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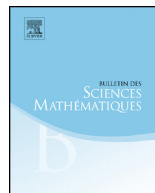


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# Hardy–Littlewood and Pitt’s inequalities for Hausdorff operators <sup>☆</sup>

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## ABSTRACT

In this paper we study transformed trigonometric series with Hausdorff averages of Fourier coefficients. We prove Hardy–Littlewood and Pitt’s inequalities for such series. The corresponding results for the Hausdorff averages of the Fourier transforms are also obtained.

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## 1. Introduction

Let

$$\frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \quad (1.1)$$

be the Fourier series of a function  $f \in L_1(\mathbb{T})$ . The well-known Hardy–Littlewood theorem reads as follows (see [34, Ch. XII, §5]).

**Theorem A.** (i) *Let  $f \in L_p(\mathbb{T})$ ,  $1 < p \leq 2$ . Then we have*

$$J_p \equiv \left( |a_0(f)|^p + \sum_{k=1}^{\infty} (k+1)^{p-2} (|a_k(f)|^p + |b_k(f)|^p) \right)^{\frac{1}{p}} \leq C(p) \|f\|_p, \quad (1.2)$$

where  $C(p) > 0$  depends only on  $p$ .

(ii) *Let  $p \geq 2$  and the sequences  $\{a_k\}$  and  $\{b_k\}$  be such that  $J_p < \infty$ . Then series (1.1) is the Fourier series of a function  $f \in L_p(\mathbb{T})$ . Moreover,*

$$\|f\|_p \leq C(p) J_p, \quad (1.3)$$

where  $C(p) > 0$  depends only on  $p$ .

It is also known that inequality (1.2) cannot be extended for  $p > 2$ ; say, for example,  $f(x) = \sum_{n=1}^{\infty} n^{-\beta} \cos 2^n x$ ,  $\beta > 1/2$ . However, the following analogue of (1.2) is valid for the Cesàro means ([21]).

**Theorem B.** *Let  $f \in L_p(\mathbb{T})$ ,  $1 < p < \infty$ . Let also*

$$\bar{a}_k(f) = \frac{1}{k+1} \sum_{l=0}^k a_l(f), \quad \bar{b}_k(f) = \frac{1}{k+1} \sum_{l=1}^k b_l(f), \quad k = 1, 2, \dots$$

Then

$$\left( |a_0(f)|^p + \sum_{k=1}^{\infty} (k+1)^{p-2} (|\bar{a}_k(f)|^p + |\bar{b}_k(f)|^p) \right)^{\frac{1}{p}} \leq C(p) \|f\|_p, \quad (1.4)$$

where  $C(p) > 0$  depends only on  $p$ .

Note that one actually has  $C(p) \rightarrow \infty$  as  $p \rightarrow \infty$ .

On the other hand, the Hardy–Littlewood inequality is closely related to Hardy’s theorem for the series of the type

$$\frac{\bar{a}_0(f)}{2} + \sum_{k=1}^{\infty} (\bar{a}_k(f) \cos kx + \bar{b}_k(f) \sin kx). \quad (1.5)$$

In particular, Hardy [15] proved that if series (1.1) is the Fourier series of a function  $f \in L_p$ ,  $1 < p < \infty$ , then series (1.5) is the Fourier series of a function  $F \in L_p$ . Goldberg [14] extended this result for the Hausdorff averages, see, e.g., [12, 18].

**Theorem C.** *Let  $1 < p < \infty$  and series (1.1) be the Fourier series of a function  $f \in L_p(\mathbb{T})$ . Let  $\psi$  be a function of bounded variation supported on  $[0, 1]$ . Then the series*

$$\frac{\bar{a}_0^\psi}{2} + \sum_{k=1}^{\infty} (\bar{a}_k^\psi(f) \cos kx + \bar{b}_k^\psi(f) \sin kx), \quad (1.6)$$

where

$$\bar{a}_k^\psi(f) = \frac{1}{k+1} \sum_{l=0}^k a_l(f) \psi\left(\frac{l}{k+1}\right)$$

and

$$\bar{b}_k^\psi(f) = \frac{1}{k+1} \sum_{l=1}^k b_l(f) \psi\left(\frac{l}{k+1}\right)$$

for  $k = 1, 2, \dots$ , is the Fourier series of a function  $F \in L_p$ .

Our main goal in this paper is to obtain the Hardy–Littlewood and Pitt type inequalities for the Hausdorff averages  $\bar{a}_k^\psi(f)$  and  $\bar{b}_k^\psi(f)$ . In Section 2, we obtain an inequality similar to (1.4) for  $\bar{a}_k^\psi(f)$  and  $\bar{b}_k^\psi(f)$  but with the constant independent of  $p$  unlike in (1.4). In particular, this result generalizes Theorems B and C.

