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On the integrability of Degasperis-Procesi equation: control of the Sobolev norms and Birkhoff resonances.

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Abstract

We consider the *dispersive* Degasperis-Procesi equation $u_t - u_{xxx} - cu_{xxx} + 4cu_x - uu_{xxx} - 3u_x u_{xx} + 4uu_x = 0$ with $c \in \mathbb{R} \setminus \{0\}$. In [12] the authors proved that this equation possesses infinitely many conserved quantities. We prove that there are infinitely many of such constants of motion which control the Sobolev norms and which are analytic in a neighborhood of the origin of the Sobolev space H^s with $s \geq 2$, both on \mathbb{R} and \mathbb{T} . By the analysis of these conserved quantities we deduce a result of global well-posedness for solutions with small initial data and we show that, on the circle, the *formal* Birkhoff normal form of the Degasperis-Procesi at any order is action-preserving.

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1 Introduction

In 1999 Degasperis and Procesi [13] applied a method of asymptotic integrability to the family of third-order dispersive PDE conservation laws

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \quad (1.1)$$

where $\alpha, c_0, c_1, c_2, c_3$ are real constants and subindices denote the partial derivatives.

Only three equations within this family result to satisfy the asymptotic integrability condition, the KdV equation ($\alpha = c_2 = c_3 = 0$), the Camassa-Holm equation ($c_1 = -3c_3/2\alpha^2, c_2 = c_3/2$) and the Degasperis-Procesi equation

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = -\frac{4c_2}{\alpha^2} u u_x + 3c_2 u_x u_{xx} + c_2 u u_{xxx}. \quad (1.2)$$

In [12] Degasperis-Holm-Hone showed that the equation (1.2) is *integrable* by constructing its Lax pair. They also proposed a bi-Hamiltonian structure and a recursive method to generate infinitely many constants of motion (see Section 4 in [12]).

Later Constantin and Lannes showed in [6] that the Degasperis-Procesi equation, as well as the Camassa-Holm equation, can be regarded as a model for nonlinear shallow water dynamics, and that it accomodates wave-breaking phenomena.

We observe that the equation (1.2) is a quasi-linear PDE, namely the linear and the nonlinear terms contain the same order of derivatives. We also remark that (1.2) is not translation invariant and the linear dispersion (see for instance (1.14) for the case $x \in \mathbb{T}$) is related to the chosen frame.

By taking $c_0 = -\gamma$ and $\alpha^2 = 1$ in (1.2) the linearized equation at $u = 0$ transforms into the following transport equation

$$u_t = \gamma u_x. \quad (1.3)$$

Hence, in this case, all the solutions are travelling waves and there are no dispersive effects. In particular, by choosing $c_0 = \gamma = 0, c_2 = 1$ and $\alpha^2 = 1$, the equation (1.2) can be transformed into the *dispersionless* form

$$u_t - u_{xxt} + 4u u_x = 3u_x u_{xx} + u u_{xxx}. \quad (1.4)$$

The family of equations (1.2) is covariant under the group of transformations

$$u \mapsto \lambda u(t, \xi x + \eta t) + \delta, \quad \lambda, \xi, \eta, \delta \in \mathbb{R}, \quad (1.5)$$

which are compositions of translations and Galilean boosts, and all the parametrized equations in (1.2) can be obtained from (1.4) by applying such changes of coordinates (see [14]). As we said above, in order to consider the dispersive effects of (1.2) we have to impose that $c_0 \neq -\gamma$ and $c_0, \gamma \neq 0$; if we let the coefficient in front of u_{xxt} be -1 , we can obtain an equation with this feature from (1.4) if and only if we apply a transformation of the form (1.5) with $\delta \neq 0$, namely we have to consider translations of the variable u .

We will consider the *dispersive* Degasperis-Procesi equation

$$u_t - u_{xxt} - c u_{xxx} + 4c u_x - u u_{xxx} - 3u_x u_{xx} + 4u u_x = 0, \quad (1.6)$$

obtained from (1.4) by translating $u \mapsto u + c$, for some real parameter $c \neq 0$. We remark that the same equation can be obtained from (1.2) by setting $\alpha^2 = 1, \gamma = -c, c_0 = 4c, c_2 = c_3 = 1$.

We note that the mass $\int u \, dx$ is a constant of motion of the Degasperis-Procesi equation (1.4), hence the subsets of functions with fixed finite average

$$\mathcal{L}_c := \{u : \int u \, dx = c\} \quad (1.7)$$

are left invariant by the flow of (1.4). On the subspace \mathcal{L}_0 it is possible to define a non-degenerate symplectic structure for which (1.4) can be seen as a Hamiltonian PDE. Our purpose is to investigate the dynamics of (1.4) on the invariant subsets \mathcal{L}_c with $c \neq 0$, which is equivalent to the one of (1.6) when $u \in \mathcal{L}_0$.

The equation (1.4) has been widely investigated by many authors since it presents wave breaking phenomena, peakon and soliton solutions and blow-up scenarios (see for instance [5], [19], [7], [9], [30], [25], [27], [26], [12] and [8]).

Lundmark and Szmigielski [28] presented an inverse scattering approach for computing n-peakon solutions to equation (1.4). Vakhnenko and Parkes [37] investigated traveling wave solutions. Holm and Staley [23] studied stability of solitons and peakons numerically.

Regarding the well-posedness of the equation (1.4) we cite the results of Yin [39], [40] on the local well-posedness with initial data $u_0 \in H^s$, $s > 3/2$, both on the line and on the circle, and the result by Himonas-Holliman [22], who showed that for (1.4) the data-to-solution map is not uniformly continuous. We also mention the paper of Coclite-Karlsen [10] for the well-posedness in classes of discontinuous functions and Wu [38] for the periodic generalized Degasperis-Procesi equation.

Although the bi-Hamiltonian structure of equation (1.4) provides an infinite number of conservation laws ([12]), there is no way to find conservation laws controlling the H^1 -norm of $u \in \mathcal{L}_0$ ([19]), which for instance represents an important difference with respect to the Camassa-Holm equation. We want to show that if we consider the dynamics on \mathcal{L}_c with $c \neq 0$, then the situation, close to the origin, is truly different.

In this paper we construct infinitely many conserved quantities K_n for the dispersive Degasperis-Procesi equation (1.6), starting from the ones proposed by Degasperis-Holm-Hone in [12].

We prove that the K_n 's are analytic functions in a ball centered at the origin of H^{n+1} with radius depending on the parameter c , but independent of n (see (1.6)), such that

(F1) for any $s \geq 2$ there exists $n = n(s) \in \mathbb{N}$ such that K_n controls the H^s -norm of solutions u of (1.6) with initial datum u_0 sufficiently close to 0 in H^s ,

(F2) the quadratic part of $K_n(u)$ is given by

$$K_n^{(0)} = \int (\partial_x^{n-1} w)^2 dx, \quad w := u - u_{xx}$$

and so, if $x \in \mathbb{T}$, it reads

$$K_n^{(0)} = \sum_{j \in \mathbb{Z}} (1 + j^2)^2 j^{2(n-1)} |u_j|^2.$$

These facts are actually proved in Theorem 1.4, which is the main result of the paper.

We provide also two applications of this result. The first one is a result of global in time well-posedness and stability in a neighborhood of the origin.

(A1) Fixed $s \geq 2$, there exists a ball centered at the origin B_s of H^s in which the equation (1.6) is *globally well-posed* (see [34]), namely any solution with initial datum belonging to B_s exists for all times and, moreover, it remains inside a slightly bigger ball centered at the origin (hence the origin is a stable fixed point).

This is the content of Theorem 1.5, which in turn is due to (F1).

The use of the conserved quantities to prove existence results for solutions of integrable PDE's is a classical argument. We mention for instance the celebrated result by Bona-Smith [3] on the initial value problem for the KdV equation.

We also mention [21], [35], [36], in which the properties of conserved quantities of other integrable PDE's, such as the DNLS and the Benjamin-Ono equation on \mathbb{T} , are used to construct functional Gibbs measures and study the long time existence of regular solutions. It would be interesting to investigate if similar results could be applied to equation (1.6).

Another application of Theorem 1.4 concerns the Birkhoff normal form analysis for the Degasperis-Procesi equation on the circle. First we briefly recall some facts and literature about Birkhoff normal form.

The Birkhoff normal form theory for PDE's has been widely developed since the 1990s, in order to extend to the infinite dimensional dynamical case the classical results which hold for finite dimensional “nearly” integrable Hamiltonian systems.

Let us consider a Hamiltonian function $H(p, q) = H = H^{(0)} + P$, with $(p, q) \in \mathbb{R}^{2n}$ with

$$H^{(0)} = \sum_{j=1}^n \omega_j \frac{p_j^2 + q_j^2}{2}, \quad (1.8)$$

and P is a smooth function with a zero of order at least 3 at the origin. The origin is an elliptic fixed point and the Hamiltonian system can be seen as a system of harmonic oscillators with frequencies ω_j coupled by the nonlinearities. Classical theory guarantees that, for any $N \geq 1$, there is a real analytic and symplectic map Φ_N such that

$$H \circ \Phi_N = H^{(0)} + Z_N + R_N,$$

where R_N is a function with a zero of order $N+3$ and Z_N is a polynomial of degree $N+2$ which Poisson-commutes with $H^{(0)}$. In particular, if the frequencies ω_j are “non resonant”, namely

$$\sum_{j=1}^n \omega_j k_j \neq 0, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \quad (1.9)$$

then Z_N depends only on the actions $I_j = (p_j^2 + q_j^2)/2$. This implies that if the initial datum has size $\varepsilon \ll 1$ then the solution remains in a neighborhood of radius 2ε for times of order $\varepsilon^{-(N+1)}$. The key idea to obtain such a result is to remove from the nonlinearity P all the monomials which do not commutes with $H^{(0)}$. This can be done iteratively by means of symplectic transformations. More precisely, one uses a sequence of maps Φ_p generated as the Hamiltonian flow at time one of an auxiliary Hamiltonian F_p of degree of homogeneity $p+2$. The F_p is chosen in such a way the equation

$$\{H^{(0)}, F_p\} + G_p = Z \quad (1.10)$$

where G_p is some homogeneous Hamiltonian of degree $p+2$ and $\{H^{(0)}, Z\} = 0$, is satisfied. It turns out that, in order to solve it, one needs some non-resonance conditions on the frequencies ω_j , for instance the relation (1.9).

In the case $n < \infty$ the number of monomials which have to be canceled out is finite. Many PDE's (NLS, KdV, Klein-Gordon...) $u_t = L(u) + f(u)$, with L possibly an unbounded linear operator and f some nonlinear function, can be written, on compact manifolds, as Hamiltonian systems whose quadratic part has a form similar to (1.8) with $n = \infty$.

There are several difficulties in extending the theory to the infinite dimensional case:

- (i) one needs suitable *non-resonance conditions* which replace (1.9) in infinite dimension;
- (ii) one needs to cancel out an *infinite* number of monomials;
- (iii) one needs to check that the normal form Z_N is *action-preserving*;
- (iv) the Birkhoff transformations are flows of possibly ill-posed PDEs.

The first difficulty has been overcome in [1] and in [2]. To deal with the second one, a good definition of the class of formal polynomials on which one works is required. For instance, one at least needs that the Hamiltonian F_p are continuous functions on the phase space. In the framework of Hamiltonian PDEs typical phase spaces are Sobolev spaces of functions (or weighted spaces of sequences). We mention [17] and [18], in which the authors introduce a suitable classes of multilinear forms.

Concerning item (iii), we say that the normal form Z_k is **action-preserving** if it depends only on the actions $|u_j|^2$. This implies that the flow generated by the Hamiltonian function Z_k leave the actions invariant. In the PDE context this means that the Sobolev norms H^s are preserved for any $s > 0$. Proving that the obtained normal form is action-preserving is a problem concerning the specific equation one is studying.

Regarding item (iv), if the nonlinearity depends upon some derivatives the Birkhoff transformations could be not well-defined. Especially for quasi-linear PDEs, the problem of constructing rigorous symplectic transformations in order to apply a Birkhoff normal form procedure is delicate.

The Birkhoff normal form methods have been used by many authors to prove long time existence of solutions with data in a small neighborhood of a fixed point. We quote for instance the papers by Delort [15], [16], in which the

author studied quasi-linear perturbations of Klein-Gordon (K-G) and the recent paper by Berti-Delort [4], where the authors applied a suitable Birkhoff normal form procedure to the capillarity-gravity water waves (WW) equation.

In the latter papers the linear frequencies depends on some “external” parameters (the “mass” for the K-G, the capillarity for the WW). This fact has been used in order to prove very strong non-resonance conditions (which hold for “most” values of parameters) and, as a consequence, that the normal forms are action-preserving.

When the equation does not depend on physical parameters, the problem of showing the integrability of the normal form relies on an analysis of the algebraic structure of the resonances. We mention the paper by Craig-Worfolk [11] in which the authors study (at a formal level) the Birkhoff normal form for pure-gravity water waves at order four. It is known that for this equation there are non trivial resonances (called “Benjamin-Feir”), which could prevent the normal forms to be action-preserving. Actually, they show that there are suitable cancellations in the coefficients of the Hamiltonian which allows to obtain an action preserving normal form.

Our result focuses only on (iii), by formulating the Birkhoff normal form procedure only at a formal level, and it is similar to the study performed in [11], since we do not deal with (i), (ii) and (iv) (which concern convergence problem).

The main differences are that:

- we use the integrability of the equation in order to overcome the problem of non trivial resonances;
- we are able to prove the integrability of the normal form at any order, since we exploit the algebraic structure of the resonances.

More precisely we prove the following:

(A2) at a purely formal level, it is possible to put the Hamiltonian (1.11) in a action-preserving Birkhoff normal form at any order.

This result is achieved thanks to (F2) and it is the content of Theorem 1.6.

One of the main applications of the Birkhoff normal form methods concerns the KAM theory for PDEs. It is well known that in order to apply perturbative arguments to construct periodic and quasi-periodic solutions for perturbed autonomous integrable equations one needs to control the frequencies of the expected invariant tori. In the infinite dimensional context, this requires the presence of parameters which modulate the frequencies, since the non-resonance conditions to be imposed are quite complicated. When the considered system does not present external parameters, one has to extract them from the equation itself: a way to do that is to perform a Birkhoff normal form procedure. Actually, the result presented in Theorem 1.6 has been motivated by the study of quasi-periodic solutions for perturbations of the Degasperis-Procesi equation [20].

1.1 Preliminaries

In order to state the main result of the paper we introduce the Hamiltonian setting and the space of formal polynomials and power series. When there is no specification of the spatial domain we mean that the arguments hold for both cases $x \in \mathbb{T}$ or $x \in \mathbb{R}$.

Hamiltonian setting. The equation (1.6) can be formulated as a Hamiltonian PDE $u_t = J \nabla_{L^2} H(u)$, where $\nabla_{L^2} H$ is the L^2 gradient of the Hamiltonian

$$H(u) = \int \frac{c u^2}{2} - \frac{u^3}{6} dx. \quad (1.11)$$

The Hamiltonian (1.11) is defined on the real phase space (recall (1.7))

$$H_0^1 := H^1 \cap \mathfrak{L}_0$$

endowed with the non-degenerate symplectic form

$$\Omega(u, v) := \int (J^{-1} u) v dx, \quad \forall u, v \in H_0^1, \quad J := (1 - \partial_{xx})^{-1} (4 - \partial_{xx}) \partial_x. \quad (1.12)$$

The Poisson bracket induced by Ω between two functions $F, G: H_0^1 \rightarrow \mathbb{R}$ is

$$\{F(u), G(u)\} := \Omega(X_F, X_G) = \int \nabla F(u) J \nabla G(u) dx, \quad (1.13)$$

where X_F and X_G are the vector fields associated to the Hamiltonians F and G , respectively. On the circle the *dispersion law* of the Degasperis-Procesi equation is given by

$$j \mapsto \omega(j) := j \frac{4+j^2}{1+j^2} = j + \frac{3j}{1+j^2}, \quad j \in \mathbb{Z} \setminus \{0\}, \quad (1.14)$$

where $\omega(j)$ are the *linear frequencies of oscillations* or the eigenvalues of the operator J on the circle (see (1.12)).

Let us define

$$w := (1 - \partial_{xx})u, \quad m := c + u - u_{xx}, \quad p = -m^{\frac{1}{3}}. \quad (1.15)$$

One can easily check that the functions

$$H(u) = \int \frac{c u^2}{2} - \frac{u^3}{6} dx, \quad M_0(u) = \frac{1}{2} \int (J^{-1} u_x) u dx, \quad M_1(u) = \int m^{\frac{1}{3}} dx, \quad (1.16)$$

are constant of motions for equation (1.6), i.e. if $u(t, x)$ solves (1.6) then

$$\frac{d}{dt} M_0(u) = \{M_0, H\}(u) = 0, \quad \frac{d}{dt} M_1(u) = \{M_1, H\}(u) = 0. \quad (1.17)$$

We will consider the Sobolev spaces

$$H^s(\mathbb{T}; \mathbb{R}) := \{u(x) \in H_0^1(\mathbb{T}; \mathbb{R}) : \|u\|_{H^s}^2 := \sum_{j \in \mathbb{Z} \setminus \{0\}} |u_j|^2 \langle j \rangle^{2s} < \infty, \bar{u}_j = u_{-j}\} \quad (1.18)$$

where $\langle j \rangle := \sqrt{1+j^2}$ and

$$H^s(\mathbb{R}; \mathbb{R}) := \{u(x) \in H_0^1(\mathbb{R}; \mathbb{R}) : \|u\|_{H^s}^2 := \sum_{k=0}^s \int_{\mathbb{R}} (\partial_x^k u)^2 dx\}. \quad (1.19)$$

We will denote both the spaces (1.18) and (1.19) with H^s in Section 2, since all the arguments hold independently by the x -space. We denote by

$$|u|_{L^\infty} := \sup_x |u(x)| \quad (1.20)$$

the L^∞ -norm either on \mathbb{R} or on \mathbb{T} .

Given a Banach space $(E, \|\cdot\|_E)$ and $r \geq 0$, we denote by

$$B_E(v, r) := \{u \in E : \|u - v\|_E < r\}$$

the open ball centered at $v \in E$ with radius r .

