = 4$, $p(4) = 10$, $p(5) = 26$, $p(6) = 80$, etc.

3. THE PERMANENT AND THE DETERMINANT OF A CIRCULANT MATRIX

In this section we consider the determinant and the permanent of generic matrices, i.e. the entries are indeterminates.

Problem 3.1. *To determine the integers $N \geq 1$ such that $d(N) = p(N)$.*

From the definition, it is clear that

$$d(N) \leq p(N)$$

since every term which appears in $\det(\text{Circ}(x_0, \dots, x_{N-1}))$ also appears in $\text{per}(\text{Circ}(x_0, \dots, x_{N-1}))$. However, due to cancellations, some terms appearing in $\text{per}(\text{Circ}(x_0, \dots, x_{N-1}))$ could be absent in $\det(\text{Circ}(x_0, \dots, x_{N-1}))$, i.e it could be $d(N) \not\leq p(N)$ (for example, $d(6) = 68 < 80 = p(6)$). To analyze whether $d(N) = p(N)$ we would like to have an efficient formula for the coefficients and decide whether they are zero or not. Let us start determining the non-zero coefficients of $\text{per}(\text{Circ}(x_0, \dots, x_{N-1}))$ and $p(N)$. The function $p(N)$ was studied in [3] by Brualdi and Newman, they showed that $p(N)$ coincides with the number of solutions to

$$(3) \quad \begin{aligned} \alpha_0 + 2\alpha_1 + \dots + N\alpha_{N-1} &\equiv 0 \pmod{N} \\ \alpha_0 + \dots + \alpha_{N-1} &= N \end{aligned}$$

in non-negative integers. They also proved by a generating function argument that

$$(4) \quad p(N) = \frac{1}{N} \sum_{k|N} \phi\left(\frac{N}{k}\right) \binom{2k-1}{k}$$

where $\phi(n)$ is the Euler's function that counts the positive integers up to n that are relatively prime to d . Unfortunately, a formula for the coefficients in the expansion of determinant, $\det(\text{Circ}(x_0, \dots, x_{N-1}))$, is not known; a criterium to decide whether a coefficient is non-zero is not available and, hence, the value of $d(N)$ is out of reach despite the fact that its definition seems at least as natural. Let us now summarize what is known about the coefficients in the left hand side of the equation (1). To this end we express the determinant of an $N \times N$ circulant matrix as follows:

$$\det(\text{Circ}(x_0, \dots, x_{N-1})) = \sum_{0 \leq a_0 \leq \dots \leq a_{N-1} \leq N-1} c_{a_0 \dots a_{N-1}} x_{a_0} \dots x_{a_{N-1}}.$$

This sum can also be written as

$$\det(\text{Circ}(x_0, \dots, x_{N-1})) = \sum_{0 \leq M_0, \dots, M_{N-1} \leq N-1} d_{M_0 \dots M_{N-1}} x_0^{M_0} \dots x_{N-1}^{M_{N-1}}$$

where $d_{M_0 \dots M_{N-1}} = c_{a_0 \dots a_{N-1}}$, if $M_0 + \dots + M_{N-1} = N$ and M_i is the multiplicity of i , the number of times the integer i occurs in the index set $[a_0, \dots, a_{N-1}]$.

Proposition 3.2. *With the above notation, if $\alpha_0 + 2\alpha_1 + \dots + N\alpha_{N-1} \equiv 0 \pmod{N}$, then:*

- (1) *If $a_0 + a_1 + \dots + a_{N-1} \not\equiv 0 \pmod{N}$, then $c_{a_0 \dots a_{N-1}} = 0$.*
- (2) *If N is prime and $a_0 + a_1 + \dots + a_{N-1} \equiv 0 \pmod{N}$, then $c_{a_0 \dots a_{N-1}} \neq 0$.*

Proof. (1) See [13]; Theorem 1 or [15]; Proposition 10.4.3.