In Section 3, we study Pitt’s inequality. Recall that the classical form of the Pitt inequality reads as follows ([24, 27]): for a  $2\pi$ -periodic function  $f$  with the Fourier series (1.1) we have

$$\left( \sum_{k=0}^{\infty} (k+1)^{-\lambda q} \left( |a_k(f)| + |b_k(f)| \right)^q \right)^{\frac{1}{q}} \leq C(p, q, \lambda) \left( \int_{\mathbb{T}} |f(x)|^p |x|^{p\alpha} dx \right)^{\frac{1}{p}}, \quad p \leq q, \quad (1.7)$$

whenever  $\max\{0, 1 - (1/q + 1/p)\} \leq \alpha < 1/p'$  and  $\lambda = 1/q + 1/p - 1 + \alpha$ . Note that the condition  $\alpha < 1/p'$  and convergence of the integral on the right-hand side of (1.7)

guarantee that  $f$  is integrable. Pitt's inequality, in particular, implies the Hausdorff–Young inequality (for  $q = p' \geq 2$  and  $\alpha = 0$ ) and Hardy–Littlewood's theorem (for  $1 < p = q \leq 2$  and  $\alpha = 0$ ).

Theorem 3.1 below extends Pitt's inequality for the Fourier averages  $\bar{a}_k^\psi(f)$  and  $\bar{b}_k^\psi(f)$  with a less restrictive condition on  $\alpha$ :

$$\left( \sum_{k=0}^{\infty} (k+1)^{-\lambda q} \left( |\bar{a}_k^\psi(f)| + |\bar{b}_k^\psi(f)| \right)^q \right)^{\frac{1}{q}} \leq C(p, q, \lambda, \psi) \left( \int_{\mathbb{T}} |f(x)|^p |x|^{p\alpha} dx \right)^{\frac{1}{p}},$$

$$p \leq q$$

whenever  $0 \leq \alpha < 1/p'$  and  $\lambda = 1/q + 1/p - 1 + \alpha$ . It is worth mentioning the works [6,10] which provide Pitt's inequality (1.7) with wider ranges of  $\alpha$  than in the classical case, but subject to additional conditions on the behavior of the Fourier coefficients.

In Section 4, we obtain similar results for the Fourier transforms.

## 2. Hardy–Littlewood inequalities for Fourier averages with absolute constants

### 2.1. The case $p \geq 2$

First, we give a simple proof of the next result, which extends Theorems B and C.

**Theorem 2.1.** *Let  $f \in L_p(\mathbb{T})$ ,  $2 \leq p < \infty$ . Let  $\psi$  be a continuous function on  $(0, 1)$ .*

(i) *Assume that  $\psi$  satisfies the condition*

$$\int_0^\infty \left| \int_0^1 \psi(t) \cos tx dt \right| dx < \infty, \quad (2.1)$$

*then*

$$\left( \sum_{k=0}^{\infty} (k+1)^{p-2} |\bar{a}_k^\psi(f)|^p \right)^{\frac{1}{p}} \leq C \|f\|_p,$$

*where  $C > 0$  depends only on  $\psi$ .*

(ii) *Assume that  $\psi$  satisfies the condition*

$$\int_0^\infty \left| \int_0^1 \psi(t) \sin tx dt \right| dx < \infty, \quad (2.2)$$

then

$$\left( \sum_{k=0}^{\infty} (k+1)^{p-2} |\bar{b}_k^\psi(f)|^p \right)^{\frac{1}{p}} \leq C \|f\|_p, \quad (2.3)$$

where  $C > 0$  depends only on  $\psi$ .

**Remark 2.1.** (i) Taking into account inequality (1.3), Theorem 2.1 generalizes Theorem C as follows: there exists a function  $F \in L_p$  with the Fourier series (1.6) and moreover

$$\|F\|_p \leq C(p, \psi) \|f\|_p.$$

(ii) Note that using M. Riesz' theorem on conjugate function, one can obtain inequality (2.3) under condition (2.1) in place of (2.2) and the similar result for part (i). However, in this case, a crucial difference is that the constant  $C$  depends also on  $p$ . See [23].

We will need the following result by E. Belinsky (see [2] and [30, 4.1.1., p. 106]).

**Lemma 2.1.** *Let a continuous function  $\varphi$  with compact support be such that*

$$\widehat{\varphi}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} \varphi(t) e^{-itx} dt \in L_1(\mathbb{R}).$$

Then

$$\sup_{\varepsilon > 0} \int_{\mathbb{T}} \left| \sum_{k \in \mathbb{Z}} \varphi(\varepsilon k) e^{ikx} \right| dx = (2\pi)^{1/2} \|\widehat{\varphi}\|_{L_1(\mathbb{R})}.$$

Moreover, if a function  $\varphi$  is of bounded variation, then ([30, 4.1.3., p. 109])

$$\int_{\mathbb{T}} \left| \sum_{k \in \mathbb{Z}} \varphi(\varepsilon k) e^{ikx} \right| dx = (2\pi)^{1/2} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} |\widehat{\varphi}| dx + cV_{\mathbb{R}}(\varphi),$$

where  $c \in (0, 4\pi)$ .

**Proof.** (i) For any integrable function  $f$ , consider the operator

$$A: f \rightarrow A(f) = (\bar{a}_0^\psi(f), 2\bar{a}_1^\psi(f), \dots, (k+1)\bar{a}_k^\psi(f), \dots)$$

and define the measure  $\nu(\{k\}) = \frac{1}{(k+1)^2}$  on  $k \in \{0\} \cup \mathbb{N}$ .