Space of polynomials. When $x \in \mathbb{T}$ it is convenient to introduce a class of polynomials which describes the Hamiltonians in terms of their Fourier coefficients. These definitions will be used in Section 4.

We use the multi-index notation $\alpha \in \mathbb{N}^{\mathbb{Z}}$, $|\alpha| := \sum_j \alpha_j$. We define

- the monomial associated to α : $u^\alpha := \prod_j u_j^{\alpha_j}$.
- the momentum associated to α : $\mathcal{M}(\alpha) = \sum_j j \alpha_j$.
- the divisor associated to α (recall the linear frequencies (1.14)): $\Omega(\alpha) := \sum_j \omega(j) \alpha_j$.

We define the set of indices with zero momentum of order $n \in \mathbb{N}$

$$\mathcal{I}_n := \{\alpha \in \mathbb{N}^{\mathbb{Z}} : |\alpha| = n + 2, \mathcal{M}(\alpha) = 0\}. \quad (1.21)$$

Definition 1.1. We say that $P := (P_\alpha)_{\alpha \in \mathcal{I}_n}$, $P_\alpha \in \mathbb{C}$ for any $\alpha \in \mathcal{I}_n$, is a *formal homogenous polynomial of degree $n + 2$* and we write the (formal) expression $P(u) = \sum_{\alpha \in \mathcal{I}_n} P_\alpha u^\alpha$.

We call $\mathcal{P}^{(n)}$ the space of the formal homogenous polynomial of degree n .

Definition 1.2. We define the space product

$$\mathcal{F} := \prod_{n \geq 0} \mathcal{P}^{(n)}.$$

If $P \in \mathcal{F}$ then we write the (formal) expression $P = \sum_{n=0}^{\infty} P^{(n)}$ where $P^{(n)} \in \mathcal{P}^{(n)}$.

There exists a obvious inclusion of $\mathcal{P}^{(n)}$ into \mathcal{F} given by

$$(P_\alpha)_{\alpha \in \mathcal{I}_n} \mapsto (\underbrace{0, \dots, 0}_{(n-1)\text{times}}, (P_\alpha)_{\alpha \in \mathcal{I}_n}, 0, \dots)$$

and we denote by $\Pi^{(n)} : \mathcal{F} \rightarrow \mathcal{P}^{(n)}$ the projection

$$(\dots, (P_\alpha)_{\alpha \in \mathcal{I}_n}, \dots) \mapsto (P_\alpha)_{\alpha \in \mathcal{I}_n}.$$

We call $\mathcal{P}^{(\leq n)} := \prod_{k=0}^n \mathcal{P}^{(k)}$ the space of the formal polynomials of degree (at most) $n + 2$. As above, $\mathcal{P}^{(\leq n)}$ can be embedded into the space of formal power series \mathcal{F} . We denote by $\Pi^{(\leq n)}$ the projection of \mathcal{F} onto $\mathcal{P}^{(\leq n)}$.

We define $\mathcal{F}^{\geq n} := \prod_{k \geq n} \mathcal{P}^{(k)}$. We denote by $\Pi^{(\geq n)}$ the projection of \mathcal{F} onto $\mathcal{P}^{(\geq n)}$.

In particular, if G is a formal power series we write

$$\Pi^{(n)} G = G^{(n)}, \quad \Pi^{(\leq n)} G = G^{(\leq n)}, \quad \Pi^{(\geq n)} G = G^{(\geq n)}.$$

Birkhoff resonances. Now we introduce the notion of Birkhoff resonances.

Definition 1.3. We say that $\alpha \in \mathcal{I}_n$ is **resonant** if its associated divisor $\Omega(\alpha) = 0$ and we write $\alpha \in \mathcal{N}_n$.

We say that α is **trivially resonant** if $\alpha_j = \alpha_{-j}$ for all j and we write $\alpha \in \mathcal{N}_n^*$. By the fact that $\omega(-j) = -\omega(j)$, if α is trivially resonant then it is resonant and its associated monomials depend only on the **actions** $I_j := |u_j|^2 = u_j u_{-j}$

$$u^\alpha = \prod_{j>0} (|u_j|^2)^{\alpha_j} = \prod_{j>0} I_j^{\alpha_j}.$$

We say that a polynomial which depends only upon the actions I_j is **action-preserving**.

1.2 Main result and applications

The main result of the paper is the following.

Theorem 1.4 (Constants of Motion). Let $c \in \mathbb{R} \setminus \{0\}$. For any $n \geq 1$ there exist a decreasing sequence of positive numbers $(r_n)_{n \geq 1}$, and a sequence of functions $K_n : H^{n+1} \rightarrow \mathbb{R}$ with the following properties:

(0) **Involution:** if we set $K_0 := H$ (see (1.11)) then for any $n \geq 0$ one has that

$$\{H(u), K_n(u)\} = 0. \quad (1.22)$$

- (i) **Analyticity:** the function K_n is analytic on $B_{H^{n+1}}(0, |\mathbf{c}|/2)$; more precisely, there exists a function $\Psi_n: \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ analytic on $B_{\mathbb{C}}(0, |\mathbf{c}|) \times \cdots \times B_{\mathbb{C}}(0, |\mathbf{c}|)$ such that

$$K_n(u) = \int \Psi_n(u, u_x, \dots, \partial_x^{n+1}u) dx.$$

Moreover, Ψ_n admits the following Taylor expansion

$$\Psi_n(u, u_x, \dots, \partial_x^{n+1}u) = \sum_{k \geq 0} \sum_{\substack{\alpha \in \mathbb{N}^{\{0, \dots, n+1\}}, \\ \sum \alpha_i = k, \\ \sum i \alpha_i \leq n+1}} \Psi_\alpha u^{\alpha_0} (\partial_x u)^{\alpha_1} \dots (\partial_x^{n+1}u)^{\alpha_{n+1}}, \quad \Psi_\alpha \in \mathbb{C}. \quad (1.23)$$

- (ii) **Characterization of quadratic parts:** the Taylor polynomial of order 2 of K_n at $u = 0$ has the form

$$\begin{aligned} K_n^{(0)}(u) &= \int_{\mathbb{R}} (\partial_x^{n-1}(u - u_{xx}))^2 dx \quad x \in \mathbb{R}, \\ K_n^{(0)}(u) &= \sum_{j \in \mathbb{Z} \setminus \{0\}} |j|^{2(n-1)} (1 + j^2)^2 |u_j|^2 \quad x \in \mathbb{T}; \end{aligned} \quad (1.24)$$

- (iii) **Control of Sobolev norms:** there exist positive constants $C = C(n, \mathbf{c})$ and $\tilde{c} = \tilde{c}(n, \mathbf{c})$ such that for any $n \geq 1$

$$|K_n^{(0)}(u)| \leq \|u\|_{H^{n+1}}^2 \leq \|u\|_{L^2}^2 + \tilde{c} |K_n^{(0)}(u)| \quad \forall u \in B_{H^{n+1}}(0, |\mathbf{c}|/2) \quad (1.25)$$

and

$$|K_n^{(\geq 1)}(u)| \leq C \|u\|_{H^{n+1}}^3 \quad \forall u \in B_{H^{n+1}}(0, r_n). \quad (1.26)$$

Let us make some comments.

- (1.25) and (1.26) imply that K_n is equivalent to the H^{n+1} -norm in a neighborhood of the origin, and this does not hold as the parameter \mathbf{c} goes to zero (see for instance Remark 2.10 and Remark 2.12).
- We remark that the radius of analyticity of the K_n 's depends only on the parameter \mathbf{c} .
- Our result is based on an explicit computation of the coefficients of the quadratic part of the constructed conserved quantities. We point out that the radii r_n in (1.26) decrease to zero as $n \rightarrow \infty$. It may be possible to improve (1.26) by studying the higher order expansions of the constants of motion.
- By the expression (1.23) the function Ψ_n is affine in the variable $\partial_x^{n+1}u$ (see Remark 2.5). This is a key point to prove the bounds in item (iii).

Let us discuss the applications we obtain by the result above.

We prove the following stability result.

Theorem 1.5 (Stability and Global existence). *Let X be \mathbb{R} or \mathbb{T} . For any $s \geq 2$ there is $r = r(s) > 0$ such that for any $u_0 \in B_{H^s}(0, r)$ there exists a unique solution $u(t, x)$ of (1.6), defined for all times, belonging to $C^0(\mathbb{R}; H^s(X; \mathbb{R}))$ such that*

$$\sup_{t \in \mathbb{R}} \|u(t, x)\|_{H^s} \leq C' r,$$

for some constant $C' = C'(s, \mathbf{c}) > 0$.

The above theorem is in turn based on a local well-posedness result for the equation (1.6) (the proof, which follows [22], is deferred to the Appendix).

The second application concerns the study of the Birkhoff normal form of the equation (1.6).

Theorem 1.6 (Formal Birkhoff normal form). *Let H be the Hamiltonian (1.11) with $x \in \mathbb{T}$. For any $N \in \mathbb{N}$ there exist, at least formally, a symplectic transformation Φ_N such that*

$$H \circ \Phi_N = H^{(0)} + Z_N + R_N \quad (1.27)$$

where $Z_N \in \mathcal{P}^{(\leq N)}$ (recall Definition 1.1) is action-preserving, hence it Poisson commutes with $H^{(0)}$ and depends only on the actions $I_j := |u_j|^2$. The function $R_N \in \mathcal{F}^{(\geq N+1)}$ (recall Definition 1.2).

Definition 1.7. We say that a Hamiltonian $G \in \mathcal{F}$ is in a Birkhoff normal form of order N if it has the form (1.27) described in Theorem 1.6.

In the proof of such result will be fundamental the explicit form of the quadratic part of the constant of motion that we give in (1.24). The proof of Theorem 1.6 is based on the following classical result (see, for instance, [29]):

- two commuting Hamiltonians $H, K \in \mathcal{F}$ can be put in Birkhoff normal form, up to order N , by the same change of coordinates (at least at the formal level).

This fact will be proved in Lemma 4.4, which is a variation of Theorem G.2 in [29], since we do not assume that the linear frequencies are non resonant.

Plan of the paper The paper is organized as follows. In Section 2 we prove Theorem 1.4. In Section 3 we first state a local-well posedness result, which is proved in the Appendix A.1, and then we prove 1.5 by using the bounds (1.25), (1.26) and a bootstrap argument. In Section 4 we give a proof of Theorem 1.6.

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2 Constants of motion

In Section 3 of [12] Degasperis-Holm-Hone give the Lax pair for the equation (1.1), which we write in the following with the choice of the parameters that leads to consider the equation (1.6) (recall the definition of m in (1.15)),

$$\begin{cases} (1 - \partial_{xx})\Psi_x &= m\Psi \\ \Psi_t + \frac{1}{\lambda}\Psi_{xx} + (u + c)\Psi_x - u_x\Psi &= 0 \end{cases}, \quad (2.1)$$

for a real parameter $\lambda \neq 0$. In Section 4 of [12] the authors derive many conservation laws by considering the following relations,

$$(1 - \partial_{xx})\rho = 3\rho\rho_x + \rho^3 + \lambda m, \quad (2.2)$$

and

$$\rho_t = j_x, \quad j = u_x - \frac{1}{\lambda}(\rho_x + \rho^2) - (u + c)\rho, \quad (2.3)$$

for the quantity

$$\rho := \left(\log(p\Psi) \right)_x,$$

where (2.2) comes from the spatial part of the Lax pair (2.1) and (2.3) comes from the time part of the Lax pair (2.1). By (2.3), for any $u(t, x)$ solution of (1.6) defined on some time interval $I \subseteq \mathbb{R}$,

$$\frac{d}{dt} \int \rho(u(t, x)) dx = 0, \quad t \in I. \quad (2.4)$$

In [12] ρ is written as a formal series in powers of the spectral parameter $\lambda = \zeta^{-3}$, $\zeta \in \mathbb{R}$, with the coefficients determined recursively from (2.2). One of the possible expansions is

$$\rho = p\zeta^{-1} + \sum_{n=0}^{\infty} \rho^{(n)}\zeta^n, \quad (2.5)$$

and we are interesting in studying the constants of motion

$$\Gamma^{(n)} := \int \rho^{(n)} dx, \quad n \geq 0, \quad (2.6)$$

given by the series in (2.5). By using (2.5) we have that (2.2) is equivalent to

$$\begin{aligned} 0 &= \rho^{(0)} p^2 + p p_x, \\ p - p_{xx} &= \rho^{(1)} p^2 + 2p(\rho^{(0)})^2 + 3\partial_x(\rho^{(0)} p), \end{aligned} \quad (2.7)$$

$$\begin{aligned} \rho^{(n)} - \rho_{xx}^{(n)} &= \rho^{(n+2)} p^2 + 3 \sum_{k_1+k_2=n+1} p \rho^{(k_1)} \rho^{(k_2)} + \sum_{k_1+k_2+k_3=n} \rho^{(k_1)} \rho^{(k_2)} \rho^{(k_3)} \\ &\quad + 3\partial_x(\rho^{(n+1)} p) + 3 \sum_{k_1+k_2=n} \rho^{(k_1)} \rho_x^{(k_2)}, \quad n \geq 0. \end{aligned} \quad (2.8)$$

From (2.7) we get

$$\rho^{(0)} = -\frac{p_x}{p}, \quad \rho^{(1)} = -\frac{p_x^2}{p^3} + \frac{2p_{xx}}{3p^2} + \frac{1}{3p}. \quad (2.9)$$

Now we want to prove that $\rho^{(n)}(w)$ can be expressed as a power series in the variables w and its derivatives in a small neighborhood of the origin of some H^s Sobolev space. We refer to the Appendix to recall some definitions and facts on analytic functions on Banach spaces.

2.1 Analyticity of composition operators on Sobolev spaces

Lemma 2.1. *Let $n \geq 1$ and $f: \mathbb{C}^n \rightarrow \mathbb{C}$ be an analytic function on $B_{\mathbb{C}}(0, r) \times \cdots \times B_{\mathbb{C}}(0, r)$ for some $r > 0$. Then the composition operator*

$$T_f[u_1, \dots, u_n] = f(u_1, \dots, u_n): B_{H^s(\mathbb{T}, \mathbb{C})}(0, \rho) \times \cdots \times B_{H^s(\mathbb{T}, \mathbb{C})}(0, \rho) \rightarrow H^s, \quad \forall 0 < \rho < r, \quad s > 1/2 \quad (2.10)$$

is weakly analytic on $B_{H^s(\mathbb{T}, \mathbb{C})}(0, r/2) \times \cdots \times B_{H^s(\mathbb{T}, \mathbb{C})}(0, r/2)$ for $s > 1/2$.

Proof. First we want to show that, given $0 < \rho < r$, T_f maps $B_{H^s(\mathbb{T}, \mathbb{C})}(0, \rho) \times \cdots \times B_{H^s(\mathbb{T}, \mathbb{C})}(0, \rho)$ into H^s for $s > 1/2$. Since f is analytic we can write

$$f(z_1, \dots, z_n) = \sum_{k \geq 0} \sum_{\substack{\beta \in \mathbb{N}^n, \\ |\beta|=k}} f_{\beta} z^{\beta}, \quad z^{\beta} := \prod_{i=1}^n z_i^{\beta_i}$$

for some coefficients $f_{\beta} \in \mathbb{C}$ satisfying

$$\sum_{k \geq 0} \sum_{\substack{\beta \in \mathbb{N}^n, \\ |\beta|=k}} |f_{\beta}| \rho^k \leq C, \quad \forall 0 < \rho < r$$

for some constant $C > 0$ depending only on ρ . By using the algebra property of the Sobolev spaces H^s with $s > 1/2$, we have

$$\|T_f[u_1, \dots, u_n]\|_{H^s(\mathbb{T}, \mathbb{C})} \leq \sum_{k \geq 0} \sum_{\substack{\beta \in \mathbb{N}^n, \\ |\beta|=k}} |f_{\beta}| \|u_1\|_{H^s(\mathbb{T}, \mathbb{C})}^{\beta_1} \cdots \|u_n\|_{H^s(\mathbb{T}, \mathbb{C})}^{\beta_n}$$

and the claim follows. In order to prove the weak analyticity of the operator T_f , we have to show that for all $w_i \in B_{H^s(\mathbb{T}, \mathbb{C})}(0, r/2)$, $h_i \in H^s(\mathbb{T}, \mathbb{C})$, $i = 1, \dots, n$ and $L \in (H^s(\mathbb{T}, \mathbb{C}))^*$ the function $(z_1, \dots, z_n) \mapsto$

$LT_f(w_1 + z_1 h_1, \dots, w_n + z_n h_n)$ is analytic in a neighborhood of the origin of \mathbb{C}^n . By Riesz theorem, for every $L \in (H^s(\mathbb{T}, \mathbb{C}))^*$ there exists a function $g \in H^s(\mathbb{T}, \mathbb{C})$ such that

$$\begin{aligned} LT_f(w_1 + z_1 h_1, \dots, w_n + z_n h_n) &= \sum_{m=0}^s \int \partial_x^m \left(f(w_1(x) + z_1 h_1(x), \dots, w_n(x) + z_n h_n(x)) \right) \partial_x^m g(x) dx \\ &= \int f(w_1(x) + z_1 h_1(x), \dots, w_n(x) + z_n h_n(x)) g(x) dx \\ &+ \sum_{m=1}^s \sum_{k=1}^m \sum_{\substack{\mathbf{v} \in \mathcal{A}_{m,k}, \\ \mathbf{p} \in \mathcal{B}_m(\mathbf{v})}} C_{\mathbf{p}} \int (D^{\mathbf{v}} f) \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \partial_x^m g(x) dx \end{aligned} \quad (2.11)$$

where $C_{\mathbf{p}}$ are combinatorial factors,

$$\begin{aligned} \mathcal{A}_{m,k} &:= \{ \mathbf{v} \in \{0, \dots, m\}^n : |\mathbf{v}| := \sum_{i=1}^n \mathbf{v}_i = k \}, \\ \mathcal{B}_r(\mathbf{v}) &:= \{ \mathbf{p} = (\mathbf{p}^{(1)}, \dots, \mathbf{p}^{(n)}), \mathbf{p}^{(i)} \in \{1, \dots, m\}^{|\mathbf{v}_i|}, \sum_{i=1}^n |\mathbf{p}^{(i)}| = r \} \quad \text{for } \mathbf{v} \in \mathcal{A}_{m,k} \end{aligned}$$

and

$$D^{\mathbf{v}} f := \partial_{\mathbf{v}_1 \dots \mathbf{v}_n} f = \frac{\partial^k}{\partial_{z_1}^{\mathbf{v}_1} \dots \partial_{z_n}^{\mathbf{v}_n}}, \quad \partial_i := \partial_{z_i}.$$