(2) See [13]; Corollary 4 or [15]; Chapter 11. □

Remark 3.3. (1) It is worthwhile to point out that expressed in terms of the set of multiplicities, $[M] = [M_0, M_1, \dots, M_{N-1}]$, $0 \leq M_0, M_1, \dots, M_{N-1} \leq N$, the condition $a_0 + a_1 + \dots + a_{N-1} \equiv 0 \pmod{N}$ becomes $0M_0 + 1M_1 + \dots + (N-1)M_{N-1} \equiv 0 \pmod{N}$ subject to the restriction $M_0 + M_1 + \dots + M_{N-1} = N$.

(2) Proposition 3.2 (2) is not true if N is not prime. Indeed, for $N = 6$ we have seen that $c_{0,0,1,3,3,5}$, $c_{0,0,1,2,4,5}$, $c_{0,0,2,3,3,4}$, $c_{0,1,1,2,4,4}$, $c_{0,1,1,2,3,5}$, $c_{0,1,2,2,3,4}$, $c_{0,1,3,4,4,5}$, $c_{0,2,3,4,4,5}$, $c_{0,2,2,4,5,5}$, $c_{1,2,2,3,5,5}$, $c_{1,1,3,4,4,5}$, $c_{1,2,3,3,4,5}$ are zero in the determinant expansion, but they satisfy the previous conditions and they appear as non-zero coefficients in the permanent expansion. Also, for $N = 10$ we have $c_{0,0,0,0,1,1,1,1,3,6,8} = 0$ (see, for instance, [15]; pag. 123). More generally, we have

Proposition 3.4. *For $N = M_0 + M_1 + 3$ with $M_0, M_1 \geq 1$, the coefficient $c_{0 \dots 01 \dots 1 a_{N-3} a_{N-2} a_{N-1}}$ with $M_1 + a_{N-3} + a_{N-2} + a_{N-1} \equiv 0 \pmod{N}$ is zero if N divides $(M_1 + 2)(M_1 + 1)$ and either*

- $a_{N-3} \leq a_{N-2} < N - M_1$, $a_{N-3} + a_{N-2} = N + 1 - \frac{(M_1+2)(M_1+1)}{N}$ and $a_{N-1} = M_0 + 2 + \frac{(M_1+2)(M_1+1)}{N}$, or
- $N - M_1 \leq a_{N-2} \leq a_{N-3}$, $a_{N-2} + a_{N-1} = N + 1 + \frac{(M_0+2)(M_0+1)}{N}$ and $a_{N-3} = M_0 + 2 - \frac{(M_0+2)(M_0+1)}{N}$.

Proof. See [13]; Corollary 6. □

Using the theory of symmetric functions H. Thomas proved in [14] that for any prime integer $p \geq 1$ and for any integer $n \geq 1$ it holds: $d(p^n) = p(p^n)$ but he left open Problem 3.1. In section 2 and Remark 3.3 we have seen examples proving that equality is not always true (for instance, $d(6) < p(6)$). We are now ready to state the main result of this paper and explicitly determine when $d(N) = p(N)$. Indeed, we have

Theorem 3.5. *Fix $N \geq 1$, then $d(N) = p(N)$ if and only if $N = p^r$, with p a prime integer.*

Proof. If $N = p^r$ with p a prime integer and $r \geq 1$ then by [14] we have $d(p^r) = p(p^r)$. Let us prove the converse. Write $N = nm$ with $1 < n < m$ and $\gcd(m, n) = 1$. In order to prove that $p(N) \not\leq d(N)$ it is enough to exhibit an N -tuple $(a_0, a_1, \dots, a_{N-1})$ verifying the equations (3), i.e. $a_0 + 2a_1 + \dots + Na_{N-1} \equiv 0 \pmod{N}$, $a_0 + \dots + a_{N-1} = N$ and such that the coefficient $c_{a_0 \dots a_{N-1}} = 0$. To this end, we apply Bezout's theorem and we write $\lambda m = 1 + \mu n$ with $1 \leq \lambda, \mu$ and $\lambda m \leq N$.