If  $f \in L_2(\mathbb{T})$ , then

$$\begin{aligned} \|A(f)\|_{l_2(\nu)}^2 &= \sum_{k=0}^{\infty} \left( (k+1) \bar{a}_k^\psi(f) \right)^2 \frac{1}{(k+1)^2} \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} \left( \sum_{l=0}^k a_l(f) \psi\left(\frac{l}{k+1}\right) \right)^2 \\ &\leq C \sum_{k=0}^{\infty} \left( \frac{1}{k+1} \sum_{l=0}^k |a_l(f)| \right)^2, \end{aligned}$$

where  $\max |\psi| = C$ . Using the Hardy inequality for averages, we have

$$\|A(f)\|_2^2 \leq 4C \sum_{l=0}^{\infty} |a_l(f)|^2 \leq 4C \|f\|_2^2.$$

If  $f \in L_\infty(\mathbb{T})$ , then

$$\begin{aligned} \|A(f)\|_\infty &= \sup_{k \geq 0} |(k+1) \bar{a}_k^\psi(f)| = \sup_{k \geq 0} \left| \sum_{l=0}^k a_l(f) \psi\left(\frac{l}{k+1}\right) \right| \\ &= \sup_{k \geq 0} \frac{1}{\pi} \left| \int_{\mathbb{T}} f(x) \left( \sum_{l=0}^k \psi\left(\frac{l}{k+1}\right) \cos lx \right) dx \right| \\ &\leq \|f\|_\infty \sup_{\varepsilon > 0} \frac{1}{\pi} \int_{\mathbb{T}} \left| \sum_{l=0}^k \psi(\varepsilon l) \cos lx \right| dx. \end{aligned} \tag{2.4}$$

Using Lemma 2.1 and condition (2.1), we obtain

$$\|A(f)\|_\infty \leq C \|f\|_\infty,$$

where  $C$  depends only on  $\psi$ .

Finally, the Riesz interpolation theorem (see [34, Ch. XII, (1.11)]) implies, for any  $p \in [2, \infty)$ ,

$$C \|f\|_p^p \geq \|A(f)\|_p^p = \sum_{k=0}^{\infty} (k+1)^{p-2} |\bar{a}_k^\psi(f)|^p,$$

where  $C$  depends on  $\psi$  but not on  $p$ .

The proof of part (ii) follows the same lines as above, taking into account condition (2.2).  $\square$

**Remark 2.2.** Various averages of the Fourier series and transforms were considered earlier in [9], [21], [22]. The case  $\psi(t) = (1 - t)\chi_{(0,1)}$  was also discussed in [8]. It would be interesting to consider similar inequalities for different complete orthogonal systems, Walsh series, sinc functions (see [31,32]), etc.

## 2.2. Around conditions (2.1) and (2.2)

First we remark that the expression  $\int_{\mathbb{T}} \left| \sum_{l=0}^k \psi(\varepsilon l) \cos lx \right| dx$  in (2.4) is the Lebesgue constant of the approximation method generated by the function  $\psi$ . Estimates of the Lebesgue constant is a well-developed topic, see, e.g., [17,28]. One of the simplest conditions for which the Lebesgue constants are bounded is written as follows:

$$\sum_{l=0}^k (l+1) |\Delta^2 \lambda_l^k| < \infty, \quad \lambda_l^k := \psi\left(\frac{l-1}{k+1}\right) - 2\psi\left(\frac{l}{k+1}\right) + \psi\left(\frac{l+1}{k+1}\right),$$

that is,  $\{\lambda_l^k\}$  is quasi-convex for any  $k$ .

More general conditions can be given using the various Hardy spaces, see [17]. In particular, if a function  $\psi$  of bounded variation satisfies

$$\int_0^{x/2} \frac{\psi'(x-t) - \psi'(x+t)}{t} dt \in L_1(0, \infty),$$

then condition (2.1) is valid. If, additionally,

$$\int_0^1 \psi(t) \frac{dt}{t} < \infty,$$

condition (2.2) is also valid. More function spaces which are subspaces of  $\{\psi \in BV : \widehat{\psi} \in L_1\}$  were recently considered in [17]. This topic is closely related to a study of the Wiener algebra  $W_0$ , see [20]. Below we list several examples.

1.  $\psi(x) = 1 - x$ ,  $x \in [0, 1]$ . It is clear that  $\|\widehat{\psi}_c\|_{L^1(\mathbb{R}_+)} < \infty$ , and therefore the part (i) of Theorem 2.1 is valid. On the other hand,  $\widehat{\psi}_s \notin L^1(\mathbb{R}_+)$ .
2.  $\psi(x) = (1 - x^2)^\alpha$ ,  $x \in [0, 1]$ ,  $\alpha > 0$ . Then  $\widehat{\psi}_c(t) = C(\alpha) \frac{J_{\alpha+1/2}(t)}{t^{1/2}}$  and the known asymptotic relations for the Bessel function  $J_{\alpha+1/2}(t)$  give that  $\|\widehat{\psi}_c\|_{L^1(\mathbb{R}_+)} < \infty$ .
3. More generally, if  $P$  is a nontrivial and nonnegative algebraic polynomial on  $[0, 1]$  and  $P(1) = 0$ , then the function  $\psi(x) = P^\alpha(|x|)\chi_{(0,1)}(x)$  with  $0 < \alpha$  satisfies  $\|\widehat{\psi}_c\|_{L^1(\mathbb{R}_+)} < \infty$  ([20]).



