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Title: *Doubling condition at the origin for non-negative positive definite functions*

Journal Information: *Proceedings of the American Mathematical Society*

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Volume, pages: 609–618, DOI:[www.doi.org/10.1090/proc/14191]

DOUBLING CONDITION AT THE ORIGIN FOR NON-NEGATIVE POSITIVE DEFINITE FUNCTIONS

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ABSTRACT. We study upper and lower estimates as well as the asymptotic behavior of the sharp constant $C = C_n(U, V)$ in the doubling-type condition at the origin

$$\frac{1}{|V|} \int_V f(x) dx \leq C \frac{1}{|U|} \int_U f(x) dx,$$

where $U, V \subset \mathbb{R}^n$ are 0-symmetric convex bodies and f is a non-negative positive definite function.

1. INTRODUCTION

Very recently, answering the question posed by Konyagin and Shteinikov related to a problem from number theory [13], the first author proved [1] that for any positive definite function $f: \mathbb{Z}_q \rightarrow \mathbb{R}_+$ and for any $n \in \mathbb{Z}_+$ one has

$$\sum_{0 \leq k \leq 2n} f(k) \leq C \sum_{0 \leq k \leq n} f(k),$$

where the positive constant C does not depend on n , f , and q . More precisely, it was proved that $C \leq \pi^2$.

In this paper we study similar inequalities for a non-negative positive definite function f defined on \mathbb{R}^n , $n \geq 1$, i.e.,

$$(1.1) \quad \int_{|x| \leq 2R} f(x) dx \leq C \int_{|x| \leq R} f(x) dx, \quad R > 0,$$

for some $C > 1$. The latter is the well-known doubling condition at the origin. The doubling condition plays an important role in harmonic and functional analysis, see, e.g., [14]. Note that very recently inequality (1.1) in the one-dimensional case was studied in [3].

Definition 1. A positive definite function $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called double positive definite function (denoted $f \succeq 0$).

Date: March 8, 2018.

1991 *Mathematics Subject Classification.* 42A82, 42A38.

Key words and phrases. non-negative positive definite functions, doubling property, Wiener property.

The first author was partially supported by the RFBR (no. 16-01-00308) and the Ministry of Education and Science of the Russian Federation (no. 5414GZ). The second author was partially supported by MTM 2014-59174-P and 2014 SGR 289.

As usual [11, Chap. 1], a continuous function $f \in C(\mathbb{R}^n)$ is positive definite if for every finite sequence $X \subset \mathbb{R}^n$ and every choice of complex numbers $\{c_a : a \in X\}$, we have

$$\sum_{a,b \in X} c_a \overline{c_b} f(a-b) \geq 0.$$

By Bochner's theorem [11, Chap. 1], $f \in C(\mathbb{R}^n)$ is positive definite if and only if there is a non-negative finite Borel measure μ such that

$$(1.2) \quad f(x) = \int_{\mathbb{R}^n} e(\xi x) d\mu(\xi), \quad \xi \in \mathbb{R}^n,$$

where $e(t) = \exp(2\pi i t)$. For $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ it is equivalent to the fact that the Fourier transform of f

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e(-\xi x) dx$$

is non-negative. Note also that since any positive definite f satisfies $f(-x) = \overline{f(x)}$, a double positive definite function is even.

Throughout the paper we assume that $U, V \subset \mathbb{R}^n$ be 0-symmetric closed convex bodies. For any function $f \geq 0$ we study the inequality

$$(1.3) \quad \frac{1}{|V|} \int_V f(x) dx \leq C \frac{1}{|U|} \int_U f(x) dx,$$

where $|A|$ is the volume of A or the cardinality of A if A is a finite set. By $C_n(U, V)$ we denote the sharp constant in (1.3), i.e.,

$$C_n(U, V) := \sup_{f \geq 0, f \neq 0} \frac{\frac{1}{|V|} \int_V f(x) dx}{\frac{1}{|U|} \int_U f(x) dx}.$$

The fact that $C_n(U, V) < \infty$ for any U and V will follow from Theorem 1 below.