We define

$$M_1 := \mu n - 1.$$

We first observe that $N = nm$ divides $(M_1 + 1)(M_1 + 2)$ (Indeed, $(M_1 + 1)(M_1 + 2) = \mu n(\mu n + 1) = (\mu n)(\lambda m) = \lambda \mu N$). Let us check that $M_1 \leq N - 4$. Notice that $M_1 = \mu n - 1 = \lambda m - 2 \leq N - 2$. Therefore, we only need to prove that the case $M_1 = N - 3$ is not possible. If $M_1 = N - 3$ then $\mu n + 1 = M_1 + 2 = N - 1$, i.e. $\mu n = N - 2 = nm - 2$. So, n divides 2. Since $1 < n$, we get $n = 2$ and $\mu = m - 1$. Using the equalities $\lambda m = 1 + \mu n = 1 + 2(m - 1) = 2m - 1$ we obtain that m divides 1 which is a contradiction.

Let us now define

$$\begin{aligned} M_0 &:= nm - \mu n - 2, \\ A_2 &:= nm - \mu n, \\ A_1 &:= \mu n - \mu \lambda + 1, \text{ and} \\ A_3 &:= nm - \mu n + \lambda \mu. \end{aligned}$$

We easily check that

- $1 \leq M_0 = N - M_1 - 3$,
- $M_1 + A_1 + A_2 + A_3 \equiv 0 \pmod{N}$, and
- $A_1 \leq A_2 < N - M_1$, $A_1 + A_2 = N + 1 - \frac{(M_1+1)(M_1+2)}{N}$ and $A_3 = M_0 + 2 + \frac{(M_1+1)(M_1+2)}{N}$.

Therefore, we can apply Proposition 3.4 and conclude that $c_{0 \dots 01 \dots 1 A_1 A_2 A_3} = 0$. \square

As an immediate consequence of Theorem 3.5 we can slightly generalize Proposition 3.2 and we get:

Corollary 3.6. *With the notation of Proposition 3.2, it holds: If N is a power of a prime and $a_0 + a_1 + \dots + a_{N-1} \equiv 0 \pmod{N}$, then $c_{a_0 \dots a_{N-1}} \neq 0$.*

4. GALOIS-TOGLIATTI SYSTEMS AND GALOIS COVERS

In this section, we will apply the above result to study the minimality of certain Galois-Togliatti systems (GT-systems, for short). So, let us start this section recalling the notion of GT-systems and relating their minimality with the problem of whether $d(N) = p(N)$. To this end, we fix k an algebraically closed field and we set $R := k[x_0, \dots, x_n]$

Definition 4.1. Let $I \subset R$ be a homogeneous artinian ideal. We say that I has Weak Lefschetz Property (WLP) if there is a $L \in [R/I]_1$ such that, for all integers j , the multiplication map

$$\times L : [R/I]_{j-1} \rightarrow [R/I]_j$$

has maximal rank.

To establish whether an ideal $I \subset R$ has the WLP is a difficult and challenging problem and even in simple cases, such as complete intersections, much remains unknown about the presence of the WLP. Recently the failure of the WLP has been connected to a large number of problems, that appear to be unrelated at first glance. For example, in [5], Mezzetti, Miró-Roig and Ottaviani proved that the failure of the WLP is related to the existence of varieties satisfying at least one Laplace equation of order greater than 2 and they proved:

Theorem 4.2. *Let $I \subset R$ be an artinian ideal generated by r homogeneous polynomials F_1, \dots, F_r of degree d and let I^{-1} be its Macaulay inverse system. If $r \leq \binom{n+d-1}{n-1}$, then the following conditions are equivalent:*

- (1) *the ideal I fails the WLP in degree $d-1$;*
- (2) *the homogeneous forms F_1, \dots, F_r become k -linearly dependent on a general hyperplane H of \mathbb{P}^n ;*
- (3) *the n -dimensional variety $X = \overline{\text{Im}(\varphi)}$ where $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}$ is the rational map associated to $(I^{-1})_d$, satisfies at least one Laplace equation of order $d-1$.*