4.  $\psi(x) = (x - x^2)^\alpha$ ,  $x \in [0, 1]$ ,  $\alpha > 0$ . Then

$$\widehat{\psi}_s(t) = C(\alpha) \frac{J_{\alpha+1/2}(t/2)}{t^{\alpha+1/2}} \sin \frac{t}{2}$$

and hence  $\|\widehat{\psi}_s\|_{L^1(\mathbb{R}_+)} < \infty$ . By Example 3,  $\|\widehat{\psi}_c\|_{L^1(\mathbb{R}_+)} < \infty$  as well.

5.  $\psi(x) = \sin \sqrt{1 - x^2}$ ,  $x \in [0, 1]$ . Then  $\widehat{\psi}_c(t) = \frac{\pi}{2} \frac{J_1(\sqrt{1+t^2})}{(1+t^2)^{1/2}}$  and  $\|\widehat{\psi}_c\|_{L^1(\mathbb{R}_+)} < \infty$ .
6.  $\psi(x) = (1 - x^2)P_{2n}^{(\alpha, \alpha)}(x)\chi_{(0,1)}(x)$ , where  $P_{2n}^{(\alpha, \alpha)}(x)$  is the Jacobi polynomial, is such that  $\|\widehat{\psi}_c\|_{L^1(\mathbb{R}_+)} < \infty$  ([11]).
7. In general, for any even function  $\psi \in C^\infty(\mathbb{R})$ ,  $\psi(0) = 1$  we have  $\|(\psi(x)|x|^\gamma)^\wedge\|_{L^1(\mathbb{R}_+)} < \infty$ ,  $\gamma > 0$ . This follows for example from Theorem 4.1 in [25]. Many different examples of such  $\psi$  can be found in [20].

### 2.3. A remark regarding the case $p \leq 2$

Taking  $\psi(t) = (1 - t)\chi_{(0,1)}(t)$  in Theorem 2.1, we obtain that

$$\left( \sum_{k=0}^{\infty} (k+1)^{p-2} |\bar{a}_k(f)|^p \right)^{\frac{1}{p}} \leq C \|f\|_p, \quad \bar{a}_n = \frac{1}{n+1} \sum_{k=1}^n a_k(f) \left( 1 - \frac{k}{n+1} \right),$$

where  $C > 0$  is an absolute constant and  $2 \leq p < \infty$ . The next result is the counterpart for the sine coefficients in the case  $1 < p \leq 2$ .

**Theorem 2.2.** *Let  $1 < p \leq 2$ . We have*

$$\left( \sum_{k=0}^{\infty} (k+1)^{p-2} |\bar{b}_k|^p \right)^{\frac{1}{p}} \leq C \|f\|_p, \quad \bar{b}_n = \frac{1}{n+1} \sum_{k=1}^n b_k(f) \left( 1 - \frac{k}{n+1} \right),$$

where  $C > 0$  is an absolute constant.

The proof follows the same lines as that of Theorem 2.1 with the aid of the following Hardy-type theorem of its own interest. Recall that the classical Hardy inequality states that

$$\sum_{n=1}^{\infty} \frac{|c_n|}{n} \leq C \left\| \sum_{n=1}^{\infty} c_n e^{inx} \right\|_1.$$

**Theorem 2.3.** *There exists a positive constant  $C$  such that for any odd function  $f \in L_1(\mathbb{T})$  we have*

$$\sum_{n=1}^{\infty} \frac{|\bar{b}_n|}{n} \leq C \|f\|_1,$$

where  $\{b_n(f)\}_{n=1}^{\infty}$  is the Fourier sine coefficients of  $f$ .

**Remark 2.3.** As the example

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{\ln(n+1)} \in L_1$$

shows, the previous theorem does not hold for the cosine coefficients.

**Proof.** For any integer  $n$ , we denote by  $\tilde{K}_n$  the conjugate Fejér kernel, that is,

$$\begin{aligned} \tilde{K}_n(t) &:= \frac{1}{n+1} \sum_{k=1}^n \tilde{D}_k(t) = \frac{1}{n+1} \sum_{k=1}^n \frac{\cos \frac{t}{2} - \cos(k + \frac{1}{2})}{2 \sin \frac{t}{2}} \\ &= \frac{1}{n+1} \frac{(n+1) \sin t - \sin(n+1)t}{4 \sin^2 \frac{t}{2}}. \end{aligned}$$

Note that  $\tilde{K}_n(t) \geq 0$  if  $t \in [0, \pi]$ . We have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|\bar{b}_n|}{n} &\leq \sum_{n=1}^{\infty} \frac{1}{n^2} \left| \frac{2}{\pi} \int_0^{\pi} f(t) \tilde{K}_n(t) dt \right| \\ &\leq \frac{2}{\pi} \int_0^{\pi} |f(t)| \sum_{n=1}^{\infty} \frac{1}{n^2} \tilde{K}_n(t) dt \\ &\leq \frac{1}{\pi} \|f\|_1 \max_t \sum_{n=1}^{\infty} \frac{1}{n^2} \tilde{K}_n(t). \end{aligned}$$

It is enough to show that there exists a positive  $C$  such that for any  $t \in [0, \pi]$  we have

$$S \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} \tilde{K}_n(t) \leq C.$$