The last summand in (2.11) can be written as

$$\begin{aligned} &\sum_{m=1}^{s-1} \sum_{k=1}^m \sum_{\substack{\mathbf{v} \in \mathcal{A}_{m,k}, \\ \mathbf{p} \in \mathcal{B}_m(\mathbf{v})}} C_{\mathbf{p}} \int (D^{\mathbf{v}} f) \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \partial_x^m g(x) dx \\ &+ \sum_{k=2}^s \sum_{\substack{\mathbf{v} \in \mathcal{A}_{m,k}, \\ \mathbf{p} \in \mathcal{B}_s(\mathbf{v})}} C_{\mathbf{p}} \int (D^{\mathbf{v}} f) \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \partial_x^s g(x) dx \\ &+ \sum_{i=1}^n \int (\partial_i f) (\partial_x^s w_i + z_i \partial_x^s h_i) \partial_x^s g(x) dx. \end{aligned} \quad (2.12)$$

We want to prove that there exist the derivatives in the complex variable z_i of the function $LT_f(w_1 + z_1 h_1, \dots, w_n + z_n h_n)$: to do that it is sufficient to prove that the derivative in z_i of the integrands in (2.11) is L^1 uniformly in the parameter z , since by dominated convergence we can pass the derivative inside the integral and use the analyticity

of f . Hence we now show that the following sum

$$\begin{aligned}
& \int |(\partial_{z_\xi} f) h_\xi g(x)| dx + \sum_{m=1}^{s-1} \sum_{k=1}^m \sum_{\substack{\mathbf{v} \in \mathcal{A}_{m,k}, \\ \mathbf{p} \in \mathcal{B}_m(\mathbf{v})}} C_{\mathbf{p}} \int |(D^{\mathbf{v}+\mathbf{e}_\xi} f) h_\xi \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \partial_x^m g(x)| dx \\
& + \sum_{m=1}^{s-1} \sum_{k=1}^m \sum_{\substack{\mathbf{v} \in \mathcal{A}_{m,k}, \\ \mathbf{p} \in \mathcal{B}_m(\mathbf{v})}} C_{\mathbf{p}} \int |(D^{\mathbf{v}} f) \prod_{i=1, i \neq \xi}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \\
& \quad \times \left(\sum_{j=1}^{|\mathbf{v}_\xi|} \prod_{b \neq j} (\partial_x^{\mathbf{p}_b^{(\xi)}} w_\xi + z_\xi \partial_x^{\mathbf{p}_b^{(\xi)}} h_\xi) \partial_x^{\mathbf{p}_j^{(\xi)}} h_\xi \right) \partial_x^m g(x)| dx \\
& + \sum_{k=2}^s \sum_{\substack{\mathbf{v} \in \mathcal{A}_{m,k}, \\ \mathbf{p} \in \mathcal{B}_s(\mathbf{v})}} C_{\mathbf{p}} \int |(D^{\mathbf{v}+\mathbf{e}_\xi} f) h_\xi \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \partial_x^s g(x)| dx \\
& + \sum_{k=2}^s \sum_{\substack{\mathbf{v} \in \mathcal{A}_{m,k}, \\ \mathbf{p} \in \mathcal{B}_s(\mathbf{v})}} C_{\mathbf{p}} \int |(D^{\mathbf{v}} f) \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \\
& \quad \times \left(\sum_{j=1}^{|\mathbf{v}_\xi|} \prod_{b \neq j} (\partial_x^{\mathbf{p}_b^{(\xi)}} w_\xi + z_\xi \partial_x^{\mathbf{p}_b^{(\xi)}} h_\xi) \partial_x^{\mathbf{p}_j^{(\xi)}} h_\xi \right) \partial_x^s g(x)| dx \\
& + \sum_{i=1}^n \int |(\partial_{z_i} f) h_\xi (\partial_x^s w_i + z_i \partial_x^s h_i) \partial_x^s g| dx + \int |(\partial_\xi f) \partial_x^s h_\xi \partial_x^s g| dx
\end{aligned} \tag{2.13}$$

is finite for some $\xi \in \{1, \dots, n\}$. Fix $|z_i| < \min\{r/2 \|h_i\|_{H^s(\mathbb{T}, \mathbb{C})}, r/2\}$ for $i = 1, \dots, n$. First we bound the derivatives of f

$$|(D^{\mathbf{v}} f)(w_1 + z_1 h_1, \dots, w_n + z_n h_n)|_{L^\infty} < \infty \quad \forall w_i \in B_{H^s(\mathbb{T}, \mathbb{C})}(0, r/2), \quad h_i \in H^s(\mathbb{T}, \mathbb{C})$$

since the derivatives of f are analytic on $B_{\mathbb{C}}(0, r) \times \dots \times B_{\mathbb{C}}(0, r)$ and

$$|w_i + z_i h_i|_{L^\infty} < r, \quad i = 1, \dots, n.$$

Since w_i and h_i belong to $H^s(\mathbb{T}, \mathbb{C})$ then, for $1 \leq k \leq m$, $1 \leq m \leq s$, $\mathbf{v} \in \mathcal{A}_{m,k}$, $\mathbf{p} \in \mathcal{B}_m(\mathbf{v})$ we have by Cauchy-Schwarz and Sobolev embeddings

$$\begin{aligned}
& \int \left| \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) h_\xi \partial_x^m g(x) \right| dx \leq \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} \|(\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i)\|_{L^\infty(\mathbb{T}, \mathbb{C})} \|h_\xi\|_{L^2(\mathbb{T}, \mathbb{C})} \|\partial_x^m g\|_{L^2(\mathbb{T}, \mathbb{C})} \\
& \leq \prod_{i=1}^n \prod_{j=1}^{|\mathbf{v}_i|} (\|\partial_x^{\mathbf{p}_j^{(i)}} w_i\|_{H^1(\mathbb{T}, \mathbb{C})} + |z_i| \|\partial_x^{\mathbf{p}_j^{(i)}} h_i\|_{H^1(\mathbb{T}, \mathbb{C})}) \|h_\xi\|_{L^2(\mathbb{T}, \mathbb{C})} \|\partial_x^m g\|_{L^2(\mathbb{T}, \mathbb{C})} \leq r^m \|h_\xi\|_{L^2(\mathbb{T}, \mathbb{C})} \|g\|_{H^s(\mathbb{T}, \mathbb{C})}
\end{aligned}$$

and similarly

$$\begin{aligned}
& \int \left| \prod_{i=1, i \neq \xi}^n \prod_{j=1}^{|\mathbf{v}_i|} (\partial_x^{\mathbf{p}_j^{(i)}} w_i + z_i \partial_x^{\mathbf{p}_j^{(i)}} h_i) \left(\sum_{j=1}^{|\mathbf{v}_\xi|} \prod_{b \neq j} (\partial_x^{\mathbf{p}_b^{(\xi)}} w_\xi + z_\xi \partial_x^{\mathbf{p}_b^{(\xi)}} h_\xi) \partial_x^{\mathbf{p}_j^{(\xi)}} h_\xi \right) \partial_x^m g(x) \right| dx \\
& \leq m r^{m-1} \|h_\xi\|_{L^2(\mathbb{T}, \mathbb{C})} \|g\|_{H^s(\mathbb{T}, \mathbb{C})}.
\end{aligned}$$

We bounded the first three terms in (2.13). The fourth and the fifth terms in (2.13) have similar bounds and the proof follows the arguments above; we remark only that the Cauchy-Schwarz inequality has to be applied to the

L^2 -product of $\partial_x^s g$ with $\partial_x^{\mathbf{P}_j^{(\xi)}} h_\xi$. For the last two summands of (2.13) we have

$$\begin{aligned} \int |h_r(\partial_x^s w_i + z_i \partial_x^s h_i) \partial_x^s g| dx &\leq |h_r|_{L^\infty(\mathbb{T}, \mathbb{C})} \|\partial_x^s w_i + z_i \partial_x^s h_i\|_{L^2(\mathbb{T}, \mathbb{C})} \|\partial_x^s g\|_{L^2(\mathbb{T}, \mathbb{C})} \\ &\leq |h_r|_{L^\infty(\mathbb{T}, \mathbb{C})} (\|w_i\|_{H^s(\mathbb{T}, \mathbb{C})} + |z_i| \|h_i\|_{H^s(\mathbb{T}, \mathbb{C})}) \|g\|_{H^s(\mathbb{T}, \mathbb{C})} \leq r \|h_r\|_{H^1(\mathbb{T}, \mathbb{C})} \|g\|_{H^s(\mathbb{T}, \mathbb{C})} \end{aligned}$$

and

$$\int |\partial_x^s h_\xi \partial_x^s g| dx \leq \|h_\xi\|_{H^s(\mathbb{T}, \mathbb{C})} \|g\|_{H^s(\mathbb{T}, \mathbb{C})}.$$

□

Lemma 2.2. *Let $\sigma \in \mathbb{N}$ and $f: \mathbb{C} \rightarrow \mathbb{C}$ be analytic on a ball $B_{\mathbb{C}}(0, r)$. Then there exists a function $g: \mathbb{C}^{\sigma+1} \rightarrow \mathbb{C}$ analytic on $B_{\mathbb{C}}(0, r) \times \cdots \times B_{\mathbb{C}}(0, r)$ such that $(\partial_x^\sigma \circ T_f)(w)$ is the restriction to*

$$w_0 = w, \dots, w_\sigma = \partial_x^\sigma w$$

of the composition operator $T_g[w_0, \dots, w_\sigma]: H^s(\mathbb{T}, \mathbb{C}) \times \cdots \times H^s(\mathbb{T}, \mathbb{C}) \rightarrow H^s(\mathbb{T}, \mathbb{C})$ for $s > 1/2$. Moreover T_g is analytic on $B_{H^s(\mathbb{T}, \mathbb{C})}(0, r/2) \times \cdots \times B_{H^s(\mathbb{T}, \mathbb{C})}(0, r/2)$ for $s > 1/2$.

Proof. By the chain rule

$$\partial_x^\sigma f(w) = \sum_{k=1}^{\sigma} \sum_{p_1 + \cdots + p_k = \sigma} C_k f^{(k)}(w) (\partial_x^{p_1} w) \cdots (\partial_x^{p_k} w), \quad (2.14)$$

hence the function $\partial_x^\sigma \circ T_f$ is the restriction of the composition operator T_g on $w_0 = w, \dots, w_\sigma = \partial_x^\sigma w$ for a function $g = g(z_0, \dots, z_\sigma): \mathbb{C}^{\sigma+1} \rightarrow \mathbb{C}$ which has the form

$$\sum_{k=1}^{\sigma} \sum_{p_1 + \cdots + p_k = \sigma} C_k f^{(k)}(z_0) z_{p_1} \cdots z_{p_k} = \sum_{k=1}^{\sigma} \tilde{C}_k f^{(k)}(z_0) z_1^{\alpha_1^{(k)}} \cdots z_\sigma^{\alpha_\sigma^{(k)}}$$

for some $\alpha_i^{(k)} \in \mathbb{N}$ and some positive constants \tilde{C}_k .

The function g is clearly analytic on $B_{\mathbb{C}}(0, r) \times \cdots \times B_{\mathbb{C}}(0, r)$ and we have the weakly analyticity of T_g by using Lemma 2.1. The fact that T_g is locally bounded as operator from $H^s(\mathbb{T}, \mathbb{C}) \times \cdots \times H^s(\mathbb{T}, \mathbb{C})$ to $H^s(\mathbb{T}, \mathbb{C})$ follows trivially by the following estimate, obtained by exploiting the algebra property of $H^s(\mathbb{T}, \mathbb{C})$ with $s > 1/2$ and the analyticity of f ,

$$\|\partial_x^\sigma f(w)\|_{H^s(\mathbb{T}, \mathbb{C})} \leq \sum_{k=1}^{\sigma} \sum_{p_1 + \cdots + p_k = \sigma} C_k \|f^{(k)}(w_0)\|_{H^s(\mathbb{T}, \mathbb{C})} \|w_{p_1}\|_{H^s(\mathbb{T}, \mathbb{C})} \cdots \|w_{p_k}\|_{H^s(\mathbb{T}, \mathbb{C})}.$$

□

The function $p(y) = -(c + y)^{1/3}$ is analytic in $\{y \in \mathbb{C} : |y| < |c|\}$, hence by Lemma 2.1 the map $p(w) = T_p[w] = -(c + w)^{1/3}$ defined in (1.15) is weakly analytic in $B_{H^s(\mathbb{T}, \mathbb{C})}(0, |c|/2)$. Moreover T_p is locally bounded, hence $p(w)$ is analytic in $B_{H^s(\mathbb{T}, \mathbb{C})}(0, |c|/2)$ and it can be represented by its Taylor expansion at the origin

$$p(w) = \sum_{n \geq 0} \frac{p^{(n)}(0)}{n!} w^n. \quad (2.15)$$

Remark 2.3. We note that the function $p(y) = -(c + y)^{1/3}$ is real on real, namely it assumes real valued when it is restricted to the real line. Then its restriction to \mathbb{R} is a real analytic function.

As a consequence, it is easy to see that the composition operator T_p is real on real and then it is analytic on $H^s := H^s(X, \mathbb{R})$, $X = \mathbb{T}, \mathbb{R}$.

2.1.1 Class of differential polynomials

We introduce a class of differential polynomials to which the Taylor expansion of the $\rho^{(n)}$ belongs. The particular form of these polynomial results to be fundamental for the Sobolev estimates on the constants of motion which we construct.

We define

$$\mathcal{J}_n^q := \{\alpha \in \mathbb{N}^{\{0, \dots, n\}} : \sum_{i=0}^n \alpha_i = q, \sum_{i=0}^n i\alpha_i \leq n\} \quad (2.16)$$

and for $\alpha \in \mathcal{J}_n^q$, $w = (w_0, \dots, w_n)$, $w_i := \partial_x^i w$ the monomial

$$w^\alpha = \prod_{i=0}^n w_i^{\alpha_i} = \prod_{i=0}^n (\partial_x^i w)^{\alpha_i}. \quad (2.17)$$

We denote by \mathcal{P}_n^q the class of formal homogenous polynomials of degree q and order n of the form

$$f = \sum_{\alpha \in \mathcal{J}_n^q} f_\alpha w^\alpha, \quad f_\alpha \in \mathbb{C}.$$

We denote by $\mathcal{P}_n^{\leq q}$ the class of formal polynomials of degree at most q and order n of the form

$$f = \sum_{k=0}^q f_k, \quad f_k \in \mathcal{P}_n^k.$$

We denote by Σ_n^q the class of formal power series of degree at least q of the form

$$f = \sum_{k=q}^{\infty} f_{n,k}, \quad f_{n,k} \in \mathcal{P}_n^k.$$

The Taylor series (2.15) is an element of Σ_0^0 .

Lemma 2.4. *Let $\sigma, n, m, q, r \in \mathbb{N}$. Then*

1. *If $f \in \mathcal{P}_n^q$, $g \in \mathcal{P}_m^r$ then*

$$f + g \in \mathcal{P}_{\max\{n, m\}}^{\leq \max\{q, r\}}, \quad f g \in \mathcal{P}_{\max\{n, m\}}^{\leq q+r}.$$

2. *The operator ∂_x^σ maps \mathcal{P}_n^q into $\mathcal{P}_{n+\sigma}^q$.*

Proof. *Proof of (1):* for the sum the proof is trivial. For the product, suppose that $m \geq n$, the claim follows by the fact that

$$w_0^{\alpha_0} \dots w_n^{\alpha_n} w_0^{\beta_0} \dots w_m^{\beta_m} = \prod_{i=0}^n w_i^{\alpha_i + \beta_i} w_{n+1}^{\beta_{n+1}} \dots w_m^{\beta_m}$$

where $|\alpha| = q$ and $|\beta| = r$.

Proof of (2): clearly it is sufficient to look at the action of ∂_x^σ on the monomials. Fixed $i \in \mathbb{N}$, we have that

$$\partial_x^p w_i = w_{i+p} \quad \text{for } p \in \mathbb{N}$$

and by the chain rule

$$\partial_x^j w_i^{\alpha_i} = \sum_{k=1}^j \sum_{p_1 + \dots + p_k = j} C_k w_i^{\alpha_i - k} (\partial_x^{p_1} w_i) \dots (\partial_x^{p_k} w_i) = \sum_{k=1}^j \sum_{p_1 + \dots + p_k = j} C_k w_i^{\alpha_i - k} w_{i+p_1} \dots w_{i+p_k}$$

is a function of variables w_i, \dots, w_{i+j} . It is easy to see that, in these variables, the degree of homogeneity has not been changed, namely it is already α_i . Hence

$$\partial_x^\sigma w^\alpha = \sum_{j_0 + \dots + j_n = \sigma} C_{j_0 \dots j_n} (\partial_x^{j_0} w_0^{\alpha_0}) \dots (\partial_x^{j_n} w_n^{\alpha_n})$$

is a function of the variables $w_0, \dots, w_{n+\sigma}$ with homogeneity degree $\alpha_0 + \dots + \alpha_n = q$. \square

The following remark is fundamental for getting bounds on the Sobolev norms of the constants of motion.

Remark 2.5. Let $f \in \Sigma_n^q$ for some $n, q \geq 0$. We note that f is necessarily affine in the variable $w_n = \partial_x^n w$, namely in the highest order derivative. Indeed, $\mathcal{M}(\alpha) = n = \sum_{i=0}^n i\alpha_i$ for $\alpha \in \mathcal{J}_n^q$, hence

$$\alpha_0 = q - 1, \quad \alpha_i = 0, \quad \alpha_n = 1, \quad i = 1, \dots, n-1, \quad \text{or} \quad \alpha_n = 0.$$

So

$$f = w_n \sum_{k \geq q} f_{(k-1, 0, \dots, 0, 1)} w^k + \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^q \\ \alpha_n = 0}} f_\alpha w^\alpha.$$

2.2 The structure of the conserved quantities $\rho^{(n)}$

The functions $\rho^{(n)}$ are given by sums and products of p and its x -derivatives. We want to show that the $\rho^{(n)}$ are composition operators for analytic and real on real functions of w and its derivatives and that the Taylor expansions of these operators belong to some Σ_n^q (recall the definitions given in Section 2.1.1). This fact will allow to prove the bound (2.21) and consequently the estimates in Theorem 1.4-(iii).