Proof. See [5, Theorem 3.2]. □

The above result motivated the following definitions:

Definition 4.3. Let $I \subset R$ be an artinian ideal generated by r forms of degree d , and $r \leq \binom{n+d-1}{n-1}$. We will say:

- (i) I is a *Togliatti system* if it satisfies one of three equivalent conditions in Theorem 4.2.
- (ii) I is a *monomial Togliatti system* if, in addition, I can be generated by monomials.
- (iii) I is a *smooth Togliatti system* if, in addition, the rational variety X is smooth.
- (iv) A monomial Togliatti system I is *minimal* if there is no proper subset of the set of generators defining a monomial Togliatti system.

These definitions were introduced in [6] and the names are in honor of Eugenio Togliatti who proved that for $n = 2$ the only smooth Togliatti system of cubics is

$$I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset k[x_0, x_1, x_2]$$

(see [2], [16] and [17]). The systematic study of Togliatti systems was initiated in [5] and for recent results the reader can see [6], [7], [8], [1] and [10].

In this paper, we will restrict our attention to a particular case of Togliatti systems, the so-called GT-systems introduced, for the case $N = 3$, in [7]. To define them we need to fix some extra notation. Fix $N \geq 3$, $d \geq 3$ and e a primitive d -th root of the unity. Since any representation of $\mathbb{Z}/d\mathbb{Z}$ in $GL(N, \mathbb{Z})$ can be diagonalized, we can assume that it is represented by a matrix of the form

$$M := M(\alpha_0, \alpha_1, \dots, \alpha_{N-1}) = \begin{pmatrix} e^{\alpha_0} & 0 & \dots & 0 \\ 0 & e^{\alpha_1} & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & e^{\alpha_{N-1}} \end{pmatrix}$$

with $\gcd(\alpha_0, \alpha_1, \dots, \alpha_{N-1}, d) = 1$.

Definition 4.4. Fix integers $3 \leq d \in \mathbb{Z}$ and $3 \leq N \in \mathbb{Z}$, e a primitive d -th root of 1 and $M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$ a representation of $\mathbb{Z}/d\mathbb{Z}$ in $GL(N, \mathbb{Z})$. A *GT-system* will be an ideal

$$I_{\alpha_0, \dots, \alpha_{N-1}}^d \subset k[x_0, x_1, \dots, x_{N-1}]$$

generated by all forms of degree d invariant under the action of a $M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$.

Example 4.5. Fix an odd integer $3 \leq d = 2k + 1$. The monomial artinian ideal $I = (x_0^d, x_1^d, x_2^d, x_0 x_1^{d-2} x_2, x_0^2 x_1^{d-4} x_2^2, \dots, x_0^k x_1 x_2^k) \subset K[x_0, x_1, x_2]$ defines a monomial GT-system. Indeed, let e be a primitive root of order d of 1 and consider the representation of the cyclic Galois group $\mathbb{Z}/d\mathbb{Z}$ on $GL(3, \mathbb{Z})$ given by the diagonal matrix $M(e^0, e^1, e^2)$. It is easy to check that I is generated by all forms of degree d invariant under the action of $M(e^0, e^1, e^2)$. Therefore, I is a GT-system (Indeed, $I = I_{0,1,2}^d$). Note that for $d = 3$ we recover the smooth Togliatti system of cubics.

We easily check that a GT-system $I := I_{\alpha_0, \dots, \alpha_{N-1}}^d$ is always an artinian ideal since $x_i^d \in I$ for $i = 0, \dots, N-1$. So, it defines a regular map

$$\psi_I : \mathbb{P}^{N-1} \longrightarrow \mathbb{P}^{\mu(I)}$$

where $\mu(I)$ denotes the minimal number of generators of I . The morphism ψ_I is a Galois cover of degree d of the $N-1$ dimensional rational variety $\psi_I(\mathbb{P}^{N-1})$ with cyclic Galois group $\mathbb{Z}/d\mathbb{Z}$ represented by the matrix $M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$. Generalizing [7], Theorem 3.1 and Proposition 3.2 from $N = 2$ to arbitrary $N \geq 2$, we get:

Proposition 4.6. Fix integers $N \geq 3$ and $d \geq N$, e a primitive d -th root of 1 and $M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$ a representation of $\mathbb{Z}/d\mathbb{Z}$. Let I denote the ideal $I_{\alpha_0, \dots, \alpha_{N-1}}^d \subset K[x_0, x_1, \dots, x_{N-1}]$ generated by all forms of degree d invariant under the action of $M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$. Assume $\mu(I) \leq \binom{N+d-2}{N-2}$. Then I is a monomial artinian ideal which fails WLP in degree $d-1$ (i.e GT-systems are Togliatti systems).

Proof. To check that $I := I_{\alpha_0, \dots, \alpha_{N-1}}^d$ is a monomial ideal is a straightforward computation that is left to the reader. Let us check that I fails WLP from degree $d-1$ to degree d . Since $\mu(I) \leq \binom{N+d-1}{N-1}$, we have

$$\begin{aligned} \dim R_{d-1} &= \dim[R/I]_{d-1} = \binom{N+d-2}{N-1} \\ &= \binom{N+d-1}{N-1} - \binom{N+d-2}{N-2} \\ &\leq \dim R_d - \dim \mu(I) = \dim[R/I]_d. \end{aligned}$$

So, to see that I fails WLP from degree $d-1$ to degree d , we have to show that for any linear form $\ell \in R$ the induced map $\times \ell : [R/I]_{d-1} \rightarrow [R/I]_d$ is not injective. By [9], Proposition 2.2, it is enough to check it for $\ell = x_0 + x_1 + \dots + x_{N-1}$. This is equivalent to prove that there exists a form $F_{d-1} \in R$ of degree $d-1$ such that $(x_0 + x_1 + \dots + x_{N-1}) \cdot F_{d-1} \in I$. Consider $F_{d-1} = (e^{\alpha_0} x_0 + e^{\alpha_1} x_1 + \dots + e^{\alpha_{N-1}} x_{N-1})(e^{2\alpha_0} x_0 + e^{2\alpha_1} x_1 + \dots + e^{2\alpha_{N-1}} x_{N-1}) \dots (e^{(d-1)\alpha_0} x_0 + e^{(d-1)\alpha_1} x_1 + \dots + e^{(d-1)\alpha_{N-1}} x_{N-1})$. The homogeneous form of degree d , $F = (x_0 + x_1 + \dots + x_{N-1}) \cdot F_{d-1}$ is invariant under the action of $M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$. Hence, it belongs to I which proves our result. \square

Remark 4.7. The hypothesis $\mu(I) \leq \binom{N+d-2}{d-2}$ in Proposition 4.6 is often satisfied and can be dropped. For instance we will see in Theorem 4.8 that it is verified when $d = N$ and $(\alpha_0, \alpha_1, \dots, \alpha_{N-1}) = (0, 1, \dots, N-1)$.

To determine the minimality of a GT-system $I_{\alpha_0, \dots, \alpha_{N-1}}^d$ is a subtle problem that for $N = 3$ was related in [7] to the determinant of certain circulant matrices. This relation works for arbitrary $N \geq 3$. We quickly recall/generalize it (from $N = 3$ to $N \geq 3$). Finally, as application of this relationship and of the results obtained in the previous section, we will be able to prove or disprove the minimality of $I_{0,1,\dots,N-1}^N$. Proving the minimality of the GT-system $I_{\alpha_0, \dots, \alpha_{N-1}}^d$ is equivalent to proving that the monomials of degree d invariant under the action of $M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$ all appear with non-zero coefficient in the development of the product of linear forms $(x_0 + x_1 + \dots + x_{N-1})(e^{\alpha_0} x_0 + e^{\alpha_1} x_1 + \dots + e^{\alpha_{N-1}} x_{N-1})(e^{2\alpha_0} x_0 + e^{2\alpha_1} x_1 + \dots + e^{2\alpha_{N-1}} x_{N-1}) \dots (e^{(d-1)\alpha_0} x_0 + e^{(d-1)\alpha_1} x_1 + \dots + e^{(d-1)\alpha_{N-1}} x_{N-1})$.