First, we list the following simple properties of $C_n(U, V)$.

(1) A trivial lower bound

$$(1.4) \quad C_n(U, V) \geq 1,$$

since $1 \geq 0$;

(2) The homogeneity property

$$(1.5) \quad C_n(\lambda U, \lambda V) = C_n(U, V), \quad \lambda > 0,$$

since $f_\lambda(x) = f(\lambda x) \geq 0$ if and only if $f \geq 0$;

(3) The homogeneity estimate

$$(1.6) \quad C_n(U, \lambda V) \geq \lambda^{-n} C_n(U, V), \quad \lambda \geq 1,$$

since $V \subset \lambda V$;

(4) $C_n(U, U) = 1$ and if $V \subset U$, then

$$C_n(U, V) \leq \frac{|U|}{|V|};$$

(5) The multiplicative estimate

$$C_n(U, V) \leq C_n(\lambda^k U, V) (C_n(U, \lambda U))^k, \quad \lambda \geq 1, \quad k \in \mathbb{Z}_+,$$

which follows from the chain of inequalities

$$\begin{aligned} C_n(U, V) &\leq C_n(\lambda U, V) C_n(U, \lambda U) \\ &\leq C_n(\lambda^2 U, V) C_n(\lambda U, \lambda^2 U) C_n(U, \lambda U) \\ &= C_n(\lambda^2 U, V) (C_n(U, \lambda U))^2 \leq \dots \\ &\leq C_n(\lambda^k U, V) (C_n(U, \lambda U))^k; \end{aligned}$$

- (6) A trivial upper bound for the doubling constant: for fixed $\lambda > 1$ and any $r > \lambda$

$$(1.7) \quad C_n(U, rU) \leq (C_n(U, \lambda U))^{\log_\lambda r}.$$

which follows from the multiplicative estimate.

Bellow we will obtain the upper bound for the constant $C_n(U, rU)$, which depends only on n .

We will use the following notation. Let $A + B$ be the Minkowski sum of sets A and B , λA be the product of A and the number λ , and $B_R := \{x \in \mathbb{R}^n : |x| \leq R\}$ be the Euclidean ball.

2. THE UPPER ESTIMATES

In what follows, we set

$$H := \frac{1}{2}U \quad \text{and} \quad K := V + H.$$

Theorem 1. *Let $X \subset \mathbb{R}^n$ be a finite set of points such that*

$$(2.8) \quad K \subseteq H + X.$$

Then

$$C_n(U, V) \leq \frac{|X||U|}{|V|}.$$

From the geometric point of view, condition (2.8) means that the translates $\{H + a : a \in X\}$ of the set H covers the set K .

Example 1 ([3]). If $n = 1$ and $r \in \mathbb{N}$, then

$$C_1(r) := C_1([-1, 1], [-r, r]) \leq 2 + \frac{1}{r}.$$

Indeed, take $H = [-\frac{1}{2}, \frac{1}{2}]$, $X = \{-r, -r+1, \dots, r-1, r\}$, and $K = [-r - \frac{1}{2}, r + \frac{1}{2}] = H + X$.

Let $n \in \mathbb{N}$. There holds ([10, (6)])

$$(2.9) \quad N(K, H) \leq \frac{|K - H|}{|H|} \theta(H).$$

Here $N(K, H)$ denotes the smallest number of translates of H required to cover K and

$$(2.10) \quad \theta(H) = \inf_{X \subset \mathbb{R}^n} \theta(H, X),$$

where $\theta(H, X)$ is the covering density of \mathbb{R}^n by translates of H [9, p.16]. In other words, for a discrete set X such that $\mathbb{R}^n \subseteq H + X$ one has $|X \cap A| |H| / |A| = \theta(H, X)(1 + o(1))$ for a convex body A such that $|A| \rightarrow \infty$.

