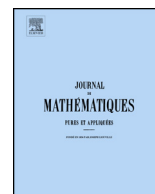




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Embeddings and characterizations of Lipschitz spaces [☆]

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ABSTRACT

In this paper we give a thorough study of Lipschitz spaces. We obtain the following new results:

- (i) Sharp Jawerth-Franke-type embeddings between the Besov and Lipschitz spaces extending the classical results for Besov and Sobolev spaces.
- (ii) Sharp embeddings between Lipschitz spaces with different parameters extending the Brézis-Wainger result.
- (iii) Characterizations for Lipschitz norms via wavelets.

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R É S U M É

Dans cet article, nous proposons une étude approfondie des espaces de Lipschitz. Nous obtenons les résultats suivants :

- (i) Les injections optimales de type Jawerth-Franke entre les espaces de Besov et de Lipschitz en étendant les résultats classiques pour les espaces de Besov et Sobolev.
- (ii) Les injections optimales entre des espaces de Lipschitz avec différents paramètres en étendant le théorème de Brézis-Wainger.
- (iii) Des caractérisations des normes de Lipschitz par les ondelettes.

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1. Introduction

The classical Hölder condition

$$\|f(\cdot + h) - f\|_{L_p(\mathbb{R}^d)} \leq C|h|^\alpha \quad (1.1)$$

is known to be extremely useful in a variety of questions in analysis and the PDE's. Embedding properties of the corresponding function space for a non-limiting parameter $0 < \alpha < 1$ – the Lipschitz/Besov space – are well known and can be found in many textbooks (see, e.g., [55–57]). In the limiting case $\alpha = 1$ the Lipschitz space has been much less investigated (see [20]). Moreover, there is a need to study not only the classical Lipschitz spaces but also the logarithmic Lipschitz spaces, which are obtained by introducing an additional logarithmic majorant in (1.1). The latter are motivated in part by the celebrated Brézis-Wainger inequality [6], which claims that every function f in the Sobolev space $H_p^{1+d/p}(\mathbb{R}^d)$, $1 < p < \infty$, is “almost” Lipschitz-continuous in the sense that

$$|f(x+h) - f(x)| \leq C|h| \log |h|^{1-1/p} \|f\|_{H_p^{1+d/p}(\mathbb{R}^d)}$$

for all $x \in \mathbb{R}^d$ and $0 < |h| < 1/2$. Taking into account that Sobolev spaces are particular cases of the Lipschitz spaces, namely $H_p^\alpha(\mathbb{R}^d) = \text{Lip}_{p,\infty}^{(\alpha,0)}(\mathbb{R}^d)$, $\alpha > 0$, the previous inequality can be interpreted in terms of the embeddings

$$\text{Lip}_{p,\infty}^{(1+d/p,0)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(1,-1+1/p)}(\mathbb{R}^d), \quad 1 < p < \infty, \quad (1.2)$$

(the precise definitions of function spaces will be given in Section 2). This embedding has been extensively studied in the framework of Triebel-Lizorkin and Besov spaces [24,25], paving the way for the nowadays well-established theory of envelopes of function spaces (see [57,37] and the references given there).

Besides their intrinsic interest, embedding theorems for function spaces of logarithmic Lipschitz type are proving useful in the theory of differential equations. For instance, in virtue of the well-known Osgood condition, the embedding (1.2), as well as its extensions to vector fields [64], is the key result to establish well-posedness of solutions of initial value problems in critical cases. On the other hand, as can be seen in [3, Section 2.11], [16, Proposition 3.5], [17, Lemma 3.13], [18, Proposition 3.3] (see also [49]), the embeddings

$$B_{\infty,\infty}^1(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(1,-1)}(\mathbb{R}^d) \hookrightarrow B_{\infty,\infty}^{1,-1}(\mathbb{R}^d) \quad (1.3)$$

and

$$B_{\infty,\infty}^{1,-1}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(1,-2)}(\mathbb{R}^d) \quad (1.4)$$

play an important role in the analysis of the incompressible Euler equations and hyperbolic operators. In particular, the left-hand side embedding in (1.3) sharpens the widely used criterion in the PDE's that $f \in \text{Lip}_{\infty,\infty}^{(1,-1)}(\mathbb{R}^d)$ whenever ∇f is a function of bounded mean oscillation (see [2, Lemma 1] and [8, Theorem 3]).

Let us also mention some important problems in PDE's which are formulated in terms of the integral log-Lipschitz spaces $\text{Lip}_{p,q}^{(\alpha,-b)}$ (cf. [15,18,30,51]). In this direction, Fanelli and Zuazua [30] studied the control problem in wave equations under lower smoothness assumptions. More precisely, they show the fulfilment of observability estimates in the classical sense for coefficients in the Zygmund class $B_{1,\infty}^1$ (in their notation \mathcal{Z}_1) and with a finite loss of derivatives for coefficients in $\text{Lip}_{1,\infty}^{(1,-1)}$ and $B_{1,\infty}^{1,-1}$ (in their notation, the spaces \mathcal{LL}_1 and \mathcal{LZ}_1 , respectively). This extends the well-known observability result for BV coefficients [32]. The precise interrelations between these function spaces will be given later.