For any integer $d \geq N$ and $0 \leq \alpha_0 < \alpha_1 < \dots < \alpha_{N-1} \leq d$, we consider the $d \times d$ circulant matrix

$$A_{\alpha_0, \alpha_1, \dots, \alpha_{N-1}}^d = \text{Circ}(0, \dots, 0, x_0, 0, \dots, 0, x_1, 0, \dots, 0, x_i, 0, \dots, 0, x_{N-1}, 0, \dots, 0)$$

where x_i is in the position of index α_i . According to (1) we have

$$\det(A_{\alpha_0, \alpha_1, \dots, \alpha_{N-1}}^d) = \prod_{j=0}^{d-1} (e^{j\alpha_0} x_0 + e^{j\alpha_1} x_1 + \dots + e^{j\alpha_{N-1}} x_{N-1}).$$

The determinant of $A_{\alpha_0, \alpha_1, \dots, \alpha_{N-1}}^d$ is therefore exactly the product we are interested in and we want to prove that all monomials of degree d invariant under the action of

$$M(\alpha_0, \alpha_1, \dots, \alpha_{N-1})$$

appear with non-zero coefficient in $\det(A_{\alpha_0, \alpha_1, \dots, \alpha_{N-1}}^d)$.

Theorem 4.8. *Fix an integer $N \geq 3$. The GT-system $I_{0,1,\dots,N-1}^N \subset K[x_0, x_1, \dots, x_{N-1}]$ is a monomial Togliatti system. It is minimal if and only if N is power of a prime integer.*

Proof. In view of Proposition 4.6, to prove that $I_{0,1,\dots,N-1}^N$ is a Togliatti system we have to check that $\mu(I_{0,1,\dots,N-1}^N) \leq \binom{2N-2}{N-2}$. But $\mu(I_{0,1,\dots,N-1}^N) \leq d(N) \leq p(N)$, so by (4) it is enough to check that

$$\frac{1}{N} \sum_{k|N} \varphi\left(\frac{N}{k}\right) \binom{2k-1}{k} \leq \binom{2N-2}{N}.$$

First we assume that N is not prime and we consider its prime decomposition $N = q_1^{r_1} \cdots q_s^{r_s}$ with $q_1 < q_2 < \cdots < q_s$. So, $N' = \frac{N}{q_1}$ is the greatest divisor of N different from N , $N \geq 2N'$ and $N' \geq 2$. Since $\varphi(1) = 1$ and $\sum_{k|N} \varphi\left(\frac{N}{k}\right) = N$, we can write:

$$\begin{aligned} \frac{1}{N} \sum_{k|N} \varphi\left(\frac{N}{k}\right) \binom{2k-1}{k} &= \frac{1}{N} \left(\sum_{k|N, k \neq N} \varphi\left(\frac{N}{k}\right) \binom{2k-1}{k} + \binom{2N-1}{N} \right) \leq \\ &\leq \frac{N-1}{N} \binom{2N'-1}{N'} + \frac{1}{N} \binom{2N-1}{N} = \\ &= \frac{(2N'-1)(N-1)}{N(N'-1)} \binom{2N'-2}{N'} + \frac{2N-1}{N(N-1)} \binom{2N-2}{N}. \end{aligned}$$

Therefore,

$$\frac{1}{N} \sum_{k|N} \varphi\left(\frac{N}{k}\right) \binom{2k-1}{k} \leq \binom{2N-2}{N}$$

which is equivalent to

$$\frac{2N'-1}{N'-1} \binom{2N'-2}{N'} \leq \frac{N^2-3N+1}{(N-1)^2} \binom{2N-2}{N}.$$