Remark 2.6. Given two composition operators T_f and T_g we have that $T_f + T_g = T_{f+g}$ and $T_f T_g = T_{fg}$. Hence if f, g are analytic we can apply to $T_f + T_g = T_{f+g}$ and $T_f T_g = T_{fg}$ Lemmata 2.1, 2.2.

Lemma 2.7. Fix $n \in \mathbb{N}$. Then there exists a function $f_n : \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ real on real, analytic on $B_{\mathbb{C}}(0, |c|) \times \dots \times B_{\mathbb{C}}(0, |c|)$ such that

$$\rho^{(n)}(w) = T_{f_n}[w, w_x, \dots, \partial_x^{n+1} w].$$

Moreover the Taylor series of T_{f_n} restricted to $w_0 = w, \dots, w_{n+1} = \partial_x^{n+1} w$ belongs to Σ_{n+1}^0 .

Proof. Let us start from $\rho^{(0)}$ and $\rho^{(1)}$, and then we argue by induction on n .

Recalling that p is analytic as function of the variable w , by Lemma 2.2 p_x is analytic as function of the two variables w, w_x . By (2.9), we have that $\rho^{(0)} = \frac{1}{3}(-c - w)^{-1} w_x$, and since the function $f_0 : \mathbb{C}^2 \rightarrow \mathbb{C}$ given by

$$f_0(z_0, z_1) := \frac{1}{3}(-c - z_0)^{-1} z_1$$

is real on real and analytic in $B_{H^s}(0, |c|) \times B_{H^s}(0, |c|)$, by Lemma 2.1 and by local boundedness of the operator T_{f_0} we get that $\rho^{(0)}$ is analytic in $B_{H^s}(0, |c|/2) \times B_{H^s}(0, |c|/2)$. From the explicit formula of f_0 one can deduce that the Taylor series of T_{f_0} restricted to $w_0 = w, w_1 = w_x$ belongs to Σ_1^0 .

Similarly, we obtain that $\rho^{(1)}$ is real on real and analytic in the variables w, w_x and w_{xx} , since it can be written as a composition operator for the following analytic function

$$f_1(z_0, z_1, z_2) := -\frac{1}{9}(-c - z_0)^{-7/3} z_1^2 + \frac{2}{3} \left(-\frac{2}{9} \frac{z_1^2}{(-c - z_0)^{5/3}} - \frac{1}{3}(-c - z_0)^{-2/3} z_2 \right) + \frac{1}{3}(-c - z_0)^{-1/3}.$$

Furthermore, from the explicit formula of f_1 one can deduce that the Taylor series of T_{f_1} restricted to $w_0 = w, \dots, w_2 = w_{xx}$ belongs to Σ_2^0 .

Now we assume that the thesis holds for $\rho^{(k)}$, $k \leq n+1$, and we only have to control that $\rho^{(n+2)}$ depends only on $w, w_x, \dots, \partial_x^{n+3} w$. But by recalling (2.8), we just observe that

- $\rho^{(n)}(w) = T_{f_n}[w, \dots, \partial_x^{n+1} w]$ for some $f_n : \mathbb{C}^{n+2} \rightarrow \mathbb{C}$ analytic on $\times_{i=1}^{n+2} B_{\mathbb{C}}(0, |c|)$, by inductive hypothesis;

- $\rho_{xx}^{(n)}(w) = T_{g_n}[w, \dots, \partial_x^{n+3}w]$ for some $g_n : \mathbb{C}^{n+4} \rightarrow \mathbb{C}$ analytic on $\times_{i=1}^{n+4} B_{\mathbb{C}}(0, |c|)$, by Lemma 2.2 and by inductive hypothesis;
- $p(w)\rho^{(k_1)}(w)\rho^{(k_2)}(w) = T_{h_{k_1, k_2}}[w, \dots, \partial_x^{\max(k_1, k_2)+1}w]$ (where $k_1 + k_2 = n + 1$), for some $h_{k_1, k_2} : \mathbb{C}^{\max(k_1, k_2)+2} \rightarrow \mathbb{C}$ analytic on $\times_{i=1}^{\max(k_1, k_2)+2} B_{\mathbb{C}}(0, |c|)$, by inductive hypothesis and by the above remark;
- $\rho^{(k_1)}(w)\rho^{(k_2)}(w)\rho^{(k_3)}(w) = T_{l_{k_1, k_2, k_3}}[w, \dots, \partial_x^{\max(k_1, k_2, k_3)+1}w]$ ($k_1 + k_2 + k_3 = n$), for some $l_{k_1, k_2, k_3} : \mathbb{C}^{\max(k_1, k_2, k_3)+2} \rightarrow \mathbb{C}$ analytic on $\times_{i=1}^{\max(k_1, k_2, k_3)+2} B_{\mathbb{C}}(0, |c|)$, by inductive hypothesis and by the above remark;
- $\partial_x(\rho^{(n+1)}p)(w) = (\partial_x \rho^{(n+1)})(w)p(w) + \rho^{(n+1)}(w)p_x(w) = T_{m_{n+1}}[w, \dots, \partial_x^{n+3}w]$ for some $m_{n+1} : \mathbb{C}^{n+4} \rightarrow \mathbb{C}$ analytic on $\times_{i=1}^{n+4} B_{\mathbb{C}}(0, |c|)$, by inductive hypothesis and by the above remark;
- $\rho^{(k_1)}(w)\rho_x^{(k_2)}(w) = T_{v_{k_1, k_2}}[w, \dots, \partial_x^{\max(k_1, k_2+1)+1}w]$ ($k_1 + k_2 = n$), for some $v_{k_1, k_2} : \mathbb{C}^{\max(k_1, k_2+1)+2} \rightarrow \mathbb{C}$ analytic on $\times_{i=1}^{\max(k_1, k_2+1)+2} B_{\mathbb{C}}(0, |c|)$, by inductive hypothesis and by the above remark.

Furthermore, again by using formula (2.8), we have that

- the Taylor series of T_{f_n} restricted to $w_0 = w, \dots, w_{n+1} = \partial_x^{n+1}w$ belongs to Σ_{n+1}^0 , by inductive hypothesis;
- the Taylor series of T_{g_n} restricted to $w_0 = w, \dots, w_{n+3} = \partial_x^{n+3}w$ belongs to Σ_{n+3}^0 , by inductive hypothesis and by Lemma 2.4;
- the Taylor series of $T_{h_{k_1, k_2}}$ ($k_1 + k_2 = n + 1$) restricted to $w_0 = w, \dots, w_{\max(k_1, k_2)+1} = \partial_x^{\max(k_1, k_2)+1}w$ belongs to Σ_{n+2}^0 , by inductive hypothesis and by Lemma 2.4;
- the Taylor series of $T_{l_{k_1, k_2, k_3}}$ ($k_1 + k_2 + k_3 = n$) when restricted to $w_0 = w, \dots, w_{\max(k_1, k_2, k_3)+1} = \partial_x^{\max(k_1, k_2, k_3)+1}w$ belongs to Σ_{n+1}^0 , by inductive hypothesis and by Lemma 2.4;
- the Taylor series of $T_{m_{n+1}}$ restricted to $w_0 = w, \dots, w_{n+3} = \partial_x^{n+3}w$ belongs to Σ_{n+3}^0 , by inductive hypothesis and by Lemma 2.4;
- the Taylor series of $T_{v_{k_1, k_2}}$ ($k_1 + k_2 = n$) restricted to $w_0 = w, \dots, w_{\max(k_1, k_2+1)+1} = \partial_x^{\max(k_1, k_2+1)+1}w$ belongs to Σ_{n+2}^0 , by inductive hypothesis and by Lemma 2.4.

This implies that the Taylor series of $T_{f_{n+2}}$ restricted to $w_0 = w, \dots, w_{n+3} = \partial_x^{n+3}w$ belongs to Σ_{n+3}^0 . \square

By Remark 2.3 the composition operators $\rho^{(n)}$ are real analytic and by Theorem A.11 $\rho^{(n)}(w)$ can be represented by their Taylor expansion at the origin as functions of $w_0 := w, \dots, w_n := \partial_x^n w$ if w belongs to a sufficiently small ball of H^{s+n} centered at the origin. For instance we can write

$$p = -c^{1/3} - \frac{1}{3c^{2/3}}w + \frac{1}{9c^{5/3}}w^2 + g_0(w), \quad (2.18)$$

$$\rho^{(0)} = -\frac{w_x}{3c} + \frac{ww_x}{3c^2} + g_1(w, w_x), \quad (2.19)$$

$$\rho^{(1)} = -\frac{1}{3c^{1/3}} + \frac{1}{9c^{4/3}}w - \frac{2}{9c^{4/3}}w_{xx} - \frac{2}{27c^{7/3}}w^2 + \frac{8}{27c^{7/3}}ww_{xx} + \frac{7}{27c^{7/3}}w_x^2 + g_2(w, w_x, w_{xx}), \quad (2.20)$$

where g_0, g_1 and g_2 have a zero of order 3 at the origin.

We remark that $\Gamma^{(n)}$ defined in (2.6) is the integral (on the torus \mathbb{T} or on \mathbb{R}) of elements of Σ_n^q . In the following lemma we prove an estimate on Sobolev spaces for this kind of functions.

Proposition 2.8. Fix $n \in \mathbb{N}$ and let $F(w) := \int f(w, \dots, \partial_x^n w) dx$, where $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is real on real, analytic on $B_{\mathbb{C}}(0, r) \times \dots \times B_{\mathbb{C}}(0, r)$ for some $r > 0$ and the Taylor expansion of $f(w, \dots, \partial_x^n w)$ at the origin belongs to Σ_n^q . Then $F : H^n \rightarrow \mathbb{C}$ is analytic on $B_{H^n}(0, r/2)$ and the following estimate holds

$$|F(w)| \leq C(n, r) \|w\|_{H^n}^q \quad (2.21)$$

Proof. First we prove the bound (2.21), which implies also that F is locally bounded on $B_{H^n}(0, r)$. We note that, since the Taylor series of $f(w, \dots, \partial_x^n w)$ belongs to Σ_n^q , we can write by Remark 2.5

$$F(w) = \int \sum_{k \geq q} \sum_{\alpha \in \mathcal{J}_n^k} f_\alpha w^\alpha dx = \int_{\mathbb{T}} w_n \sum_{k \geq q} f_{(k-1, 0, \dots, 0, 1)} w^k dx + \int \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^k, \\ \alpha_n = 0}} f_\alpha w^\alpha dx.$$

Hence by Cauchy-Schwarz and Sobolev embeddings

$$\begin{aligned} |F(w)| &\leq \sum_{k \geq q} |f_{(k-1, 0, \dots, 0, 1)}| \int |\partial_x^n w| |w^{k-1}| dx + \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^k, \\ \alpha_n = 0}} |f_\alpha| \int |w^\alpha| dx \\ &\leq \sum_{k \geq q} |f_{(k-1, 0, \dots, 0, 1)}| \|w\|_{L^\infty}^{k-2} \int |\partial_x^n w| |w| dx + \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^k, \\ \alpha_n = 0}} |f_\alpha| \prod_{i=0}^{k-2} \|w\|_{H^{i+1}} \|w\|_{H^{n-1}}^2 \\ &\leq \sum_{k \geq q} |f_{(k-1, 0, \dots, 0, 1)}| \|w\|_{L^\infty}^{k-2} \|w\|_{H^n} \|w\|_{L^2} + \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^k, \\ \alpha_n = 0}} |f_\alpha| \|w\|_{H^n}^k \\ &\leq \sum_{k \geq q} |f_{(k-1, 0, \dots, 0, 1)}| \|w\|_{H^1}^{k-1} \|w\|_{H^n} + \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^k, \\ \alpha_n = 0}} |f_\alpha| \|w\|_{H^n}^k \\ &\leq \|w\|_{H^n}^q \left(\sum_{k \geq q} |f_{(k-1, 0, \dots, 0, 1)}| \|w\|_{H^n}^{k-q} + \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^k, \\ \alpha_n = 0}} |f_\alpha| \|w\|_{H^n}^{k-q} \right) \\ &\leq \|w\|_{H^n}^q \sum_{k \geq q} \sum_{\alpha \in \mathcal{J}_n^k} |f_\alpha| r^{k-q} \leq r^{-q} \|w\|_{H^n}^q \sum_{k \geq q} \sum_{\alpha \in \mathcal{J}_n^k} |f_\alpha| r^k \leq \frac{C(f, n, r)}{r^q} \|w\|_{H^n}^q \end{aligned} \quad (2.22)$$

Now we prove the weakly analyticity of F . Since $\mathbb{C}^* = \mathbb{C}$ it is sufficient to show that for every $w \in B_{H^s}(0, r/2)$ and $h \in H^s$ there exists $a = a(h) > 0$ such that the function $z \mapsto F(w + zh)$ is analytic as function of a complex variable in $B_{\mathbb{C}}(0, a)$.

If $|z| < \min\{\frac{r}{2\|h\|_{H^n}}, \frac{r}{2}\}$ then

$$\sup_x (|\partial_x^i w| + |z \partial_x^i h|) < r, \quad i = 0, \dots, n-1 \quad (2.23)$$

since $w \in H^n$. The proof follows the strategy adopted in the proof of Lemma 2.1, namely we isolate the terms with the pair of functions in the integrands with the highest order of derivatives and we apply the Cauchy-Schwarz inequality to their L^2 -scalar product. We remark that is fundamental, as for obtaining the bound (2.22), that $\partial_x^n w$ appears linearly in the Taylor expansion of F . Indeed by this fact it is sufficient to require the condition (2.23) for $i \leq n-1$ and the loss of regularity due to the Sobolev embedding does not force us to require more smoothness on w than $w \in H^n$. Eventually we have

$$\begin{aligned} \frac{d}{dz} F(w + zh) &= \int (w_n + zh_n) \frac{d}{dz} \sum_{k \geq q} f_{(k-1, 0, \dots, 0, 1)} (w + zh)^k dx \\ &\quad + \int h_n \sum_{k \geq q} f_{(k-1, 0, \dots, 0, 1)} (w + zh)^k dx \\ &\quad + \int \frac{d}{dz} \sum_{k \geq q} \sum_{\substack{\alpha \in \mathcal{J}_n^k, \\ \alpha_n = 0}} f_\alpha (w + zh)^\alpha dx \end{aligned}$$

and these derivatives exist by the analyticity of f on $B_{\mathbb{C}}(0, r) \times \dots \times B_{\mathbb{C}}(0, r)$. \square

Now we prove two facts (recall (2.6) for the definition of $\Gamma^{(n)}$):

- the quadratic terms in the expansion of $\Gamma^{(n)}$ have a particular form ;
- the cubic remainder of the expansion of $\Gamma^{(n)}$ does not contain derivatives of w of order greater than the ones appearing in the quadratic part (see Section 2.3). We will do that by showing that the coefficient of the quadratic part associated to the monomials containing the highest number of derivatives is non-zero.

Remark 2.9. We point out the following properties of differential polynomial in \mathcal{P}_n^k .

(i) Let $g_0 \in \mathcal{P}_n^1$ for some n , and recall that $\int w dx = 0$. Then one has

$$\int g_0 dx = 0;$$

(ii) let f be a polynomial of degree 2 depending on the derivatives of w of order exactly $n = 2k + 1$ for $k \in \mathbb{N}$, then

$$\int f dx = 0.$$

Indeed, by (2.16), (2.17) we need to show that

$$\int (\partial_x^{k_1} w)(\partial_x^{k_2} w) dx = 0,$$

when $k_1 + k_2 = 2k + 1$ and at least one between k_1 and k_2 is ≥ 1 . Assume $k_2 \geq 1$. Hence by integrating by parts we have, for $\sigma = 1$ or $\sigma = -1$

$$\int (-1)^\sigma (\partial_x^k w)(\partial_x^{k+1} w) dx = \int (-1)^\sigma \partial_x [(\partial_x^k w)^2] dx = 0;$$

(iii) by the above computations for $n = 2k$, $k \geq 0$,

$$\Gamma^{(n)}(w) = \int g^{(n)}(w) dx, \tag{2.24}$$

for some g belonging to the class Σ_{n+1}^3 , namely $\Gamma^{(n)}$ has a zero of order three at the origin. On the other hand for $n = 2k + 1$, $k \geq 0$, we simply have that

$$\Gamma^{(n)}(w) = \int f_2^{(n)} + h^{(n)}(w) dx, \quad h^{(n)}(w) \in \Sigma_{n+1}^3, \quad f_2^{(n)} \in \mathcal{P}_{n+1}^2. \tag{2.25}$$

More precisely $f_2^{(n)}$ has the form

$$f_2^{(n)} = \sum_{p=0}^{n+1} \sum_{k_1+k_2=p} (\partial_x^{k_1} w)(\partial_x^{k_2} w) c_n^{k_1, k_2}. \tag{2.26}$$

In the following Sections we analyse precisely the form of $c_n^{k_1, k_2}$.

2.3 Computation of $\Gamma^{(n)}$ for n odd

Now we want to derive some explicit expression for the coefficients of the quadratic part of the functions $\Gamma^{(n)}$ ($n \in \mathbb{N}$ is odd) introduced in (2.6); more precisely, by recalling the definitions of Section 2.1.1 and (2.25)-(2.26), we want to compute the coefficients $c_n^{k_1, k_2}$ of $f_2^{(n)}$.