From (2.9), taking into account that $H = -H$, $K - H = V + 2H = V + U$, and $|U| = 2^n |H|$, we obtain that

$$N(K, H) \leq 2^n \frac{|V + U|}{|U|} \theta(H).$$

Moreover, it is clear that the best possible result in Theorem 1 is when X is such that $|X| = N(K, H)$. Therefore, we have

Corollary 1. *For $n \geq 1$ and any U and V , we have*

$$C_n(U, V) \leq 2^n \frac{|V + U|}{|V|} \theta(H).$$

In particular, for $r \geq 1$

$$(2.11) \quad C_n(U, rU) \leq 2^n (1 + r^{-1})^n \theta(H).$$

Estimate (2.11) substantially improves (1.7). For $n = 1$ and $r \geq 1$, we have that $\theta([- \frac{1}{2}, \frac{1}{2}]) = 1$ and $C_1(r) \leq 2(1 + r^{-1})$, which is similar to the estimate from Example 1.

Note that Rogers [8] proved that

$$(2.12) \quad \theta(H) \leq n \ln n + n \ln \ln n + 5n, \quad n \geq 2.$$

Estimate (2.12) was slightly improved in [4] as follows

$$\theta(H) \leq n \ln n + n \ln \ln n + n + o(n) \quad \text{as } n \rightarrow \infty.$$

Therefore, we obtain

Corollary 2. *We have*

$$C_n(U, V) \leq 2^n (n \ln n + n \ln \ln n + n + o(n)) \frac{|V + U|}{|V|} \quad \text{as } n \rightarrow \infty.$$

In particular, taking $V = rU$, $r \geq 1$, we arrive at the following example.

Example 2. We have

$$(2.13) \quad C_n(U, rU) \leq 2^n (n \ln n + n \ln \ln n + n + o(n)) (1 + r^{-1})^n \quad \text{as } n \rightarrow \infty.$$

Proof of Theorem 1. Consider the function

$$\varphi := \varphi_H = |H|^{-1} \cdot \chi_H * \chi_H,$$

where χ_H is the characteristic function of H and $(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y) dy$ is the convolution of f and g .

Since $\varphi \geq 0$, $\text{supp } \varphi \subset U$, and $\varphi \leq \varphi(0) = 1$, we have for any $f \geq 0$

$$I := \int_{\mathbb{R}^n} f(x)\varphi(x) dx = \int_U f(x)\varphi(x) dx \leq \int_U f(x) dx.$$

Let $X \subset \mathbb{R}^n$ be a finite set and

$$S(x) = \frac{1}{|X|} \sum_{a \in X} \varphi(x - a).$$

Then $S \geq 0$ and $\widehat{S} = \widehat{\varphi}D$, where

$$D(\xi) = \frac{1}{|X|} \sum_{a \in X} e(a\xi)$$

is the Dirichlet kernel with respect to X .

Let us estimate the integral I from below. Using $f(x) = f(-x)$, we get

$$\int_V f(x)S(x) dx \leq \int_{\mathbb{R}^n} f(x)S(x) dx = \int_{\mathbb{R}^n} f(x)S_0(x) dx := I_1,$$

where $S_0(x) = 2^{-1}(S(x) + S(-x))$. Taking into account that

$$\widehat{S_0}(\xi) = \widehat{\varphi}(\xi) \frac{D(\xi) + D(-\xi)}{2} = \widehat{\varphi}(\xi) \frac{1}{|X|} \sum_{a \in X} \cos(2\pi a\xi) \leq \widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n,$$

and using (1.2), we obtain

$$I_1 = \int_{\mathbb{R}^n} \widehat{S_0}(\xi) d\mu(\xi) \leq \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) d\mu(\xi) = \int_{\mathbb{R}^n} f(x)\varphi(x) dx = I,$$

provided that f and φ are even.