Uniform boundedness of this series on  $[\frac{1}{2}, \pi]$  is clear. Let  $t \in (0, \frac{1}{2})$ . Then

$$S = \sum_{n=1}^{\lfloor \frac{1}{2t} \rfloor} \frac{1}{n^2} \tilde{K}_n(t) + \sum_{n=\lfloor \frac{1}{2t} \rfloor + 1}^{\infty} \frac{1}{n^2} \tilde{K}_n(t) \equiv S_1 + S_2.$$

We have

$$\begin{aligned} S_2 &\leq \sum_{n=\lfloor \frac{1}{2t} \rfloor + 1}^{\infty} \frac{1}{n^2(n+1)} \frac{\pi^2(n+1)t}{4t^2} + \sum_{n=\lfloor \frac{1}{2t} \rfloor + 1}^{\infty} \frac{1}{n^2(n+1)} \frac{\pi^2}{4t^2} \\ &\leq 3 \left( \frac{1}{t} \sum_{n=\lfloor \frac{1}{2t} \rfloor + 1}^{\infty} \frac{1}{n^2} + \frac{1}{t^2} \sum_{n=\lfloor \frac{1}{2t} \rfloor + 1}^{\infty} \frac{1}{n^3} \right) \leq C. \end{aligned}$$

To estimate  $S_1$ , we note that  $|\sin u - u| \leq C_1 u^3$  for  $|u| \leq \frac{1}{2}$ , with  $C_1 > 0$ . Therefore we obtain for  $1 \leq n \leq [\frac{1}{2t}]$

$$|(n+1)\sin t - \sin(n+1)t| \leq C_1 \left( (n+1)t^3 + ((n+1)t)^3 \right) \leq 2C_1 ((n+1)t)^3,$$

which implies

$$S_1 \leq \sum_{n=1}^{[\frac{1}{2t}]} \frac{1}{n^2} \cdot \frac{3}{(n+1)t^2} \cdot 2C_1 ((n+1)t)^3 \leq C_2 t \sum_{n=1}^{[\frac{1}{2t}]} 1 \leq C_3,$$

where  $C_2$  and  $C_3$  are positive constants.  $\square$

### 3. Pitt's inequality for functions with transformed Fourier series

Let  $\mathbb{X} = \mathbb{T}$  or  $\mathbb{X} = \mathbb{R}$ . Let us first rewrite the Pitt's inequality (1.7) using the weighted Lebesgue spaces  $L_{\omega(p,q)}(\mathbb{X})$  defined by

$$\|f\|_{L_{\omega(p,q)}(\mathbb{X})} = \left( \int_{\mathbb{X}} \left( \omega(p,q,x) |f(x)| \right)^q dx \right)^{1/q}, \quad \omega(p,q,x) = x^{1/p-1/q},$$

which is similar to the norm of the Lorentz space  $L_{p,q}(\mathbb{X})$

$$\|f\|_{L_{\omega(p,q)}(\mathbb{X})} = \left( \int_0^{|\mathbb{X}|} \left( x^{1/p-1/q} f^*(x) \right)^q dx \right)^{1/q},$$

where  $f^*$  is the decreasing rearrangement of  $f$ .

**Pitt's Theorem.** For  $f(x) \sim \sum_{k \in \mathbb{Z}} c_k(f) e^{ikx}$ , one has

$$\left( \sum_{k=-\infty}^{\infty} \left( (|k|+1)^{1/p'} |c_k(f)| \right)^q \frac{1}{k} \right)^{\frac{1}{q}} \leq C(p,q,r) \|f\|_{L_{\omega(p,r)}(\mathbb{T})}$$

whenever

$$1 < p \leq r \leq q \leq p' < \infty.$$

The next result provides Pitt's inequality for the Hausdorff averages, which is more general with respect to the parameters  $p, r, q$ .

**Theorem 3.1.** Let  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  and  $\sum_{k \in \mathbb{Z}} c_k(f) e^{ikx}$  be the Fourier series of a function  $f \in L_{p,q}(\mathbb{T})$ . Let  $\psi$  be a function of bounded variation on  $[0, 1]$ . Then

$$\left( \sum_{k=-\infty}^{\infty} \left( (|k| + 1)^{1/p'} |\bar{c}_k^\psi(f)| \right)^q \frac{1}{k} \right)^{\frac{1}{q}} \leq C(p, q, \psi) \|f\|_{L_{p,q}(\mathbb{T})}, \quad (3.1)$$

where

$$\bar{c}_k^\psi(f) = \frac{1}{2k+1} \sum_{l=-k}^k c_l(f) \psi\left(\frac{l}{k+1}\right).$$

If, additionally,  $1 < p \leq r \leq q \leq \infty$ , we have

$$\left( \sum_{k=1}^{\infty} \left( k^{1/p'} |\bar{c}_k^\psi(f)| \right)^q \frac{1}{k} \right)^{\frac{1}{q}} \leq C(p, q, r, \psi) \|f\|_{L_{\omega(p,r)}(\mathbb{T})}. \quad (3.2)$$

**Remark 3.1.** Note that (3.2) implies the Hardy–Littlewood inequality

$$\left( \sum_{k=0}^{\infty} (k+1)^{p-2} |\bar{c}_k^\psi(f)|^p \right)^{\frac{1}{p}} \leq C(p, \psi) \|f\|_p, \quad 1 < p < \infty.$$

Thus, Hardy–Littlewood type inequality holds for any functions with the transformed Fourier series and for any  $1 < p < \infty$  but with the constant depending on  $p$  unlike Theorem 2.1.