2.3.1 Coefficients of the linear terms

We begin by computing the coefficients $c_n := c_n^{n+1}$ of the linear terms, since this will be useful for the computation of the coefficients of the quadratic terms. Consider the recursion relation (2.8) between the $\rho^{(n)}$; since $p \in \Sigma_0^0$ and that $p = -c^{1/3} + \mathcal{O}(w)$ for small $|w|$, the coefficients in front of the leading order linear term of $\rho^{(n+2)}p$ is proportional to the coefficient of maximal order of the linear term of $\rho^{(n+2)}$. Hence, if we write only the coefficients of maximal order for the linear terms, we get

$$\begin{aligned} -c_n &= c^{2/3} c_{n+2} - c^{1/3} 3c_{n+1}, \\ c_{n+2} &= -c^{-2/3} c_n + 3c^{-1/3} c_{n+1}, \quad n \geq 0. \end{aligned} \quad (2.27)$$

Now, recall that by (2.19) and (2.20) we have that $c_0 = -\frac{1}{3c}$ and $c_1 = -\frac{2}{9c^{4/3}}$. One can check that

$$c_m = d_1 a^m + d_2 b^m, \quad m \geq 0, \quad (2.28)$$

$$a := a(c) = \frac{3 + \sqrt{5}}{2c^{1/3}}, \quad b := b(c) = \frac{3 - \sqrt{5}}{2c^{1/3}}, \quad (2.29)$$

$$d_1 := d_1(c) = \frac{-3 - \sqrt{5}}{18c}, \quad d_2 := d_2(c) = \frac{-3 + \sqrt{5}}{18c}. \quad (2.30)$$

Remark 2.10. Notice that the following properties hold.

(i) From (2.30) one readily obtains that

$$\begin{aligned} \lim_{c \rightarrow 0^\pm} d_1(c) &= \mp \infty, \\ \lim_{c \rightarrow 0^\pm} d_2(c) &= \mp \infty. \end{aligned}$$

The last two limits imply that in the dispersionless limit

$$\lim_{c \rightarrow 0^\pm} c_m = -(sgn(c))^m \infty, \quad m \geq 1.$$

(ii) By direct computation one also obtains that:

(a) for any $c > 0$

$$\begin{cases} c_m < 0 & \text{for even } m \geq 2; \\ c_m < 0 & \text{for odd } m \geq 2, \end{cases}$$

(b) for any $c < 0$

$$\begin{cases} c_m > 0 & \text{for even } m \geq 2; \\ c_m < 0 & \text{for odd } m \geq 2, \end{cases}$$

(c) $\lim_{m \rightarrow \infty} |c_m| = +\infty$ (respectively, $\lim_{m \rightarrow \infty} |c_m| = 0$) for $|c| < c^* := \left(\frac{3+\sqrt{5}}{2}\right)^3$ (respectively, for $|c| > c^*$).

2.3.2 Coefficients of the quadratic terms

To determine the coefficients in front of the quadratic terms containing the maximal number of derivatives in $f_2^{(n)}$, we integrate (2.8)

$$\int \rho^{(n+2)} p^2 dx = \int \rho^{(n)} - \rho_{xx}^{(n)} - 3 \sum_{k_1+k_2=n+1} p \rho^{(k_1)} \rho^{(k_2)} - \sum_{k_1+k_2+k_3=n} \rho^{(k_1)} \rho^{(k_2)} \rho^{(k_3)} dx \quad (2.31)$$

$$- \int 3 \partial_x (\rho^{(n+1)} p) - 3 \sum_{k_1+k_2=n} \rho^{(k_1)} \rho_x^{(k_2)} dx. \quad (2.32)$$

Now, it is easy to see that the integral in (2.32) vanishes, since the term $\partial_x(\rho^{(n+1)}p)$ is a total derivative, and since

$$\sum_{k_1+k_2=n} \rho^{(k_1)} \rho_x^{(k_2)} = \sum_{\substack{k_1+k_2=n \\ k_1 > k_2}} \partial_x(\rho^{(k_1)} \rho^{(k_2)}).$$

Now, if we write $\rho^{(n)} = f_2^{(n)} + h^{(n)}$ and we consider only the coefficients in front of the quadratic terms containing the maximal number of derivatives, the relation (2.31)- (2.32) reads as

$$c^{2/3} \int \sum_{k_1+k_2=n+3} c_{n+3}^{k_1, k_2} (\partial_x^{k_1} w)(\partial_x^{k_2} w) dx = 3c^{1/3} \int \sum_{k_1+k_2=n+1} c_{k_1} c_{k_2} (\partial_x^{k_1+1} w)(\partial_x^{k_2+1} w) dx, \quad (2.33)$$

but since

$$\begin{aligned} \int \sum_{k_1+k_2=n+3} c_{n+3}^{k_1, k_2} (\partial_x^{k_1} w)(\partial_x^{k_2} w) dx &= \sum_{k=0}^{n+3} c_{n+3}^{k, n+3-k} (-1)^{q(k)} \int (\partial_x^{\frac{n+3}{2}} w)^2 dx, \\ q(k) &:= \frac{n+3}{2} - k, \\ \int \sum_{k_1+k_2=n+1} c_{k_1} c_{k_2} (\partial_x^{k_1+1} w)(\partial_x^{k_2+1} w) dx &= \sum_{k=0}^{n+1} c_k c_{n-k+1} (-1)^{\tilde{q}(k)} \int (\partial_x^{\frac{n+3}{2}} w)^2 dx, \\ \tilde{q}(k) &:= \frac{n+1}{2} - k, \end{aligned}$$

we obtain

$$\sum_{k=0}^{n+3} (-1)^{q(k)} c_{n+3}^{k, n+3-k} = 3c^{-1/3} \sum_{k=0}^{n+1} (-1)^{\tilde{q}(k)} c_k c_{n-k+1}. \quad (2.34)$$

Now we show that the coefficients in front of the quadratic terms containing the maximal number of derivatives do not vanish. We recall that by Remark 2.9-(ii) we deal only with n odd.

Proposition 2.11. *Let n be odd, then, recalling (2.34), we have*

$$S_n := \sum_{k=0}^{n+1} (-1)^{\tilde{q}(k)} c_k c_{n-k+1} \neq 0. \quad (2.35)$$

Proof. Consider the right-hand side of Eq. (2.34); by simple calculations

$$S_n = 2(-1)^{\frac{n+1}{2}} \left(\sum_{\substack{k=0, \dots, \frac{n-1}{2} \\ k \text{ even}}} c_k c_{n+1-k} - \sum_{\substack{k=0, \dots, \frac{n-1}{2} \\ k \text{ odd}}} c_k c_{n+1-k} \right) + c_{\frac{n+1}{2}}^2. \quad (2.36)$$

First, one can verify explicitly that

$$\begin{aligned} S_1 &= -2c_0 c_2 + c_1^2 = -2(d_1 + d_2)(d_1 a^2 + d_2 b^2) + (d_1 a + d_2 b)^2 \\ &= -\frac{14}{81c^{8/3}} \neq 0. \end{aligned}$$

Now we distinguish the two cases $n = 4l + 3$ ($l \geq 0$) and $n = 4l + 1$ ($l > 0$).

Case $n = 4l + 3$: first observe that

$$\begin{aligned}
c_{2s}c_{n+1-2s} - c_{2s+1}c_{n-2s} &= d_1d_2 \left[a^{2s}b^{n+1-2s} + b^{2s}a^{n+1-2s} - a^{2s+1}b^{n-2s} - b^{2s+1}a^{n-2s} \right] \\
&= d_1d_2 \left[a^{2s}b^{n-2s}(b-a) + b^{2s}a^{n-2s}(a-b) \right] \\
&= d_1d_2 \left[b^n \left(\frac{a}{b} \right)^{2s} (b-a) - a^n \left(\frac{b}{a} \right)^{2s} (b-a) \right] \\
&= d_1d_2(b-a) \left[b^n \left(\frac{a}{b} \right)^{2s} - a^n \left(\frac{b}{a} \right)^{2s} \right], \tag{2.37}
\end{aligned}$$

so in this case we have

$$\begin{aligned}
\sum_{s=0}^{\frac{n-3}{4}} (c_{2s}c_{n+1-2s} - c_{2s+1}c_{n-2s}) &= d_1d_2(b-a) \left[b^n \frac{1 - \left(\frac{a^2}{b^2} \right)^{\frac{n+1}{4}}}{1 - a^2/b^2} - a^n \frac{1 - \left(\frac{b^2}{a^2} \right)^{\frac{n+1}{4}}}{1 - b^2/a^2} \right] \\
&= d_1d_2(b-a) \left(b^{n+2-\frac{n+1}{2}} \frac{b^{\frac{n+1}{2}} - a^{\frac{n+1}{2}}}{b^2 - a^2} + a^{n+2-\frac{n+1}{2}} \frac{a^{\frac{n+1}{2}} - b^{\frac{n+1}{2}}}{b^2 - a^2} \right) \\
&= \frac{d_1d_2}{a+b} (b^{\frac{n+3}{2}} - a^{\frac{n+3}{2}})(b^{\frac{n+1}{2}} - a^{\frac{n+1}{2}}) \\
&\stackrel{n=4l+3}{=} \frac{d_1d_2}{a+b} (b^{2l+3} - a^{2l+3})(b^{2l+2} - a^{2l+2}),
\end{aligned}$$

and the thesis is equivalent to

$$2 \frac{d_1d_2}{a+b} (a^{2l+3} - b^{2l+3})(a^{2l+2} - b^{2l+2}) + (d_1a^{2l+2} + d_2b^{2l+2})^2 \neq 0 \tag{2.38}$$

In order to verify (2.38) we observe that

$$(a^{2l+3} - b^{2l+3})(a^{2l+2} - b^{2l+2}) = a^{4l+5} \left[1 - \left(\frac{b}{a} \right)^{2l+2} - \left(\frac{b}{a} \right)^{2l+3} + \left(\frac{b}{a} \right)^{4l+5} \right] =: a^{4l+5} \alpha_l,$$

where $(\alpha_l)_{l \in \mathbb{N}}$ is an increasing sequence of positive numbers (which do not depend on c) satisfying $\lim_{l \rightarrow \infty} \alpha_l = 1$; similarly, we have

$$(d_1a^{2l+2} + d_2b^{2l+2})^2 = a^{4l+4} d_1^2 \left[1 + \frac{d_2}{d_1} \left(\frac{b}{a} \right)^{2l+2} \right]^2 =: a^{4l+4} d_1^2 \beta_l,$$

where $(\beta_l)_{l \in \mathbb{N}}$ is a decreasing sequence of positive numbers (which do not depend on c) satisfying $\lim_{l \rightarrow \infty} \beta_l = 1$. Since $2a \frac{d_1d_2}{a+b} > 0$ by (2.29) and (2.30), we have that the left-hand side in (2.38) is given by

$$a^{4l+4} \left(2a \frac{d_1d_2}{a+b} \alpha_l + d_1^2 \beta_l \right) = \frac{1}{c^{2+(4l+4)/3}} \left(\frac{3+\sqrt{5}}{2} \right)^{4l+4} \left(\frac{3+\sqrt{5}}{243} \alpha_l + \frac{7+3\sqrt{5}}{162} \beta_l \right) > 0.$$

Case $n = 4l + 1$ ($l \geq 1$): by arguing as in (2.37), we get

$$\sum_{s=0}^{l-1} (c_{2s}c_{n+1-2s} - c_{2s+1}c_{n-2s}) = \frac{d_1d_2}{a+b} (b^{2l+2} - a^{2l+2})(b^{2l+1} - a^{2l+1}),$$

and the thesis is equivalent to the following inequality,

$$2 \frac{d_1d_2}{a+b} (b^{2l+2} - a^{2l+2})(b^{2l+1} - a^{2l+1}) + 2(d_1a^{2l} + d_2b^{2l})(d_1a^{2l+2} + d_2b^{2l+2}) - (d_1a^{2l+1} + d_2b^{2l+1})^2 \neq 0. \tag{2.39}$$

In order to verify (2.39) we observe that

$$\begin{aligned} & 2 \frac{d_1 d_2}{a+b} (a^{2l+2} - b^{2l+2})(a^{2l+3} - b^{2l+3}) \\ &= 2 \frac{d_1 d_2}{a+b} a^{4l+3} \left[1 - \left(\frac{b}{a}\right)^{2l+1} - \left(\frac{b}{a}\right)^{2l+2} + \left(\frac{b}{a}\right)^{4l+3} \right] =: 2 \frac{d_1 d_2}{a+b} a^{4l+3} \gamma_l, \end{aligned}$$

where $(\gamma_l)_{l \geq 1}$ is a increasing sequence of positive numbers (which do not depend on c) satisfying $\lim_{l \rightarrow \infty} \gamma_l = 1$; similarly,

$$\begin{aligned} & 2(d_1 a^{2l} + d_2 b^{2l})(d_1 a^{2l+2} + d_2 b^{2l+2}) \\ &= 2d_1^2 a^{4l+2} \left[\left(1 + \frac{d_2}{d_1} \left(\frac{b}{a}\right)^{2l}\right) \left(1 + \frac{d_2}{d_1} \left(\frac{b}{a}\right)^{2l+2}\right) \right] =: 2d_1^2 a^{4l+2} \delta_l, \end{aligned}$$

where $(\delta_l)_{l \geq 1}$ is an decreasing sequence of positive numbers (which do not depend on c) satisfying $\lim_{l \rightarrow \infty} \delta_l = 1$, and

$$(d_1 a^{2l+1} + d_2 b^{2l+1})^2 = a^{4l+2} d_1^2 \left(1 + \frac{d_2}{d_1} (b/a)^{2l+1}\right)^2 =: a^{4l+2} d_1^2 \epsilon_l,$$

where $(\epsilon_l)_{l \geq 1}$ is a decreasing sequence of positive numbers (which do not depend on c) such that $\lim_{l \rightarrow \infty} \epsilon_l = 1$. Since $2a \frac{d_1 d_2}{a+b} > 0$ by (2.29) and (2.30), we have that the left-hand side in (2.38) is given by

$$\begin{aligned} a^{4l+2} \left(2a \frac{d_1 d_2}{a+b} \gamma_l + 2d_1^2 \delta_l - d_1^2 \epsilon_l \right) &= \frac{1}{c^{2+(4l+2)/3}} \left(\frac{3+\sqrt{5}}{2} \right)^{4l+2} \left(\frac{3+\sqrt{5}}{243} \gamma_l + \frac{7+3\sqrt{5}}{81} \delta_l - \frac{7+3\sqrt{5}}{162} \epsilon_l \right) \\ &\neq 0. \end{aligned}$$

□

Remark 2.12. By arguing as in the proof of the above proposition, one can also show that for any odd number n

$$\lim_{c \rightarrow 0} |S_n| = +\infty,$$

which reflects the fact that the present approach cannot be applied to the dispersionless DP equation (1.4).

By Proposition 2.11 we have that the number of derivatives appearing in the quadratic part is greater or equal than the one appearing in the cubic remainder.

2.4 Proof of Theorem 1.4

The proof of Theorem 1.4 is based on the following reasoning. We recall that, by the discussion in Section 2.3, we have constructed sequence of constants of motion $\Gamma^{(n)}(w)$ with $n \geq 0$ of the form

$$\Gamma^{(n)}(w) = \int_{\mathbb{T}} \rho^{(n)}(w) dx, \quad (2.40)$$

where $\rho^{(n)} \in \Sigma_{n+1}^0$ (see Lemma 2.7) are defined iteratively by (2.7), (2.8). We recall also (see (2.24), (2.25)) that, for n even

$$\Gamma^{(n)}(w) = \int_{\mathbb{T}} \rho^{(n)}(w) dx = \int_{\mathbb{T}} g^{(n)}(w) dx, \quad g^{(n)} \in \Sigma_{n+1}^3, \quad (2.41)$$

while for n odd

$$\Gamma^{(n)}(w) = \int_{\mathbb{T}} \rho^{(n)}(w) dx = \int_{\mathbb{T}} f_2^{(n)}(w) + h^{(n)}(w) dx, \quad h^{(n)} \in \Sigma_{n+1}^3, \quad (2.42)$$

and $f_2^{(n)} \in \mathcal{P}_{n+1}^2$ as in (2.26).

Proposition 2.13. For any $N \in \mathbb{N}$, $N \geq 2$ there are $r = r(N) > 0$, $c_1 = c_1(N) > 0$, $c_2 = c_2(N) > 0$ and a sequence of functions

$$F_n : H^N \rightarrow \mathbb{R}, \quad 1 \leq n \leq N, \quad (2.43)$$

analytic on $B_{H^N}(0, |c|/2)$ with the following properties.

(i) For any $0 \leq n \leq N$, we have

$$F_n(w) = \int g^{(n)} dx, \quad g^{(n)} \in \Sigma_{n+1}^3, \quad n = 2k, \quad \text{for some } k \in \mathbb{N}, \quad (2.44)$$

and

$$F_n(w) = \int (q_2^{(n)} + q_3^{(n)}) dx, \quad q_3^{(n)} \in \Sigma_{n+1}^3, \quad n = 2k+1, \quad \text{for some } k \in \mathbb{N}, \quad (2.45)$$

where $q_2^{(n)} \in \mathcal{P}_{n+1}^2$ and

$$\int q_2^{(n)}(w) dx = \int (\partial_x^{k+1} w)^2 dx. \quad (2.46)$$

(ii) For $n = 2k$, $k \in \mathbb{N}$, we have

$$|F_n(w)| \leq c_1 \|w\|_{H^{n+1}}^3, \quad \forall 0 \leq n \leq N, \quad w \in B_{H^N}(0, r), \quad (2.47)$$

(iii) for $n = 2k+1$, $k \in \mathbb{N}$, we have

$$|F_n(w) - \int q_2^{(n)} dx| \leq c_2 \|w\|_{H^{n+1}}^3, \quad \forall 0 \leq n \leq N, \quad w \in B_{H^N}(0, r). \quad (2.48)$$

Finally, let $u(t, x)$ be the solution of (1.6) with $u(0, x) = u_0(x) \in H^{N+2}$ such that $\|u_0\|_{H^{N+2}} \leq \varepsilon$. Define $w(t, x) = (1 - \partial_{xx})u(t, x)$. Then, as long as $\|u(t, \cdot)\|_{H^{N+2}} \leq r/2$, we have that

$$\frac{d}{dt} F_n(w) = 0, \quad 0 \leq n \leq N. \quad (2.49)$$

Proof. First of all note that for $0 \leq s \leq N$

$$\|w\|_{H^s} \leq \|u\|_{H^s} + \|u\|_{H^{s+2}}.$$

This implies that as long as $\|u(t, \cdot)\|_{H^{N+2}} \leq r/2$ one has $\|w\|_{H^s} \leq r$.