Let $K = V + H \subseteq H + X$. This means that for any points $x \in V$ and $y \in H$ there is $a \in X$ such that $x + y \in H + a$. Hence,

$$\sum_{a \in X} \chi_H(x + y - a) \geq 1.$$

Using $H = -H$, we have

$$\varphi(x) = \frac{1}{|H|} \int_H \chi_H(x + y) dy.$$

Therefore, for any $x \in V$

$$\begin{aligned} S(x) &= \frac{1}{|X|} \sum_{a \in X} \frac{1}{|H|} \int_H \chi_H(x - a + y) dy \\ &\geq \frac{1}{|X||H|} \int_H \sum_{a \in X} \chi_H(x - a + y) dy \\ &\geq \frac{1}{|X||H|} \int_H dy = \frac{1}{|X|}. \end{aligned}$$

Thus, combining the estimates above, we arrive at the inequality

$$\frac{1}{|X|} \int_V f(x) dx \leq \int_V f(x)S(x) dx \leq I \leq \int_U f(x) dx,$$

which is the desired result. \square

3. THE LOWER ESTIMATES

Our goal is to improve the trivial lower estimate (1.4). The idea is to consider the functions $\sum_{a,b \in X \cap B_R} \delta(x + a - b)$, where X is a packing of \mathbb{R}^n by H and $R \gg 1$ (see also [2, 3]).

First we consider the one-dimensional result, partially given in Example 1.

Theorem 2 ([3]). *For $r \in \mathbb{N}$, we have*

$$2 - \frac{1}{r} \leq C_1(r) \leq 2 + \frac{1}{r},$$

and $\lim_{r \rightarrow \infty} C_1(r) = 2$.

This is one of the main results of the paper [3]. The upper bound is given in Example 1. The lower bound follows from Theorem 3 below for $U = [-1, 1]$, $V = [-r, r]$, and $\Lambda = \mathbb{Z}$. The fact that $\lim_{r \rightarrow \infty} C_1(r) = 2$ follows from estimates of $C_1(r)$ for integers r and (1.6).

Now we consider the general case $n \geq 1$. Our aim is to improve the trivial lower bound (1.4) respect to n .

Let

$$\delta_L(H) = \sup_{\Lambda \subset \mathbb{R}^n} \delta(H, \Lambda),$$

where $\delta(H, \Lambda)$ is the packing density of \mathbb{R}^n by lattice translates of H [9, Intr.]. In other words, $\Lambda = M\mathbb{Z}^n \subset \mathbb{R}^n$ is a lattice of rank n ($M \in \mathbb{R}^{n \times n}$ is a generator matrix of Λ , $\det M \neq 0$) such that $a - b \notin \text{int}(2H)$ for any $a, b \in \Lambda$, $a \neq b$, and $|\Lambda \cap A| |H| / |A| = \delta(H, \Lambda)(1 + o(1))$ for a convex body A such that $|A| \rightarrow \infty$. Note that in this case $H + \Lambda$ is a lattice packing of H [6, Sect. 30.1]. Recall that $H = \frac{1}{2}U$.

Theorem 3. *Let $H + \Lambda$ be a lattice packing of H . Then*

$$(3.14) \quad C_n(U, V) \geq \frac{|\Lambda \cap \text{int } V| |U|}{|V|}.$$

In particular,

$$(3.15) \quad C_n(U, V) \geq 2^n \delta_L(H)(1 + o(1)) \quad \text{as } |V| \rightarrow \infty.$$

Proof of Theorem 3. Let Λ be an lattice with the packing density $\delta(H, \Lambda)$. Denote $\Lambda_N = \Lambda \cap B_N$ for $N > 0$. Let B_r be the smallest ball that contained V . Assume that $R \geq r$ is sufficiently large number and ε is sufficiently small. Define $\varphi_\varepsilon = \varphi_{B_\varepsilon}$.

We consider the function

$$f(x) = \sum_{a, b \in \Lambda_R} \varphi_\varepsilon(x + a - b).$$

It is easy to see that

$$f(x) = \sum_{c \in \Lambda_{2R}} N_c \varphi_\varepsilon(x + c),$$

where

$$N_c = \sum_{a-b=c} 1 = \sum_{a \in \Lambda_R \cap (\Lambda_R + c)} 1 = |\Lambda_R \cap (\Lambda_R + c)|.$$

Since Λ is a lattice, we have $\Lambda = \Lambda + c$ for any $c \in \Lambda$. Hence, $N_0 = |\Lambda_R|$ and $N_c \geq |\Lambda_{R-r}|$ for $|c| \leq r$, provided $\Lambda_{R-r} \subset \Lambda_R \cap (\Lambda_R + c)$.