**Proof.** Let  $0 < \tau < \pi$ . For a function  $f \in L_{p,q}(\mathbb{T})$  we define

$$f_1(x) := \begin{cases} f(x), & x \in \{y : \frac{\tau}{2} < |y| \leq \pi, |f(y)| < f^*(\tau)\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_0(x) := f(x) - f_1(x).$$

We will use the following estimate of the Fourier coefficients given by (3.1):

$$\begin{aligned} |\bar{c}_k^\psi(f)| &= \left| \frac{1}{2k+1} \int_{\mathbb{T}} (f_0(x) + f_1(x)) \sum_{l=-k}^k \psi\left(\frac{l}{2k+1}\right) e^{ilx} dx \right| \\ &\leq \frac{1}{2k+1} \|f_0\|_{L_1} \sum_{l=-k}^k \left| \psi\left(\frac{l}{2k+1}\right) \right| \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2k+1} \int_{\mathbb{T}} |f_1(x)| \sum_{l=-k}^{k-1} \left| \psi\left(\frac{l}{2k+1}\right) - \psi\left(\frac{l+1}{2k+1}\right) \right| |D_{[-k,l]}(x)| dx \\
& + \frac{1}{2k+1} \int_{\mathbb{T}} |f_1(x)| \left| \psi\left(\frac{k}{2k+1}\right) \right| |D_{[-k,k]}(x)| dx,
\end{aligned}$$

where  $D_{[n,m]}(x)$  is the partial Dirichlet kernel, that is,  $D_{[n,m]}(x) = \sum_{j=n}^m e^{ijx}$ .

Noting that  $f_1(x) = 0$  for  $|x| < \tau/2$  and

$$\sup_{-k \leq l \leq k-1} |D_{[-k,l]}(x)| \leq 1 + \frac{\pi}{2|x|}, \quad 0 < |x| < \pi,$$

we get

$$\sup_{-k \leq l \leq k-1} \int_{\mathbb{T}} |f_1(x)| |D_{[-k,l]}(x)| dx \leq \frac{3\pi}{2} \int_{\tau/2 < |x| < \pi} \frac{|f_1(x)|}{|x|} dx.$$

By rearrangement inequality the latter is bounded by

$$\frac{3\pi}{2} \int_0^{2\pi} \frac{f_1^*(t)}{t+\tau} dt \leq \frac{3\pi}{2} \int_0^{2\pi-\tau} \frac{f^*(t+\tau)}{t+\tau} dt = \frac{3\pi}{2} \int_{\tau}^{2\pi} \frac{f^*(t)}{t} dt.$$

Taking into account that

$$\|f_0\|_{L_1} \leq \int_0^{2\tau} f^*(t) dt,$$

we have

$$|\bar{c}_k^\psi(f)| \leq M \int_0^{2\tau} f^*(t) dt + (V + M) \frac{1}{2k+1} \sup_{l \in [-k,k]} \int_{\mathbb{T}} |f_1(x)| |D_{[-k,l]}(x)| dx,$$

where  $M = \sup_{x \in \mathbb{T}} |\psi(x)|$  and  $V = V_0^{2\pi}(\psi)$  is the variation of the function  $\psi$  on  $\mathbb{T}$ .

Finally,

$$|\bar{c}_k^\psi(f)| \leq \frac{3\pi}{2} (V + M) \left( \int_0^{2\tau} f^*(t) dt + \frac{1}{k} \int_{\tau}^{2\pi} \frac{f^*(t)}{t} dt \right).$$

Letting  $\tau = 1/k$ , we get for  $1 \leq q < \infty$

$$\begin{aligned} & \left( \sum_{k=1}^{\infty} \left( k^{1/p'} |\bar{c}_k^\psi(f)| \right)^q \frac{1}{k} \right)^{\frac{1}{q}} \\ & \leq \frac{3\pi}{2} (V + M) \left( \sum_{k=1}^{\infty} k^{q/p'-1} \left( \int_0^{2/k} f^*(t) dt + \frac{1}{k} \int_{1/k}^{2\pi} f^*(t) \frac{dt}{t} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Taking into account known Hardy's inequalities with weights for the operator  $\int_0^t g(x) dx$  (see, e.g. [5, p. 124]), this implies inequality (3.1).

If  $q = \infty$ , then

$$\begin{aligned} \sup_{k \in \mathbb{N}} k^{1/p'} |\bar{c}_k^\psi(f)| & \leq C(\psi) \sup_{k \in \mathbb{N}} k^{1/p'} \left( \int_0^{2/k} f^*(t) dt + \frac{1}{k} \int_{1/k}^{2\pi} f^*(t) \frac{dt}{t} \right) \\ & \leq C(p) \|f\|_{L_{p,\infty}} \sup_{k \in \mathbb{N}} k^{1/p'} \left( \int_0^{2/k} t^{-1/p} dt + \frac{1}{k} \int_{1/k}^{2\pi} t^{-1/p} \frac{dt}{t} \right) \\ & = C_1(p, \psi) \|f\|_{L_{p,\infty}}. \end{aligned}$$

Thus, the proof of (3.1) is complete.