For $n \in \mathbb{N}$ even we define $F_n(w) := \Gamma^{(n)}(w)$, hence (2.44) holds by (2.41). The bound (2.47) follows by Proposition 2.8.

For n odd we construct the functions F_n iteratively, by putting the quadratic parts of the functions $\Gamma^{(n)}$ with n odd in a *triangular* form.

First we observe that for any n odd (see (2.26))

$$\int f_2^{(n)} dx = \sum_{\substack{0 \leq p \leq n+1, \\ p \equiv 0(2)}} \sum_{k_1 + k_2 = p} c_n^{k_1 k_2} \int (\partial_x^{k_1} w) (\partial_x^{k_2} w) dx = \sum_{i=0}^{(n+1)/2} d_n^i \int (\partial_x^i w)^2 dx \quad (2.50)$$

for some coefficients d_n^i obtained after the integration by parts and by Lemma 2.11

$$d_n^{(n+1)/2} \neq 0 \quad \forall n. \quad (2.51)$$

Now let us show the first step of the triangularization. We write the quadratic part of $\Gamma^{(1)}$ as

$$\int f_2^{(1)} dx = d_1^0 \int w^2 dx + d_1^1 \int w_x^2 dx.$$

Recall the definition of the constants of motion M_1 in (1.16). Its quadratic part is, by (1.15), (2.18) and Remark 2.9,

$$M_1^{(2)} = \frac{1}{9} \int w^2 dx.$$

We remark that $\Gamma^{(1)}$, M_1 and their linear combinations are constants of motion. We define (recall (2.51))

$$F_1 := \frac{1}{d_1^1} \left(\Gamma^{(1)} - 9d_1^0 M_1 \right)$$

so that its quadratic part reads as

$$\int q_2^{(1)} dx := \int w_x^2 dx \quad (2.52)$$

and for the cubic part we have

$$\int q_3^{(1)} dx = \frac{1}{d_1^1} h^{(1)} - 9d_1^0 M_1^{(\geq 3)}.$$

Since by Lemma 2.4 $q_3^{(1)} \in \Sigma_2^3$, by Proposition 2.8 the bound (2.48) holds for $F^{(1)}$. We define iteratively for $n = 2k + 1$, $k \geq 1$

$$F_{2k+1} := \frac{1}{d_{2k+1}^{k+1}} \left(\Gamma^{(2k+1)} - 9d_{2k+1}^0 M_1 - \sum_{i=1}^k d_{2k+1}^i F_{2i+1} \right) \quad (2.53)$$

and (2.45) follows easily by Lemma 2.4. Now we prove (2.46) by induction on $k \geq 0$. We just showed the basis of the induction in (2.52) ($k = 0$). Now suppose that

$$\int q_2^{(2k+1)} dx = \int (\partial_x^{k+1} w)^2 dx.$$

Then by (2.53)

$$d_{2k+3}^{k+2} \int q_2^{(2k+3)} dx = \int f_2^{(2k+3)} dx - d_{2k+3}^0 \int w^2 dx - \sum_{i=1}^{k+1} d_{2k+3}^i \int (\partial_x^i w)^2 dx \stackrel{(2.50)}{=} d_{2k+3}^{k+2} \int (\partial_x^{k+2} w)^2 dx,$$

which proved the claim. The bound (2.48) follows trivially by triangle inequality and Proposition 2.8.

Obviously the F_{2k+1} defined in (2.53) are constants of motion because they are linear combinations of conserved quantities. □

Proof of Theorem 1.4. We set $K_1(u) := M_1(u)$ where M_1 is defined in (1.16). For any $n \geq 2$ of the form $n = k + 1$ we set

$$K_n(u) = F_{2k-1}(w),$$

where $F_{2k-1}(w)$ is given in Proposition 2.13.

The item (0) is verified since the functions F_n constructed in Proposition (1.4) satisfy (2.49).

The item (i) and (ii) follows by item (i) of Proposition (1.4).

For the item (iii), (1.26) follows by (2.48) and (1.25) by (2.45), the definitions (1.18), (1.19) and by considering the equivalent norm $\|u\|_{L^2} + \|\partial_x^s u\|_{L^2} \approx \|u\|_{H^s}$. □

3 Global well-posedness near the origin

In this Section we give the proof of Theorem 1.5 by using Theorem 1.4. First we state the following local well-posedness result for Eq. (1.6).

Proposition 3.1 (Local existence). *For any $s > 3/2$ and for any $u_0 \in H^s$ there exist $\bar{T} = \bar{T}(\|u_0\|_{H^s})$ and a unique solution $u(t, x)$ of (1.6) with initial condition $u(0, x) = u_0(x)$ defined for $t \in [-\bar{T}, \bar{T}]$ belonging to the space $C([- \bar{T}, \bar{T}], H^s)$, such that*

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad |t| \leq \bar{T} \leq \frac{1}{2C_s\|u_0\|_{H^s}}, \quad (3.1)$$

for some constant $C_s > 0$ depending only on s .

The proof of the above result can be found in the Appendix A.1. It is based on a Galerkin-type approximation method, and follows closely the argument reported in [22] for the dispersionless DP equation (1.4).

Remark 3.2. Proposition 3.1 implies that for any $s > 3/2$ there exists $r(s) > 0$ such that for any $u_0 \in B_{H^s}(0, r(s))$ Eq. (1.6) with initial datum u_0 admits a solution $u(t, x)$ such that $\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}$ for $|t| \leq \frac{1}{2C_s r(s)}$.

The proof of Theorem 1.5 is based on the following bootstrap argument.

Consider the function $u(t, x)$ solution of (1.6), defined on some interval $t \in [-T, T]$ with $0 < T \leq \bar{T}$ with initial datum $\|u(0; \cdot)\|_{H^s} \leq r_0$ given by Proposition 3.1.

By choosing $n := n(s) = [s] - 1$, if r_0 small enough, we have by item (iii) of Theorem 1.4 and by Proposition 3.1 that

$$\|u(t, \cdot)\|_{H^s}^2 \leq \|u(\cdot)\|_{L^2}^2 + \tilde{c}|K_n^{(0)}(u(t, \cdot))| \leq \|u(\cdot)\|_{L^2}^2 + \tilde{c}|K_n(u(t, \cdot))| + \tilde{c}C\|u(t, \cdot)\|_{H^s}\|u(t, \cdot)\|_{H^s}^2, \quad (3.2)$$

with \tilde{c}, C given by Theorem 1.4. Since

$$\begin{aligned} K_1^{(0)}(u(t, \cdot)) &= \int (u - u_{xx})^2 dx \\ &= \int u^2 dx + 2 \int u_x^2 dx + \int u_{xx}^2 dx \end{aligned}$$

by (1.26) we have that for r_0 small enough

$$\|u(t, \cdot)\|_{L^2}^2 \leq C|K_1(u(t, \cdot))|. \quad (3.3)$$

Since $K_1(u), K_n(u)$ are constant of motion, we have

$$|K_1(u(t, \cdot))| + |K_n(u(t, \cdot))| \leq |K_1(u(0, \cdot))| + |K_n(u(0, \cdot))| \stackrel{(1.25), (1.26)}{\leq} \kappa(s)\|u(0, \cdot)\|_{H^s}^2 \leq \kappa(s)r_0^2, \quad (3.4)$$

for some $\kappa(s) > 0$ depending only on s . The bound (3.2) reads

$$\|u(t, \cdot)\|_{H^s}^2 \leq \tilde{c}\kappa(s)r_0^2 + \tilde{c}C\|u(t, \cdot)\|_{H^s}\|u(t, \cdot)\|_{H^s}^2. \quad (3.5)$$

Now let \hat{T} be the supremum of those T such that the solution $u(t, x)$ is defined on $[-T, T]$ and

$$\sup_{t \in [-T, T]} \|u(t, \cdot)\|_{H^s}^2 \leq Q(s)r_0^2, \quad (3.6)$$

where $Q(s) \geq 4\kappa(s)\tilde{c}$, with $\kappa(s)$ given in (3.4) and \tilde{c} given by Theorem 1.4. For $t \in [-\hat{T}, \hat{T}]$ we deduce, by (3.5), that

$$\|u(t, \cdot)\|_{H^s}^2 \leq \tilde{c}\kappa(s)r_0^2 + \tilde{c}C\sqrt{Q(s)}r_0\|u(t, \cdot)\|_{H^s}^2. \quad (3.7)$$

Hence, if we take r_0 sufficiently small such that $\tilde{c}C\sqrt{Q(s)}r_0 \leq 1/2$, we obtain

$$\|u(t, \cdot)\|_{H^s}^2 \leq 2\tilde{c}r_0^2 < Q(s)r_0^2. \quad (3.8)$$

Of course estimate (3.8) leads to the contradiction of the fact that \hat{T} is the supremum. Since estimate (3.8) does not depend on \hat{T} , we must have $\hat{T} = +\infty$, which implies Theorem 1.5.

4 Birkhoff resonances

In this Section we prove Theorem 1.6. First we need some preliminaries definitions and results to show the formal Birkhoff normal form procedure. Recall the definitions given in Section 1.1 for the space of formal polynomials.

Definition 4.1. (Poisson brackets) Let $P = \sum_{\alpha \in \mathcal{I}_n} P_\alpha u^\alpha$ and $Q = \sum_{\beta \in \mathcal{I}_m} Q_\beta u^\beta$ (recall (2.16)) two formal homogenous polynomials. We define $\{ \cdot, \cdot \}: \mathcal{P}^{(n)} \times \mathcal{P}^{(m)} \rightarrow \mathcal{F}$

$$\{P, Q\} := \sum_{\alpha \in \mathcal{I}_n, \beta \in \mathcal{I}_m} P_\alpha Q_\beta \{u^\alpha, u^\beta\} = \sum_{\alpha \in \mathcal{I}_n, \beta \in \mathcal{I}_m} \sum_j ((-i)\omega(j)\alpha_j \beta_{-j}) P_\alpha Q_\beta u^{\alpha+\beta-\mathbf{e}_j-\mathbf{e}_{-j}} \quad (4.1)$$

where \mathbf{e}_k is the element of $\mathbb{N}^{\mathbb{Z}}$ with all components equal to zero except for the k -th one, which is equal to 1.

In the following lemma we prove that the above definition is well posed. We point out that the assumption of zero momentum (see (2.16)) is a key ingredient for the proof.

Lemma 4.2. *Given $P \in \mathcal{P}^{(n)}$, $Q \in \mathcal{P}^{(m)}$ we have that*

- (i) *the sum $P + Q \in \mathcal{P}^{(\max\{m, n\})}$.*
- (ii) *the Poisson bracket $\{P, Q\} \in \mathcal{P}^{(n+m)}$.*

Proof. The item (i) is trivial. We prove item (ii). We write $P = \sum_{\alpha \in \mathcal{I}_n} P_\alpha u^\alpha$ and $Q = \sum_{\beta \in \mathcal{I}_m} Q_\beta u^\beta$. Recalling (4.1), we have to prove the following claim: given $\gamma \in \mathcal{I}_{n+m-2}$ there is only a finite number of α, β, j such that

$$\gamma = \alpha + \beta - \mathbf{e}_j - \mathbf{e}_{-j}, \quad \alpha_j \beta_{-j} \neq 0, \quad (4.2)$$

where we denoted by \mathbf{e}_j the element of $\mathbb{N}^{\mathbb{Z}}$ with all components zero except for the j -th, which is 1. Indeed, if this holds, there exists a sequence $(R_\gamma)_{\gamma \in \mathcal{I}_{n+m-2}}$ of complex numbers such that

$$\{P, Q\} = \sum_{\gamma \in \mathcal{I}_{n+m-2}} R_\gamma u^\gamma.$$

First we observe the following: given $\gamma \in \mathcal{I}_k$, for some $k \geq 2$, there exist only finitely many couples $(a, b) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$ such that we can decompose $\gamma = a + b$. Note that $\mathcal{M}(\gamma) = \mathcal{M}(a) + \mathcal{M}(b) = 0$. We call $\mathcal{M}^\gamma := \sum_{j>0} j\gamma_j$ and we observe that, for any choice of a and b , we have $|\mathcal{M}(a)| \leq \mathcal{M}^\gamma$.

We can choose a and b such that $\mathcal{M}(a) = j$, for instance,

$$a = \alpha - \mathbf{e}_{-j}, \quad b = \beta - \mathbf{e}_j.$$

hence $|j| \leq \mathcal{M}^\gamma$. So given γ there is only a finite number of j for which (4.2) holds.

Now we prove that, given γ and j , there is only a finite number of α and β such that (4.2) holds and so the claim is proved. We have $\gamma + \mathbf{e}_j + \mathbf{e}_{-j} = \alpha + \beta$. Hence we have to split the left-hand side in two elements of $\mathbb{N}^{\mathbb{Z}}$ with zero momentum. If $\gamma = a + b$ with $(a, b) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}$, then there is only a finite number of choices for a and b , which are

$$a = \alpha - \mathbf{e}_{\pm j}, \quad b = \beta - \mathbf{e}_{\mp j}.$$

□

Adjoint action and quadratic Hamiltonians. Let $G \in \mathcal{P}^{(m)}$, $m \geq 0$. We define the *adjoint action* of G as

$$\text{ad}_G: \mathcal{P}^{(n)} \rightarrow \mathcal{P}^{(m+n)} \subset \mathcal{F}^{(\geq m)}, \quad \text{ad}_G[P] := \{G, P\},$$

then we extend it to the entire \mathcal{F} by setting

$$\text{ad}_G[P] := (\text{ad}_G[P^{(n)}])_{n \geq 0}$$

with

$$\Pi^{(d)} \text{ad}_G[P] = \begin{cases} \sum_{n=d-m} \{G^{(m)}, P^{(n)}\} & \text{if } d \geq m, \\ 0 & \text{otherwise} \end{cases}$$

for $d \geq 0$. Note that $\Pi^{(d)} \text{ad}_G[P] \in \mathcal{P}^{(d)}$ since the above sum is finite, so ad_G is well defined on \mathcal{F} .

We define the kernel and range of ad_G as

$$\text{Ker}(G) := \{F \in \mathcal{F} : \{F, G\} = 0\}, \quad \text{Rg}(G) := \{F \in \mathcal{F} : \{F, G\} \neq 0\}.$$

Consider a quadratic Hamiltonian G in diagonal form, $G = \sum_j \lambda(j) |u_j|^2$, $\lambda(j) \in \mathbb{C}$. Given $\alpha \in \mathbb{N}^{\mathbb{Z}}$ we define the associated G -divisor as (recall (1.14))

$$\Omega_G(\alpha) = \sum_j \omega(j) \lambda(j) \alpha_j \quad (4.3)$$

and we have

$$\{G(u), u^\alpha\} = \left(-i \sum_j \omega(j) \lambda(j) \alpha_j \right) u^\alpha = -i \Omega_G(\alpha) u^\alpha.$$

Definition 4.3. We define $\Pi_{\text{Ker}(G)}$ as the projector on the kernel of the adjoint action $\text{ad}_G[\cdot]$, i.e.

$$\forall \alpha \in \mathcal{I}_n \quad \Pi_{\text{Ker}(G)}(u^\alpha) := \begin{cases} u^\alpha & \text{if } \Omega_G(\alpha) \neq 0, \\ 0 & \text{if } \Omega_G(\alpha) = 0. \end{cases}$$

We define the projector on the range of the adjoint action as $\Pi_{\text{Rg}(G)} := \text{I} - \Pi_{\text{Ker}(G)}$ where I is the identity. We define the action of $\Pi_{\text{Rg}(G)}$ and $\Pi_{\text{Ker}(G)}$ on any Hamiltonian $H \in \mathcal{P}^{(n)}$ by linearity.

Exponential map and Lie transformation. Let $G \in \mathcal{P}^{(m)}$ with $m \geq 1$. We note that for $k \geq 0$

$$\text{ad}_G^k : \mathcal{P}^{(n)} \rightarrow \mathcal{P}^{(n+km)} \subset \mathcal{F}^{(\geq n+km)}$$

We define the exponential map $e^{\{G, \cdot\}} : \mathcal{F} \rightarrow \mathcal{F}$ as

$$e^{\{G, \cdot\}} P := \left(\sum_{k \geq 0} \frac{\text{ad}_G^k[P^{(n)}]}{k!} \right)_{n \geq 0}$$

and it is well-defined since

$$\Pi^{(d)} e^{\{G, \cdot\}} P = \sum_{(n,k) : n+km=d} \frac{\text{ad}_G^k[P^{(n)}]}{k!}$$

but there is only a finite number of couples $(n, k) \in \mathbb{N}^2$ such that $n + km = d$ ($d \geq 0$ and $m \geq 1$ are fixed). Hence $\Pi^{(d)} e^{\{G, \cdot\}} P \in \mathcal{P}^{(d)}$.

Let $\chi \in \mathcal{P}^{(n)}$ and $H \in \mathcal{F}$ be two Hamiltonians. We call Φ_χ^t the flow of χ , namely

$$\begin{cases} \frac{d}{dt} \Phi_\chi^t(u) = J \nabla \chi(u), \\ \Phi_\chi^0(u) = u. \end{cases}$$

We have

$$\frac{d^k}{dt^k} (H \circ \Phi_\chi^t) = \text{ad}_\chi^k[H] \circ \Phi_\chi^t$$

Then by expanding in the (formal) Taylor series at $t = 0$ we get

$$H \circ \Phi_\chi^t := \sum_{k \geq 0} \frac{\text{ad}_\chi^k[H]}{k!} t^k.$$

We call Lie transformation the map at time one $\Phi_\chi^{t=1} =: \Phi_\chi$ and we have

$$H \circ \Phi_\chi := \sum_{k \geq 0} \frac{\text{ad}_\chi^k[H]}{k!} = e^{\{ \chi, \cdot \}} H.$$

4.1 Proof of Theorem 1.6

Let $H \in \mathcal{P}^{(\leq n)}$ be a Hamiltonian. The goal of the formal Birkhoff normal form procedure is to construct a change of coordinates Φ which puts the Hamiltonian H in a Birkhoff normal form of some order (recall (1.27)). This algorithm consists of different steps. If we denote by $\chi_k \in \mathcal{P}^{(k)}$ the generators of the Birkhoff transformation at the step k and with

$$H_0 := H, \quad H_k := e^{\{\chi_k, \cdot\}} \dots e^{\{\chi_1, \cdot\}} H \quad k > 0,$$

then we have to choose χ_{k+1} such that

$$\Pi^{(k+1)} e^{\{\chi_{k+1}, \cdot\}} H_k = \Pi_{\text{Ker}(H^{(0)})} \Pi^{(k+1)} H_k = \Pi_{\text{Ker}(H^{(0)})} H_k^{(k+1)} \quad (4.4)$$

The right-hand side of (4.4) contributes to the normal form of H_{k+1} . We want to show that this homogenous polynomial of degree $k+3$ is supported on \mathcal{N}_{k+1}^* (recall Definition 1.3).