On the one hand, since $2H = U$ and $c \notin \text{int } U$ if $c \in \Lambda \setminus \{0\}$, we have

$$\int_{(1-\varepsilon)U} f(x) dx = N_0 = |\Lambda_R|.$$

On the other hand, since $V \subset B_r$, we obtain

$$\int_{(1+\varepsilon)V} f(x) dx \geq \sum_{c \in \Lambda_{2R} \cap V} N_c \geq |\Lambda_{R-r}| |\Lambda \cap V|.$$

Therefore,

$$C_n((1-\varepsilon)U, (1+\varepsilon)V) \geq \frac{(1-\varepsilon)^n}{(1+\varepsilon)^n} \frac{|\Lambda_{R-r}|}{|\Lambda_R|} \frac{|\Lambda \cap V| |U|}{|V|}.$$

Replacing V by $\frac{1-\varepsilon}{1+\varepsilon}V$ and using (1.5) and (1.6) as above, we arrive at

$$C_n(U, V) \geq \frac{|\Lambda_{R-r}|}{|\Lambda_R|} \frac{|\Lambda \cap \frac{1-\varepsilon}{1+\varepsilon}V| |U|}{|V|}.$$

Letting $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$ concludes the proof of (3.14).

Inequality (3.15) follows easily from (3.14) and the definition of $\delta_L(H)$. \square

Example 3. We consider the balls $U = B_1$ and $V = B_r$, $r > 1$. It is known that

$$\delta_L(B_1) \geq c_n 2^{-n},$$

where $c_n \geq 1$ is the Minkowski constant. It was recently proved in [15] that $c_n > 65963n$ for every sufficiently large n and there exist infinitely many dimensions n for which $c_n \geq 0.5n \ln \ln n$.

Corollary 3. *Let $n \in \mathbb{N}$. We have*

$$(3.16) \quad C_n(B_1, B_r) \geq c_n(1 + o(1)) \quad \text{as } r \rightarrow \infty.$$

Comparing (2.13) and (3.16) for fixed n and $r \rightarrow \infty$, one observes the exponential gap between the upper and lower estimates of $C_n(B_1, B_r)$ with respect to n . Let us give examples of U for which the upper and lower estimates of $C_n(U, V)$ coincide.

Example 4. Let H be a convex body and Λ be a lattice. The set $H + \Lambda$ is lattice tiling if it is both a packing and a covering [6, Sect. 32]. In this case H is a *tile* and $\delta_L(H) = \theta_L(H) = 1$, where $\theta_L(H)$ is the lattice covering density, cf. (2.10). To define $\theta_L(H)$, we take the infimum in (2.10) over all lattices $\Lambda \subset \mathbb{R}^n$ of rank n . Note that $\theta(H) \leq \theta_L(H)$.

For example, the Voronoi polytop

$$V(\Lambda) = \{x \in \mathbb{R}^n : |x| \leq |x - a|, \forall a \in \Lambda\}$$

of a lattice Λ is a tile. In particular, $V(\mathbb{Z}^n)$ is the cube $[-\frac{1}{2}, \frac{1}{2}]^n$.

From Corollary 1 and Theorem 3, we have

Theorem 4. *Let $n \in \mathbb{N}$ and U be a tile. We have*

$$C_n(U, V) = 2^n(1 + o(1)) \quad \text{as } |V| \rightarrow \infty.$$

4. FINAL REMARKS

1. The inequality

$$\frac{1}{|V|} \int_V f(x) dx \leq C_n(U, V) \frac{1}{|U|} \int_U f(x) dx$$

holds for any 1-periodic function $f \geq 0$. In this case we assume that $U, V \subseteq \mathbb{T}^n$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Since a positive definite f is such that $f(-x) = \overline{f(x)}$, then $|f|^p \geq 0$ for any $p = 2k, k \in \mathbb{N}$. Hence, we obtain the following L^p -analogue:

$$\frac{1}{|V|} \int_V |f(x)|^p dx \leq C_n(U, V) \frac{1}{|U|} \int_U |f(x)|^p dx.$$