To obtain (3.2), we use the known embedding properties of the Lorentz spaces  $\|f\|_{L_{p,q}} \leq C \|f\|_{L_{p,r}}$  for  $r \leq q$ . Assuming also that  $p \leq r$ , we obtain  $\|f\|_{L_{p,r}} \leq \|f\|_{L_{\omega(p,r)}}$ , concluding the proof.  $\square$

#### 4. Fourier transforms inequalities for Hausdorff operators

In this section we give counterparts of the main results in Sections 2 and 3 for the functions on  $\mathbb{R}$ . As an average of the Fourier transform of  $f$

$$\widehat{f}(s) = \int_{\mathbb{R}} f(x) e^{-isx} dx$$

with respect to the function  $\psi$ , we consider the Hausdorff type operators:

$$\mathcal{F}_\psi f(t) = \int_{\mathbb{R}} \widehat{f}(ts) \psi(s) ds,$$

see, e.g., [16]. Here the classical example  $\psi(s) = \chi_{(0,1)}$  corresponds to the Cesàro operator  $\mathcal{F}_\psi f(t) = \frac{1}{t} \int_0^t \widehat{f}(s) ds$ .

Regarding the weighted Fourier inequalities, we start with the well-known analogue of Hardy–Littlewood type inequality for the Fourier transforms (see [29, Th. 80])

$$\left( \int_0^\infty t^{p-2} |\widehat{f}(t)|^p dt \right)^{\frac{1}{p}} \leq C(p) \|f\|_{L_p},$$

where  $1 < p \leq 2$ . We extend this result to  $p > 2$  with an absolute constant in the corresponding estimate.

**Theorem 4.1.** *Let  $2 \leq p < \infty$  and  $f(x) \in L_{p,\text{loc}}(\mathbb{R})$ , where  $L_{p,\text{loc}}(\mathbb{R})$  is the set of all functions from  $L_p(\mathbb{R})$  with compact support. Let  $\psi$  be such that  $\widehat{\psi} \in L_1(\mathbb{R})$  and*

$$\int_{\mathbb{R}} t^{-1/2} |\psi(t)| dt = D < \infty. \quad (4.1)$$

We have

$$\left( \int_0^\infty t^{p-2} |\mathcal{F}_\psi f(t)|^p dt \right)^{\frac{1}{p}} \leq C \|f\|_{L_p(\mathbb{R})}, \quad (4.2)$$

where  $C$  depends only on  $\psi$ .

**Proof.** Consider

$$A(f, t) = \int_{\mathbb{R}} \widehat{f}(s) \psi(s/t) ds.$$

If  $f \in L_2(\mathbb{R})$ , then

$$\begin{aligned} \|A(f, t)\|_{L_2(\mathbb{R}, \frac{1}{|t|})} &= \|\mathcal{F}_\psi f(\cdot)\|_{L_2(\mathbb{R})} \\ &= \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \widehat{f}(st) \psi(s) ds \right|^2 dt \right)^{1/2} \\ &\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\widehat{f}(st)|^2 dt \right)^{1/2} |\psi(s)| ds \\ &\leq D \|\widehat{f}\|_{L_2} = D \|f\|_{L_2(\mathbb{R})}. \end{aligned}$$

For  $f \in L_\infty(\mathbb{R})$  we have

$$\begin{aligned} \|A(f, t)\|_{L_\infty(\mathbb{R})} &= \sup_{t \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \psi(s/t) e^{-ixs} ds dx \right| \\ &= \sup_{t \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x/t) \int_{\mathbb{R}} \psi(s) e^{-ixs} ds dx \right| \\ &\leq \|f\|_{L_\infty(\mathbb{R})} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \psi(s) e^{-ixs} ds \right| dx. \end{aligned}$$

Hence, we get

$$\|A(f)\|_\infty \leq \|\widehat{\psi}\|_{L_1} \|f\|_\infty.$$

Fix any positive  $N$  and consider  $f \in L_p(\mathbb{R})$  with the support on  $[-N, N]$ . Then the interpolation theorem implies for any  $p \in [2, \infty)$

$$C \|f\|_p^p \geq \|A(f)\|_p^p = \int_{\mathbb{R}} |t|^{p-2} |\mathcal{F}_\psi f(t)|^p dt,$$

where  $C$  is a constant depending on  $\psi$  but not on  $p$ .  $\square$

Now we can define the function  $\mathcal{F}_\psi f(t)$  for all  $f \in L_p(\mathbb{R})$ . If  $f_n(x) = f(x)\chi_{(-n, n)}(x)$ , then all  $f_n \in L_{p, \text{loc}}(\mathbb{R})$  and by the previous theorem the corresponding functions  $\mathcal{F}_\psi f_n$  are fundamental in weighted space  $L_{p, |t|^{p-2}}(\mathbb{R})$ . We define their limit in this space as  $\mathcal{F}_\psi f$ . Note that for this function the inequality of Theorem 4.1 is also valid.

Our next result is a Pitt-type inequality.