First we prove that, given a finite set of Hamiltonians in involution (namely which pairwise commute) and fixed N , there exists a Birkhoff transformation Φ_N which puts all these Hamiltonians in Birkhoff normal form of order N according to Definition 1.7. It is sufficient to prove that for two commuting Hamiltonians.

Lemma 4.4. *Consider $H, K \in \mathcal{F}$ two commuting Hamiltonians. For any N there exist, at least formally, a change of coordinates Φ_N such that*

$$H \circ \Phi_N = H^{(0)} + Z_N + R_N, \quad K \circ \Phi_N = K^{(0)} + W_N + Q_N \quad (4.5)$$

where $Z_N, W_N \in \mathcal{P}^{(N)}$ commuting with $H^{(0)}$ and $K^{(0)}$. $R_N, Q_N \in \mathcal{F}^{(\geq N+1)}$.

Proof. We argue by induction on the number of steps N . For $N = 0$ it is trivial since Φ_0 is the identity map. Suppose that we have performed N steps. By the fact that $\{H, K\} = 0$ then $\{H, K\} \circ \Phi_N = 0$ and so, at each order, we have

$$\begin{aligned} \{H^{(0)}, K^{(0)}\} &= 0, \\ \{H^{(0)}, W_N\} + \{Z_N, K^{(0)}\} + \Pi^{(\leq N)} \{Z_N, W_N\} &= 0, \\ \Pi^{(N+1)} \{Z_N, W_N\} + \{H^{(0)}, Q_N^{(N+1)}\} + \{R_N^{(N+1)}, K^{(0)}\} &= 0, \\ \dots \end{aligned}$$

By the inductive hypothesis $W_N, Z_N \in \text{Ker}(H^{(0)}) \cap \text{Ker}(K^{(0)})$, hence $\{H^{(0)}, W_N\} = \{Z_N, K^{(0)}\} = 0$ and

$$\{H^{(0)}, Q_N^{(N+1)}\} + \{R_N^{(N+1)}, K^{(0)}\} = 0 \quad (4.6)$$

since $\{H^{(0)}, Q_N^{(N+1)}\} \in \text{Rg}(H^{(0)})$ and $\{R_N^{(N+1)}, K^{(0)}\} \in \text{Rg}(K^{(0)})$.

We note the following fact, which derives from the Jacobi identity: if $f \in \text{Ker}(H^{(0)})$ then $\{f, K^{(0)}\} \in \text{Ker}(H^{(0)})$.

Then we have that $\{\Pi_{\text{Ker}(H^{(0)})} R_N^{(N+1)}, K^{(0)}\} \in \text{Ker}(H^{(0)})$ and by (4.6)

$$\{\Pi_{\text{Ker}(H^{(0)})} R_N^{(N+1)}, K^{(0)}\} = -\{\Pi_{\text{Rg}(H^{(0)})} R_N^{(N+1)}, K^{(0)}\} + \{H^{(0)}, Q_N^{(N+1)}\} \in \text{Rg}(H^{(0)}).$$

Thus $\{\Pi_{\text{Ker}(H^{(0)})} R_N^{(N+1)}, K^{(0)}\} = 0$ and

$$\Pi_{\text{Ker}(H^{(0)})} R_N^{(N+1)} = \Pi_{\text{Ker}(H^{(0)})} \Pi_{\text{Ker}(K^{(0)})} R_N^{(N+1)}.$$

By symmetry $\Pi_{\text{Ker}(K^{(0)})} Q_N^{(N+1)} = \Pi_{\text{Ker}(H^{(0)})} \Pi_{\text{Ker}(K^{(0)})} Q_N^{(N+1)}$. Hence

$$\Pi_{\text{Rg}(H^{(0)})} \Pi_{\text{Ker}(K^{(0)})} Q_N^{(N+1)} = \Pi_{\text{Rg}(K^{(0)})} \Pi_{\text{Ker}(H^{(0)})} R_N^{(N+1)} = 0. \quad (4.7)$$

In order to obtain the Birkhoff normal form at order $N + 1$ we consider a Birkhoff transformation $\Phi_{\chi_{N+1}}$ with generator $\chi_{N+1} \in \mathcal{P}^{(N+1)}$ and we define $\Phi_{N+1} := \Phi_N \circ \Phi_{\chi_{N+1}}$. The function χ_{N+1} is chosen in order to solve the homological equation

$$\{H^{(0)}, \chi_{N+1}\} = -\Pi_{Rg(H^{(0)})} R_N^{(N+1)} \stackrel{(4.7)}{=} -\Pi_{Rg(K^{(0)})} \Pi_{Rg(H^{(0)})} R_N^{(N+1)}.$$

We now show that χ_{N+1} solves also the homological equation for K_N . Indeed, by the fact that $\text{ad}_{H^{(0)}}^{-1}$ commutes with $\text{ad}_{K^{(0)}}$ on the intersection $Rg(H^{(0)}) \cap Rg(K^{(0)})$

$$\{K^{(0)}, \chi_{N+1}\} = -\text{ad}_{H^{(0)}}^{-1} \{K^{(0)}, \Pi_{Rg(K^{(0)})} \Pi_{Rg(H^{(0)})} R_N^{(N+1)}\}$$

and by (4.6), (4.7) we get

$$\{K^{(0)}, \Pi_{Rg(K^{(0)})} \Pi_{Rg(H^{(0)})} R_N^{(N+1)}\} = \{H^{(0)}, \Pi_{Rg(K^{(0)})} \Pi_{Rg(H^{(0)})} Q_N^{(N+1)}\}.$$

□

Suppose that we performed N Birkhoff steps. Then the transformed Hamiltonian is

$$H_N = H^{(0)} + Z_N + R_N, \quad Z_N \in \mathcal{P}^{(\leq N)}, R_N \in \mathcal{F}^{(\geq N+1)}$$

where Z_N is action-preserving.

If we perform the $(N + 1)$ -th step then the term $\Pi_{Ker(H^{(0)})} R_N^{(N+1)} \in \mathcal{P}^{(N+1)}$ contributes to the normal form. We have to show the following claim

- The resonant term $\Pi_{Ker(H^{(0)})} R_N^{(N+1)}$ is supported on \mathcal{N}_{N+1}^* .

The identity (4.6) holds for any $N \geq 0$. Thus if we consider the set of commuting Hamiltonians K_1, \dots, K_{N+2} (defined in (1.11) and in Theorem 1.4), which are in the Birkhoff normal form after N steps

$$K_{m,N} = K_m^{(0)} + W_{m,N} + Q_{m,N} \quad m = 1, \dots, N + 2,$$

we have that $R_N^{(N+1)} \in \mathcal{P}^{(N+1)}$ satisfies

$$\{H^{(0)}, Q_{m,N}^{(N+1)}\} = \{K_m^{(0)}, R_N^{(N+1)}\}, \quad \forall m = 1, \dots, N + 2. \quad (4.8)$$

Thus by (1.24) the above relation writes as (recall (4.3))

$$\Omega(\alpha) K_{m,N,\alpha}^{(N+1)} = \Omega_{K_m^{(0)}}(\alpha) R_{N,\alpha}^{(N+1)} = \left(\sum_j j^{2(m-1)} (1 + j^2)^2 \omega(j) \alpha_j \right) R_{N,\alpha}^{(N+1)}, \quad (4.9)$$

for any $\alpha \in \mathcal{I}_{N+1}$ and any $m = 1, \dots, N + 2$.

If $\Omega(\alpha) \neq 0$ or $R_{N,\alpha}^{(N+1)} = 0$ then the resonant term $\Pi_{Ker(H^{(0)})} R_N^{(N+1)} = 0$.

If $\Omega(\alpha) = 0$ then $R_{N,\alpha}^{(N+1)} \neq 0$ if and only if

$$\sum_j j^{2(m-1)} (1 + j^2)^2 \omega(j) \alpha_j = \sum_j j^{2m-1} (1 + j^2) (4 + j^2) \alpha_j = 0, \quad \forall \alpha \in \mathcal{I}_{N+1}, \quad \forall m = 1, \dots, N + 2. \quad (4.10)$$

We have to prove that the linear $(N + 3) \times (N + 3)$ -dimensional system (4.10) has no solutions, except for $\alpha \in \mathcal{N}_{N+1}^*$.

Finding a solution of (4.10) is equivalent to prove that there are integers $j_1, \dots, j_{N+3} \in \mathbb{Z} \setminus \{0\}$ such that the following matrix M has a non trivial kernel

$$M := \text{diag}_{i=1, \dots, N+3} \left((1 + j_i^2) (4 + j_i^2) \right) V, \quad V := \begin{bmatrix} j_1 & \dots & j_N \\ j_1^3 & \dots & j_N^3 \\ \vdots & \dots & \vdots \\ j_1^{2(N+3)+1} & \dots & j_N^{2(N+3)+1} \end{bmatrix}. \quad (4.11)$$

Hence our goal is to prove that $\det M = 0$ if and only if $(j_1, \dots, j_{N+3}) = (i, -i, j, -j, \dots)$ or its permutations. Clearly $\det M = 0$ if and only if $\det V = 0$. Note that

$$\det V = \left(\prod_{i=1}^{N+3} j_i \right) \det \begin{bmatrix} 1 & \dots & 1 \\ j_1^2 & \dots & j_{N+3}^2 \\ \vdots & \dots & \vdots \\ j_1^{2N+6} & \dots & j_{N+3}^{2N+6} \end{bmatrix}$$

and by renaming $x_i = j_i^2$ and by using the well known formula for the determinant of the Vandermonde matrix we have that

$$\det V = \sqrt{\prod_{i=1}^{N+3} x_i} \prod_{i < j} (x_i - x_j).$$

So it is clear that $\det V = 0$ if and only if

$$\left((j_1 + j_2) \dots (j_{N+2} + j_{N+3}) \right) \left((j_1 - j_2) \dots (j_{N+2} - j_{N+3}) \right) = 0.$$

By the fact that the indices j_i are all distincts we deduce the claim.

A Appendix

A.1 Proof of local existence

Here we prove Proposition 3.1 about the local well-posedness of Eq. (1.6). The argument follows closely the proof of Theorem 1.1 in [22], which discusses the well-posedness of the dispersionless DP equation (1.4). We present the proof in the compact case; during the proof we point out the minor changes one has to make in order to adjust the proof to the noncompact case.

First observe that if $u \in H^s(\mathbb{T}; \mathbb{R})$, then $u \partial_x u + (1 - \partial_{xx})^{-1} \partial_{xxx} u \in H^{s-1}(\mathbb{T}; \mathbb{R})$. We handle this problem by considering the mollified version of (1.6): fix a Schwartz function $j \in \mathcal{S}(\mathbb{R})$ satysfying $0 \leq \hat{j}(\xi) \leq 1$ for all $\xi \in \mathbb{R}$, and $\hat{j}(\xi) = 1$ for $|\xi| \leq 1$. Then we define the periodic functions j_ϵ , $0 < \epsilon \leq 1$ by the following formula,

$$j_\epsilon(x) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{j}(\epsilon n) e^{inx},$$

and we define the mollifier by

$$J_\epsilon f := j_\epsilon * f. \tag{A.1}$$

By direct computation one has that $\hat{j}_\epsilon(k) = \hat{j}(\epsilon k)$; furthermore, for $0 < \sigma \leq s$ the map $Id - J_\epsilon : H^s(\mathbb{T}; \mathbb{R}) \rightarrow H^\sigma(\mathbb{T}; \mathbb{R})$ satisfies

$$\|Id - J_\epsilon\|_{L(H^s(\mathbb{T}; \mathbb{R}), H^\sigma(\mathbb{T}; \mathbb{R}))} = o(\epsilon^{s-\sigma}). \tag{A.2}$$

Now recall that Eq. (1.6) is obtained by its dispersionless version (1.4) by applying the boost $u \mapsto c + u$. This means that we can derive the mollified version of (1.6) simply by translation from the mollified dispersionless DP equation (see Eq. (110) in [22]; in the rest of this section we denote by D the operator $(1 - \partial_{xx})^{1/2}$)

$$u_t = -J_\epsilon(J_\epsilon(c + u) \partial_x J_\epsilon(c + u)) - \frac{3}{2} \partial_x D^{-2} ((c + u)^2), \quad 0 < \epsilon \leq 1, \tag{A.3}$$

with initial datum $u(0, x) = u_0(x) \in H^s$.

Now introduce the map $F_\epsilon : H^s \rightarrow H^s$,

$$\begin{aligned} F_\epsilon(u) &= -J_\epsilon(J_\epsilon(c+u)\partial_x J_\epsilon(c+u)) - \frac{3}{2}\partial_x D^{-2}((c+u)^2) \\ &= -J_\epsilon(cJ_\epsilon\partial_x J_\epsilon u) - J_\epsilon(J_\epsilon u\partial_x J_\epsilon u) - \frac{3}{2}\partial_x D^{-2}(u^2) - 3c\partial_x D^{-2}(u); \end{aligned} \quad (\text{A.4})$$

we can observe that for any ϵ the map F_ϵ is differentiable. Therefore (A.3) with initial datum $u(0, \cdot) = u_0 \in H^s$, $s > 3/2$, defines an ODE on H^s , which thus admits a unique solution u_ϵ with existence time $T_\epsilon > 0$. We now prove Proposition 3.1 after some intermediate lemmata.

Lemma A.1. *Let $s > 3/2$, then there exists $C(s) > 0$ such that the existence time T_ϵ for the solution of (A.3) satisfies*

$$T_\epsilon \geq \bar{T} := \bar{T}(\|u_0\|_{H^s}) = \frac{1}{2C(s)\|u_0\|_{H^s}}, \quad (\text{A.5})$$

while the solution u_ϵ satisfies

$$\|u_\epsilon(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad |t| \leq \bar{T}. \quad (\text{A.6})$$

Proof. First apply the operator D^s to both sides of (A.3), and multiply both sides by $D^s u_\epsilon$. By integration and (A.4) we have

$$\frac{1}{2} \frac{d}{dt} \|u_\epsilon\|_{H^s}^2 = - \int_0^{2\pi} D^s u_\epsilon D^s \left[J_\epsilon(J_\epsilon u_\epsilon \partial_x J_\epsilon u_\epsilon) + \frac{3}{2} \partial_x D^{-2}(u^2) \right] dx, \quad (\text{A.7})$$

where we exploited the fact that the first and the last term in (A.4) are linear in u and are given by the action of skew-adjoint Fourier multipliers which commute with D^s (all these facts imply that the terms one would need to add to formula (117) in [22] vanish).

In order to commute the operator D^s with $J_\epsilon u_\epsilon$ we apply the following Kato-Ponce commutator estimate (see [24]).

Lemma A.2. *Let $s > 0$, then there exists $C(s) > 0$ such that*

$$\|D^s(fg) - fD^s g\|_{L^2} \leq C(s) (\|D^s f\|_{L^2} \|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty} \|D^{s-1} g\|_{L^2}). \quad (\text{A.8})$$

By exploiting (A.8) and Sobolev embedding we have that there exists $C(s) > 0$ such that

$$\frac{d}{dt} \|u_\epsilon\|_{H^s}^2 \leq 2C(s) \|u_\epsilon\|_{H^s}^3,$$

which implies that

$$\|u_\epsilon(t)\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - C(s)\|u_0\|_{H^s} t}, \quad (\text{A.9})$$

and by setting $\bar{T} := \frac{1}{2C(s)\|u_0\|_{H^s}}$ we get the thesis. \square

Lemma A.3. *Consider Eq. (1.6) with initial datum $u_0 \in H^s$, $s > 3/2$. Then there exists a solution $u \in C([- \bar{T}, \bar{T}]; H^s)$ with \bar{T} as in (A.5) such that*

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad |t| \leq \bar{T}. \quad (\text{A.10})$$

Proof. To simplify the notation we set $I := [-\bar{T}, \bar{T}]$. The proof is divided in several steps, whose purpose is to obtain the convergence of the family $(u_\epsilon)_{0 < \epsilon \leq 1}$ by extracting subsequences $(u_{\epsilon_\nu})_\nu$; after each such extraction, we assume that the resulting sequence is relabeled as $(u_\epsilon)_\epsilon$.

Step 1: weak* convergence in $L^\infty(I; H^s)$. The family $(u_\epsilon)_{0 < \epsilon \leq 1}$ is bounded in the space $C(I; H^s) \subset L^\infty(I; H^s)$. Since $L^\infty(I; H^s)$ is the dual of the space $L^1(I; H^s)$, Alaoglu's Theorem implies that $(u_\epsilon)_\epsilon$ is

precompact with respect to the weak* topology. Hence there exists a subsequence $(u_{\epsilon_\nu})_\nu$ which converges to $u \in L^\infty(I; H^s)$ weakly*, and such that u satisfies (A.10).

Step 2: convergence in $C(I; H^{s-1})$. In order to show the strong convergence in $C(I; H^{s-1})$, we show that $(u_\epsilon)_\epsilon$ satisfies the hypotheses of Ascoli-Arzelá Theorem. Indeed, $(u_\epsilon)_\epsilon$ is equicontinuous, since for any $t_1, t_2 \in I$

$$\begin{aligned} \|u_\epsilon(t_1) - u_\epsilon(t_2)\|_{H^{s-1}} &\leq \sup_{t \in I} \|\partial_t u_\epsilon\|_{H^{s-1}} |t_1 - t_2| \\ &\stackrel{(A.3), (A.4), (A.8)}{\leq} 10C(s)(\|u_0\|_{H^s} + \|u_0\|_{H^s}^2) |t_1 - t_2|. \end{aligned}$$

Setting $U(t) := (u_\epsilon(t))_\epsilon$, we see that for any $t \in I$ the set $U(t) \subset H^s$ is bounded. On the other hand, since \mathbb{T} is a compact manifold, we have that the inclusion $i : H^s \rightarrow H^{s-1}$ is compact. Therefore $U(t)$ is precompact in H^{s-1} .