Using that $\frac{2N'-1}{N'-1} = 1 + \frac{N'}{N'-1} \leq 3$ for all $N' \geq 2$, it is enough to see the following inequality:

$$\frac{3(N^2-2N+1)}{N^2-3N+1} \leq \frac{\binom{2N-2}{N}}{\binom{2N'-2}{N'}}.$$

Since $\frac{3(N^2-2N+1)}{N^2-3N+1} \leq 6$ for $N \geq 4$, it only remains to check that $\binom{2N-2}{N} \geq 6 \binom{2N'-2}{N'}$. By the Pascal's rule $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, it suffices to verify that $\binom{4N'-2}{2N'} \geq 6 \binom{2N'-2}{N'}$. Applying consecutively the Pascal's rule, we obtain:

$$\binom{4N'-2}{2N'} = \binom{4N'-3}{2N'} + \binom{4N'-3}{2N'-1} = \cdots = \binom{4N'-5}{2N'} + 3 \binom{4N'-5}{2N'-1} + 4 \binom{4N'-5}{2N'-2}.$$

Since $2N' - 3 \geq N' - 2$, we have $\binom{4N'-5}{2N'-2} = \binom{2N'-2+2N'-3}{N'+N'-2} \geq \binom{2N'-2+N'-1}{N'} \geq \binom{2N'-2}{N'}$. Similarly, $\binom{2N'-2+2N'-3}{N'+N'-1} \geq \binom{2N'-2+N'-2}{N'} \geq \binom{2N'-2}{N'}$. Summing up, $\binom{4N'-2}{2N'} \geq 7\binom{2N'-2}{N'} + \binom{4N'-5}{2N'}$ which implies $\binom{4N'-2}{2N'} \geq 6\binom{2N'-2}{N'}$.

If N is prime, $\frac{1}{N} \sum_{k|N} \varphi\left(\frac{N}{k}\right) \binom{2k-1}{k} = \frac{N-1}{N} + \frac{1}{N} \binom{2N-1}{2N} = \frac{N-1}{N} + \frac{2N-1}{N(N-1)} \binom{2N-2}{2N}$. In this case, the expected inequality becomes $\frac{N^2-2N+1}{N^2-3N+1} \leq \binom{2N-2}{N}$, which can be easily reduced to verify that $\binom{2N-2}{N} \geq 2$, since $\frac{N^2-2N+1}{N^2-3N+1} \leq 2$ for any $N \geq 3$. The result follows directly from the growth of the binomial coefficients, or simply observing the Pascal's triangle.

It remains to prove that $I_{0,1,\dots,N-1}^N$ is a minimal Togliatti system. First of all we observe that a monomial $m = x_0^{i_0} x_1^{i_1} \cdots x_{N-1}^{i_{N-1}}$ with $i_j \geq 0$ and $\sum_{j=0}^{N-1} i_j = N$ belongs to $I_{0,1,\dots,N-1}^N$ if and only if $i_0 + 2i_1 + 3i_2 + \cdots + Ni_{N-1} \equiv 0 \pmod{N}$. Therefore, the number of generators of $I_{0,1,\dots,N-1}^N$ is equal to $d(N) = \frac{1}{N} \sum_{d|N} \phi\left(\frac{N}{d}\right) \binom{2N-1}{N}$ and the GT-system $I_{0,1,\dots,N-1}^N$ will be minimal if $d(N)$ coincides with the number of non-zero coefficients in the development of the product of linear forms $(x_0 + x_1 + \cdots + x_{N-1})(x_0 + ex_1 + \cdots + e^{N-1}x_{N-1})(x_0 + e^2x_1 + \cdots + e^{2(N-1)}x_{N-1}) \cdots (x_0 + e^{d-1}x_1 + \cdots + e^{(d-1)(N-1)}x_{N-1})$. In other words, $I_{0,1,\dots,N-1}^N$ will be minimal if and only if $d(N) = p(N)$ and, by Theorem 3.5, if and only if N is a power of a prime integer. \square

Proposition 4.9. *Fix integers $N \geq 3$ and $d \geq N$. If d is a power of a prime then the GT-system $I_{0,1,\dots,N-1}^d \subset K[x_0, x_1, \dots, x_{N-1}]$ is minimal.*

Proof. Indeed, if the coefficient of an invariant monomial is zero in

$$I_{0,1,\dots,N-1}^d \subset K[x_0, x_1, \dots, x_{N-1}]$$

it is zero also in the ideal in d variables hence d is not a power of prime. \square

We end this note with a conjecture based on Theorem 4.8, on our results in [7], and on many examples computed with Macaulay2 [4].