**Theorem 4.2.** *Let  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ , and  $f(x) \in L_{p,q}(\mathbb{R})$ . Let  $\psi \in L_1(\mathbb{R})$  be of bounded variation on  $\mathbb{R}$ . Then*

$$\|\mathcal{F}_\psi f(t)\|_{L_{\omega(p', q)}(\mathbb{R})} \leq C(\psi, p, q) \|f\|_{L_{p,q}(\mathbb{R})}.$$

If, additionally,  $1 < p \leq r \leq q < \infty$ , we have

$$\|\mathcal{F}_\psi f(t)\|_{L_{\omega(p', q)}(\mathbb{R})} \leq C(\psi, p, q) \|f\|_{L_{\omega(p, r)}(\mathbb{R})}. \quad (4.3)$$

**Proof.** Defining again for  $0 < \tau < \infty$

$$f_1(x) = \begin{cases} f(x), & x \in \{\tau < |x| < \infty : |f(x)| < f^*(\tau)\}, \\ 0, & \text{otherwise} \end{cases}$$



and

$$f_0(x) = f(x) - f_1(x),$$

we have for  $t \in \mathbb{R}$

$$\begin{aligned} |\mathcal{F}_\psi f(t)| &= \left| \frac{1}{t} \int_{\mathbb{R}} (f_0(x) + f_1(x)) \int_{\mathbb{R}} \psi(s/t) e^{-ixs} ds dx \right| \\ &= \left| \int_{\mathbb{R}} (f_0(x) + f_1(x)) \int_{\mathbb{R}} \psi(s) e^{-ixst} ds dx \right| \leq \int_{\mathbb{R}} |\psi(s)| ds \|f_0\|_{L_1(\mathbb{R})} \\ &\quad + \int_{\mathbb{R}} |f_1(x)| \left| \frac{\psi(s) e^{-ixs}}{tx} \right|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{e^{-istx}}{tx} d\psi(s) dx \\ &\leq M \|f_0\|_{L_1(\mathbb{R}_+)} + (M + V) \int_{\tau < |x| < \infty} |f_1(x)| \min(1, \frac{1}{|tx|}) dx, \end{aligned}$$

where  $M = \|\psi\|_{L_1}$  and  $V = V_{\mathbb{R}}(\psi)$  is the variation of  $\psi$  on  $\mathbb{R}$ .

Setting  $\tau = 1/|t|$ , we apply Hardy's inequality to get

$$\begin{aligned} &\left( \int_0^\infty t^{\frac{q}{p'}-1} |\mathcal{F}_\psi f(t)|^q dt \right)^{\frac{1}{q}} \\ &\leq 2(M + V) \left( \int_0^\infty t^{\frac{q}{p'}-1} \left( \int_0^{2/t} f^*(s) ds + \frac{1}{t} \int_{1/t}^\infty f^*(s) \frac{ds}{s} \right)^q dt \right)^{\frac{1}{q}} \\ &= 2(M + V) \left( \int_0^\infty \left( \frac{1}{t^{1/p'}} \int_0^{2t} f^*(s) ds + t^{1/p} \int_t^\infty f^*(s) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq C(\psi, p, q) \|f\|_{L_{p,q}(\mathbb{R})}. \end{aligned}$$

This implies inequality (4.3) taking into account that, for  $p \leq r \leq q$ ,

$$\|f\|_{L_{p,q}} \leq C(p, r) \|f\|_{L_{p,r}} \leq C(p, r) \left( \int_{\mathbb{R}} |f(x)|^r |x|^{r(1/p-1/r)} dx \right)^{1/r},$$

where the last estimate follows from the Hardy–Littlewood rearrangement inequality, see [3, p. 7].  $\square$

Note that the classical Pitt inequality [1,3] reads as follows

$$\|\widehat{f}\|_{L_{\omega(p',q)}(\mathbb{R})} \leq C(p, q, r) \|f\|_{L_{\omega(p,r)}(\mathbb{R})}$$

provided

$$1 < p \leq r \leq q \leq p' < \infty, \quad (4.4)$$

cf. (4.3). Remark that several attempts were made earlier to extend the range (4.4), with a use of some additional conditions on functions, see [4,7,13,26].

We also note that letting  $p = q = r$ , inequality (4.3) yields (4.2) with  $1 < p < \infty$  but with a constant  $C$  depending on  $\psi$  and  $p$ .

Finally, we remark that defining the Hausdorff operator

$$(\mathcal{H}_\psi f)^\wedge(t) = \int_{\mathbb{R}} \widehat{f}(ts) \psi(s) ds,$$

as in [16,19] and using Theorems 4.1 and 4.2 and Pitt's inequality we arrive at

**Corollary 4.1.** *Let  $1 < p' \leq q \leq s \leq p < \infty$ . Then  $\mathcal{H}_\psi$  with  $\psi \in L_1(\mathbb{R})$  of bounded variation is a bounded operator from  $L_{p,q}$  into  $L_{\omega(p',s)}$ . In particular,  $\mathcal{H}_\psi$  is a bounded operator in  $L_p$  for  $p \geq 2$ . The same result holds provided that  $\psi$  satisfies (4.1) and  $\widehat{\psi} \in L_1(\mathbb{R})$ .*

Note that  $L_p$  boundedness of the operator  $\mathcal{H}_\psi$  is known (see [18,33]) under the condition  $\int_0^\infty t^{-1+1/p} |\psi(t)| dt < \infty$ . Thus, Corollary 4.1 extends this with respect to behavior of  $\psi$  near the origin.

## Conflict of interest statement

There is no conflict of interest.

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