Step 3: convergence in $C(I; H^{s-\sigma})$, $\sigma \in (0, 1)$. For each $\sigma \in (0, 1)$ we have

$$\|u_\epsilon\|_{C^\sigma(I; H^{s-\sigma})} = \sup_{t \in I} \|u_\epsilon\|_{H^{s-\sigma}} + \sup_{t \neq t'} \frac{\|u_\epsilon(t) - u_\epsilon(t')\|_{H^{s-\sigma}}}{|t - t'|^\sigma}.$$

Now, the first term in the right-hand side of the above inequality is bounded, since

$$\sup_{t \in I} \|u_\epsilon(t)\|_{H^{s-\sigma}} \stackrel{(A.6)}{\leq} 2\|u_0\|_{H^s},$$

while the second term can be bounded by exploiting (A.4) and (A.8). Putting these bounds together allows us to apply Ascoli-Arzelá Theorem, since the equicontinuity condition follows from

$$\|u_\epsilon(t_1) - u_\epsilon(t_2)\|_{H^{s-\sigma}} \leq \|u_\epsilon\|_{C^\sigma(I; H^{s-\sigma})} |t_1 - t_2|^\sigma,$$

while the precompactness condition can be verified as in the previous step.

Step 4: convergence in $C(I; C^1(\mathbb{T}))$. Now fix $\sigma \in (0, 1)$ such that $s - \sigma > 3/2$, then by Sobolev embedding implies that $u_\epsilon \rightarrow u$ in $C(I; C^1(\mathbb{T}))$. Now we need to study $\partial_t u_\epsilon$. Starting with the two non-local terms of (A.4), the continuity of the operator $\partial_x D^{-2}$ implies that $\partial_x D^{-2}(u_\epsilon^2) \rightarrow \partial_x D^{-2}(u^2)$ and $\partial_x D^{-2}u_\epsilon \rightarrow \partial_x D^{-2}u$ in $C(I; C(\mathbb{T}))$. To handle the first two terms, first observe that

$$\|J_\epsilon u_\epsilon - u\|_{C(I; C(\mathbb{T}))} \leq \|J_\epsilon u_\epsilon - u_\epsilon\|_{C(I; C(\mathbb{T}))} + \|u_\epsilon - u\|_{C(I; C(\mathbb{T}))}. \quad (\text{A.11})$$

To estimate the first term in the right-hand side of (A.11), choose $r \in (1/2, s)$, and observe that for any $t \in I$ (A.2) implies that there exists $C(r) > 0$ such that

$$\|J_\epsilon u_\epsilon - u_\epsilon\|_{C(I; C(\mathbb{T}))} \leq 2C(r)\|Id - J_\epsilon\|_{L(H^s; H^r)} \|u_0\|_{H^s} = o(\epsilon^{s-r}),$$

from which we can deduce that $J_\epsilon u_\epsilon \rightarrow u$ in $C(I; C(\mathbb{T}))$. With a similar argument we can show that $J_\epsilon \partial_x u_\epsilon \rightarrow \partial_x u$ in $C(I; C(\mathbb{T}))$. Therefore one can conclude that

$$\partial_t u_\epsilon \rightarrow -(\mathfrak{c} + u)\partial_x(\mathfrak{c} + u) - \frac{3}{2}\partial_x D^{-2}((\mathfrak{c} + u)^2)$$

in $C(I; C(\mathbb{T}))$. Recalling that also $u_\epsilon \rightarrow u$ in $C(I; C^1(\mathbb{T}))$, we can deduce that $t \mapsto u(t)$ is a differentiable map such that

$$\partial_t u = -(\mathfrak{c} + u)\partial_x(\mathfrak{c} + u) - \frac{3}{2}\partial_x D^{-2}((\mathfrak{c} + u)^2).$$

Step 5: convergence in $C(I; H^s)$. Fix $t \in I$ and take a sequence $(t_n)_{n \in \mathbb{N}} \rightarrow t$. Since $u \in L^\infty(I; H^s)$, we have that $t \mapsto u(t)$ is continuous with respect to the weak topology on H^s ; thus, to verify the continuity we just need to check that the map $t \mapsto \|u(t)\|_{H^s}^2$ is continuous. We begin by introducing

$$\begin{aligned} F(t) &:= \|u(t)\|_{H^s}^2, \\ F_\epsilon(t) &:= \|J_\epsilon u(t)\|_{H^s}^2. \end{aligned}$$

Now, (A.2) implies that $F_\epsilon \rightarrow F$ pointwise as $\epsilon \rightarrow 0$. Therefore it suffices to show that each F_ϵ is Lipschitz and that the Lipschitz constants for this family are bounded. Since

$$\frac{1}{2}F'_\epsilon(t) = - \int_0^{2\pi} D^s J_\epsilon u D^s J_\epsilon [(c+u)\partial_x(c+u)] dx - \frac{3}{2} \int_0^{2\pi} D^s J_\epsilon u D^s J_\epsilon \partial_x D^{-2}((c+u)^2) dx. \quad (\text{A.12})$$

To bound the first term on the right-hand side of (A.12) we need to use the commutator estimate (A.8), in order to commute the operator D^s with u ; but since we also need to commute J_ϵ with u , we exploit the following result (see [32]):

Lemma A.4. *Let $f, g \in H^s$, then there exists $C > 0$ such that*

$$\|[f, J_\epsilon]\partial_x g\|_{L^2} \leq C\|f\|_{C^1(\mathbb{T})}\|g\|_{L^2(\mathbb{T})}.$$

By applying the Cauchy-Schwarz inequality, the algebra property, and estimate (3.1) on the size of the solution, we can conclude that there exists $C(s) > 0$ such that

$$|F'_\epsilon(t)| = \left| \frac{d}{dt} \|J_\epsilon u(t)\|_{H^s}^2 \right| \leq C(s).$$

□

Remark A.5. To adjust the proof for the noncompact case, we have to define the mollifiers J_ϵ in the following way: first we fix $j \in \mathcal{S}(\mathbb{R})$ such that $\hat{j}(\xi) = 1$ for $|\xi| \leq 1$. Then we set $j_\epsilon(x) := \epsilon^{-1}j(x/\epsilon)$; this gives again that $\|Id - J_\epsilon\|_{L(H^s, H^r)} = o(\epsilon^{s-r})$.

Moreover, we exploited the compactness of \mathbb{T} in order to satisfy the hypotheses of Ascoli's theorem; in the non-compact case the embedding $H^s \rightarrow H^{s'}$ for $s > s'$ does not define a compact operator. We handle this problem by first fixing $\phi \in \mathcal{S}(\mathbb{R})$ with $0 < \phi(x) \leq 1$. Then Rellich's Theorem implies that the operator $u_\epsilon \mapsto \phi u_\epsilon$ is compact from H^s to $H^{s'}$. Using this modification and by recalling that $\phi \neq 0$, we obtain again the existence of a solution u .

Lemma A.6. *Consider Eq. (1.6) with initial datum $u_0 \in H^s$, $s > 3/2$. Then its solution $u \in C([-T, T]; H^s)$ with T as in (A.5) is unique.*

Proof. Let $u_0 \in H^s$, and let u and w be two solutions to (1.6) with $u(0, \cdot) = w(0, \cdot) = u_0$. Consider $v := u - w$, then

$$\partial_t v = -\frac{1}{2}\partial_x [(2c + u + w)v] - \frac{3}{2}\partial_x D^{-2}((2c + u + w)v). \quad (\text{A.13})$$

Fix $\sigma \in (1/2, s-1)$; then

$$\frac{d}{dt} \|v\|_{H^\sigma}^2 = - \int_0^{2\pi} D^\sigma v [D^\sigma \partial_x ((2c + u + w)v) + 3\partial_x D^{\sigma-2}((2c + u + w)v)] dx. \quad (\text{A.14})$$

In order to bound the first term in the right-hand side of (A.14) we commute $D^\sigma \partial_x$ with $u + w$ by exploiting the following Calderon-Coifman-Meyer estimate (see Proposition 4.2 in [33])

Lemma A.7. *Let $\sigma \geq -1$, then for any $\rho > 3/2$ such that $\sigma + 1 \leq \rho$ there exists $C > 0$ such that*

$$\|[D^\sigma \partial_x, f]v\|_{L^2} \leq C\|f\|_{H^\rho}\|v\|_{H^\sigma}.$$

The nonlocal term is bounded by Plancherel and Cauchy-Schwarz inequality. Hence there exists $c(s) > 0$ such that

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{H^\sigma}^2 &\leq c(s)\|v\|_{H^\sigma}^2; \\ \|v\|_{H^\sigma} &\leq e^{c(s)T}\|v(0)\|_{H^\sigma} = 0, \end{aligned}$$

and we can conclude that $u = w$.

□

Lemma A.8. Consider Eq. (1.6) with initial datum $u_0 \in H^s$, $s > 3/2$. Then the solution map from $H^s \rightarrow C(I; H^s)$ ($I = [-\bar{T}, \bar{T}]$, with \bar{T} as in (A.5)) given by $u_0 \mapsto u$ is continuous.

Proof. Fix $u_0 \in H^s$, and let $(u_{0,n})_n \subset H^s$ be a sequence such that $\lim_{n \rightarrow \infty} u_{0,n} = u_0$. Then, if u_n is the solution of Eq. (1.6) with initial datum $u_{0,n}$, we want to show that

$$\lim_{n \rightarrow \infty} u_n = u \text{ in } C(I; H^s); \quad (\text{A.15})$$

equivalently, let $\eta > 0$, we want to show that there exists $N > 0$ such that

$$\|u - u_n\|_{C(I; H^s)} < \eta, \quad \forall n > N. \quad (\text{A.16})$$

As before, we will use the convolution operator to smooth out the initial data. Let $0 < \epsilon \leq 1$, let u^ϵ be the solution to (1.6) with initial datum $J_\epsilon u_0 = j_\epsilon * u_0$ and let u_n^ϵ be the solution of (1.6) with initial datum $J_\epsilon u_{0,n}$. Then

$$\|u - u_n\|_{C(I; H^s)} \leq \|u - u^\epsilon\|_{C(I; H^s)} + \|u^\epsilon - u_n^\epsilon\|_{C(I; H^s)} + \|u^\epsilon - u_n\|_{C(I; H^s)}. \quad (\text{A.17})$$

We will prove that each of these terms can be bounded by $\eta/3$, for suitable choices of ϵ and N . We also point out that the quantity ϵ will be independent of N and will only depend on η , while the choice of N will depend on both η and ϵ .

We start with $\|u^\epsilon - u_n^\epsilon\|_{C(I; H^s)}$. Set $v := u^\epsilon - u_n^\epsilon$, then v satisfies

$$\begin{aligned} \partial_t v &= -\frac{1}{2} \partial_x [(2c + u^\epsilon + u_n^\epsilon)v] - \frac{3}{2} \partial_x D^{-2}((2c + u^\epsilon + u_n^\epsilon)v), \\ v(0) &= u^\epsilon(0) - u_n^\epsilon(0) = J_\epsilon u_0 - J_\epsilon u_{0,n}, \end{aligned}$$

and

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^s}^2 = - \int_0^{2\pi} D^s v D^s \left[\frac{1}{2} \partial_x [(2c + u^\epsilon + u_n^\epsilon)v] + \frac{3}{2} \partial_x D^{-2}((2c + u^\epsilon + u_n^\epsilon)v) \right] dx. \quad (\text{A.18})$$

Applying (A.8) and the estimate $\|u^\epsilon\|_{H^{s+1}} \leq C/\epsilon$, (A.18) implies that there exists $c_s > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^s}^2 \leq \frac{c_s}{\epsilon} \|v(t)\|_{H^s}^2, \quad (\text{A.19})$$

which in turn leads to

$$\|v(t)\|_{H^s} \leq e^{c_s T/\epsilon} \|v(0)\|_{H^s} \leq 2e^{c_s T/\epsilon} \|u_0 - u_{0,n}\|_{H^s}. \quad (\text{A.20})$$

Notice that (A.20) does not imply any constraint on ϵ ; however, handling the first and the third term in the right-hand side of (A.17) will require ϵ to be small. After choosing ϵ , we will take N so large that $\|u_0 - u_{0,n}\|_{H^s} < \frac{\eta}{6} e^{-c_s T/\epsilon}$, which will imply that $\|u^\epsilon - u_n^\epsilon\|_{C(I; H^s)} < \eta/3$.

Now we estimate $\|u^\epsilon - u\|_{C(I; H^s)}$ and $\|u^\epsilon - u_n\|_{C(I; H^s)}$. We set $v := u^\epsilon - u$ and $v_n := u_n^\epsilon - u_n$. Since v and v_n will satisfy the same energy estimates, we will write $v_{(n)}$ to mean that an equation holds both with and without the subscript. We observe that $v_{(n)}$ solves the Cauchy problem

$$\begin{aligned} \partial_t v_{(n)} &= -\frac{1}{2} \partial_x [(2c + u^\epsilon + u_{(n)})v_{(n)}] - \frac{3}{2} \partial_x D^{-2}((2c + u^\epsilon + u_{(n)})v_{(n)}) \\ &= -\frac{1}{2} \partial_x [(2c + 2u^\epsilon + v_{(n)})v_{(n)}] - \frac{3}{2} \partial_x D^{-2}((2c + 2u^\epsilon + v_{(n)})v_{(n)}), \\ v(0) &= j_\epsilon * u_{0,(n)} - u_{0,(n)}. \end{aligned}$$

By exploiting (A.8), the Cauchy-Schwarz inequality and Sobolev embedding we get

$$\frac{1}{2} \frac{d}{dt} \|v_{(n)}(t)\|_{H^s} \leq c'_s \left[\|v_{(n)}\|_{H^s}^3 + (1 + \|u_{(n)}^\epsilon\|_{H^s}) \|v_{(n)}\|_{H^s}^2 + (1 + \|u_{(n)}^\epsilon\|_{H^{s+1}}) \|v_{(n)}\|_{H^{s-1}} \|v_{(n)}\|_{H^s} \right] \quad (\text{A.21})$$

for some $c'_s > 0$. Since $\|u_{(n)}^\epsilon(t)\|_{H^{s+1}} \leq c_1(s)/\epsilon$ and that $\|v_{(n)}(t)\|_{L^2} = o(\epsilon)$, (A.21) gives

$$\frac{dy}{dt} \leq c_2(s) (y^2 + y + \delta), \quad (\text{A.22})$$

where $\delta = \delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

The quadratic expression $y^2 + y + \delta$ has roots

$$r_{-1} = \frac{-1 - \sqrt{1 - 4\delta}}{2}, \quad r_0 = \frac{-1 + \sqrt{1 - 4\delta}}{2}. \quad (\text{A.23})$$

Restricting ϵ so that the roots given in (A.23) are real-valued, we observe that r_0 and r_{-1} are negative and, as $\delta \rightarrow 0$, we have $r_{-1} \rightarrow -1$ and $r_0 \rightarrow 0$. Setting $R := \sqrt{1 - 4\delta}$ and taking into account the constant c_s , we solve (A.22) via

$$\frac{y(t) - r_0}{y(t) - r_{-1}} \leq \gamma, \quad (\text{A.24})$$

$$\gamma := e^{c_s R T} \frac{y(0) - r_0}{y(0) - r_{-1}}. \quad (\text{A.25})$$

From here, we will treat the cases $y = \|v\|_{H^s}$ and $y = \|v_n\|_{H^s}$ separately.

Case $y = \|v\|_{H^s}$. Using (A.2) we have $y(0) \rightarrow 0$ as $\epsilon \rightarrow 0$. This implies that $\gamma \rightarrow 0$ as $\epsilon \rightarrow 0$. From (A.24), we then obtain $y(t) \leq y(t) - r_0 \leq \gamma[y(t) - r_{-1}]$. Solving for $y(t)$ gives us

$$y(t) \leq \frac{-r_{-1}}{1 - \gamma} \gamma$$

$$\xrightarrow{\gamma \rightarrow 0} 0.$$

Therefore, for sufficiently small ϵ we can bound the first term of (A.17) by $\eta/3$.

Case $y = \|v_n\|_{H^s}$. We begin by bounding $y(0)$ by

$$\|j_\epsilon * u_{0,n} - u_{0,n}\|_{H^s} \leq 2\|u_{0,n} - u_0\|_{H^s} + \|j_\epsilon * u_0 - u_0\|_{H^s},$$

which implies that

$$\gamma \leq \frac{e^{c_s R T}}{-r_{-1}} (2\|u_{0,n} - u_0\|_{H^s} + \|j_\epsilon * u_0 - u_0\|_{H^s}) + \frac{r_0 e^{c_s R T}}{r_{-1}}, \quad (\text{A.26})$$

where we may independently choose ϵ sufficiently small and N sufficiently large so that $\gamma < 1/2$. Then, arguing as in the previous case we obtain $y(t) \leq 2\gamma$. We may now further refine the choice of ϵ and N so that $y(t) < \eta/3$, completing this case. Collecting our results completes the proof. \square

Remark A.9. The proofs for uniqueness of the solution and for the continuous dependence on the initial datum do not rely on compactness properties, hence they do not need any adjustment in the noncompact case.

A.2 Analyticity on Sobolev spaces

We recall some facts about analytic functions on Banach spaces following Appendix A of [31].

Definition A.10. (Weakly analyticity) Let E, F two complex Banach spaces and U an open subset of E . A map $f: U \rightarrow F$ is said weakly analytic if for each $w \in U$, $h \in E$ and $L \in F^*$ the function

$$z \mapsto Lf(w + zh)$$

is analytic in some neighborhood of the origin in \mathbb{C} in the usual sense of one complex variable.

Theorem A.11. *Let $f: U \rightarrow F$ be a map from an open subset U of a complex Banach space E into a complex Banach space F . Then the following three statements are equivalent.*

1. *f is analytic in U .*
2. *f is locally bounded and weakly analytic in U .*
3. *f is infinitely often differentiable on U , and is represented by its Taylor series in a neighborhood of each point in U .*

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