Conjecture 4.10. Fix an integer $d \geq 3$ and integers $1 \leq n < m \leq d - 1$ such that $\gcd(n, m, d) = 1$. Then, the GT-system $I_{0,n,m}^d \subset k[x, y, z]$ is minimal.

REFERENCES

- [1] C. Almeida, Aline V. Andrade and R.M. Miró-Roig, *Gaps in the number of generators of monomial Togliatti systems*, Preprint.
- [2] H. Brenner and A. Kaid, *Syzygy bundles on \mathbb{P}^2 and the Weak Lefschetz Property*, Illinois J. Math. **51** (2007), 1299–1308.
- [3] R. Brualdi and M. Newman, *An enumeration problem for a congruence equation*, J. Res. Nat. Bur. Standards Sect. B **74B** (1970), 37–40.
- [4] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>

- [5] E. Mezzetti, R.M. Miró-Roig and G. Ottaviani, *Laplace Equations and the Weak Lefschetz Property*, Canad. J. Math. **65** (2013), 634–654.
- [6] E. Mezzetti and R.M. Miró-Roig, *The minimal number of generators of a Togliatti system*, Annali di Matematica Pura ed Applicata **195** (2016), 2077–2098. DOI:10.1007/s10231-016-0554-y.
- [7] E. Mezzetti and R.M. Miró-Roig, *Togliatti systems and Galois coverings*, J. Algebra **509** (2018), 263–291. DOI: 10.1016/j.jalgebra.2018.05.014
- [8] M. Michalek and R.M. Miró-Roig, *Smooth monomial Togliatti systems of cubics*, Journal of Combinatorial Theory, Ser. A **143** (2016), 66–87. <http://dx.doi.org/10.1016/j.jcta.2016.05.004>.
- [9] J. Migliore, R.M. Miró-Roig and U. Nagel, *Monomial ideals, almost complete intersections, and the Weak Lefschetz property*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 229–257.
- [10] R.M. Miró-Roig and M. Salat, *On the classification of Togliatti systems*, Comm. Algebra, **46:6** (2017), 2459–2475, DOI: 10.1080/00927872.2017.1388813r.
- [11] I. Kra and S.R. Simanca, *On circulant matrices*, Notices AMS **59** (3) (2012), 368–377.
- [12] N.A. Loehr, G.S. Warrington, H.S. Wil, *The Combinatorics of a Three-Line Circulant Determinant*, Israel Journal of Mathematics **143** (2004) 141–156.
- [13] J. Malenfant, *On the Matrix-Element Expansion of a Circulant Determinant*, arXiv:1502.06012v5
- [14] H. Thomas, *The number of terms in the permanent and the determinant of a generic circulant matrix*, J. Algebraic Combin. **20** (2004), no. 1, 5560.
- [15] A. Wyn-Jones, *Circulants*, Chapters 10 and 11. www.circulants.org/circ/circall.pdf.
- [16] E. Togliatti, *Alcuni esempi di superfici algebriche degli iperspazi che rappresentano un'equazione di Laplace*, Comm. Math. Helvetici **1** (1929), 255–272.
- [17] E. Togliatti, *Alcune osservazioni sulle superfici razionali che rappresentano equazioni di Laplace*, Ann. Mat. Pura Appl. (4) **25** (1946) 325–339.
- [18] L. G. Valiant, *The complexity of computing the permanent*, Theoret. Comput. Sci. **8** (1979) 189–201.

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