For $U \subset V = \mathbb{T}^n$, this inequality is the well-known Wiener estimate for positive definite periodic functions (see [12, 7, 2]):

$$(4.17) \quad \int_{\mathbb{T}^n} |f(x)|^p dx \leq W_{n,p}(U) \frac{1}{|U|} \int_U |f(x)|^p dx,$$

which is valid only for $p = 2k, k \in \mathbb{N}$. Here, $W_{n,p}(U)$ is a sharp constant in (4.17). It is clear that

$$W_{n,2k}(U) \leq C_n(U, \mathbb{T}^n).$$

It is interesting to compare the known upper bounds of $W_{n,2k}(U)$ and $C_n(U, \mathbb{T}^n)$. In [2] it was shown that

$$W_{n,2k}(rB_1) \leq 2^{(0.401\dots+o(1))n}, \quad r \in (0, 1/2).$$

On the other hand, by Corollary 2, we obtain that

$$C_n(rB_1, \mathbb{T}^n) \leq 2^{n(1+o(1))}(1+2r)^n.$$

The exponential gap in the last two bounds is related to the restriction to the class of functions under consideration.

2. If $f \succeq 0$, then $f^p \succeq 0$ for any $p \in \mathbb{N}$. This gives

$$\frac{1}{|V|} \int_V (f(x))^p dx \leq C_n(U, V) \frac{1}{|U|} \int_U (f(x))^p dx, \quad p \in \mathbb{N}.$$

It would be of interest to investigate this inequality for any positive p ; see in this direction the paper [5].

3. As we showed above, any function $f \succeq 0$ satisfies the doubling property at the origin (1.1). However, taking any nontrivial function $f \succeq 0$ such that $f|_A = 0$, where A is a ball, we can see that the doubling property may fail outside the origin.

REFERENCES

- [1] D. V. Gorbachev, *Certain inequalities for discrete, nonnegative, positive definite functions* (in Russian), Izv. Tul. Gos. Univ. Est. nauki (2015), no. 2, 5–12.
- [2] D. V. Gorbachev, S. Yu. Tikhonov, *Wiener's problem for positive definite functions*, arXiv:1604.01302.
- [3] A. Efimov, M. Gaal, Sz. Gy. Revesz, *On integral estimates of non-negative positive definite functions*, arXiv:1612.00235.
- [4] G. Fejes Tóth, *A note on covering by convex bodies*, Canad. Math. Bull. **52** (2009), no. 3, 361–365.
- [5] C. FitzGerald, R. Horn, *On fractional Hadamard powers of positive definite matrices*, J. Math. Anal. Appl. **61** (1977), no. 3, 633–642.
- [6] P. M. Gruber, *Convex and Discrete Geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 336, Springer-Verlag, Berlin, 2007.
- [7] E. Hlawka, *Anwendung einer Zahlentheoretischen Methode von C. L. Siegel auf Probleme der Analysis*, Comment Math. Helvetici **56** (1981), 66–82.
- [8] C. A. Rogers, *A note on coverings*, Mathematika **4** (1957), 1–6.
- [9] C. A. Rogers, *Packing and Covering*, Cambridge University Press, 1964.
- [10] C. A. Rogers, C. Zong, *Covering convex bodies by translates of convex bodies*, Mathematika **44** (1997), 215–218.
- [11] W. Rudin, *Fourier analysis on groups*, Interscience Publ., New York, 1962.
- [12] H. S. Shapiro, *Majorant problems for Fourier coefficients*, Quart. J. Math. Oxford Ser. (2) **26** (1975), 9–18.
- [13] Yu. N. Shteinikov, *On the set of joint representatives of two congruence Classes*, Proceedings of the Steklov Institute of Mathematics **290** (2015), no. 1, 189–196.
- [14] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, 1993.
- [15] A. Venkatesh, *A note on sphere packings in high dimension*, Int. Math. Res. Notices (2013), no. 7, 1628–1642.

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