Among many other applications of embeddings between Lipschitz spaces we highlight the Ulyanov-type inequalities for moduli of smoothness (see [23]). This topic is a cornerstone in the study of various problems in approximation theory [21,43] and functional analysis [45].

Our main goal in this paper is to present the detailed study of embedding properties and wavelet characterizations of the Lipschitz spaces to fill in the gap in the existing literature. Our results can be naturally divided into three parts.

In the first part we study embeddings between Lipschitz and the most familiar Besov spaces. In particular, in Section 4 we extend (1.3) and (1.4) to the full range of parameters. Specifically, if $\alpha > 0$, $1 < p < \infty$, $0 < q \leq \infty$, and $b > 1/q$, then

$$B_{p,q}^{\alpha,-b+1/\min\{2,p,q\}}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p,q}^{\alpha,-b+1/\max\{2,p,q\}}(\mathbb{R}^d) \quad (1.5)$$

and

$$B_{p,\min\{2,p,q\}}^{\alpha,-b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p,\max\{2,p,q\}}^{\alpha,-b+1/q}(\mathbb{R}^d). \quad (1.6)$$

It is worthwhile to mention that (1.5) and (1.6) extend the classical Besov–Sobolev embedding (see, e.g., [56, Section 2.3.2])

$$B_{p,\min\{2,p\}}^{\alpha}(\mathbb{R}^d) \hookrightarrow H_p^{\alpha}(\mathbb{R}^d) \hookrightarrow B_{p,\max\{2,p\}}^{\alpha}(\mathbb{R}^d). \quad (1.7)$$

We also investigate the limiting cases $p = 1, \infty$. In particular, we obtain the following embeddings

$$\text{BV}(\mathbb{R}^d) \hookrightarrow B_{1,\infty}^1(\mathbb{R}^d) \hookrightarrow \text{Lip}_{1,\infty}^{(1,-1)}(\mathbb{R}^d) \hookrightarrow B_{1,\infty}^{1,-1}(\mathbb{R}^d).$$

These embeddings provide the connections between observability estimates for wave equations and smoothness conditions of their coefficients in the Fanelli and Zuazua theorem discussed above.

Moreover, we derive new embeddings of Jawerth-Franke type for Lipschitz spaces, namely, if $\alpha > 0$, $1 \leq p_0 < p < p_1 \leq \infty$, $0 < q \leq \infty$, and $b > 1/q$, we have

$$B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/\min\{p,q\}}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/\max\{p,q\}}(\mathbb{R}^d) \quad (1.8)$$

and

$$B_{p_0,\min\{p,q\}}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,\max\{p,q\}}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d). \quad (1.9)$$

These extend the long-established Jawerth-Franke result for Sobolev spaces [33,41] (cf. also [47] and [61]) given by

$$B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d) \hookrightarrow H_p^{\alpha}(\mathbb{R}^d) \hookrightarrow B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d). \quad (1.10)$$

Before proceeding further, some comments are in order. The embeddings (1.5), (1.6), (1.8) and (1.9) reveal the importance of the parameters b and q in $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$. In particular, it is interesting to compare the embeddings

$$\begin{aligned} H_p^{\alpha}(\mathbb{R}^d) &= \text{Lip}_{p,\infty}^{(\alpha,0)}(\mathbb{R}^d) \hookrightarrow B_{p,\max\{2,p\}}^{\alpha}(\mathbb{R}^d), \\ \text{Lip}_{p,\infty}^{(\alpha,-b)}(\mathbb{R}^d) &\hookrightarrow B_{p,\infty}^{\alpha,-b}(\mathbb{R}^d), \quad b > 0, \end{aligned} \quad (1.11)$$

cf. (1.7) and (1.6) correspondingly. Note that both embeddings in (1.11) are sharp, which shows substantial differences between embedding theorems for the spaces $\text{Lip}_{p,\infty}^{(\alpha,0)}(\mathbb{R}^d) = H_p^\alpha(\mathbb{R}^d)$ and those for $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$, $b > 1/q$. The same comment also applies to (1.9) and (1.10).

Our second aim is to fully characterize the embeddings between various Lipschitz spaces, i.e.,

$$\text{Lip}_{p_0,q_0}^{(\alpha_0,-b_0)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p_1,q_1}^{(\alpha_1,-b_1)}(\mathbb{R}^d).$$

This is addressed in Section 5. Among various other results, we obtain the Sobolev type embedding with the parameters $1 < p_0 < p_1 < \infty$ such that $\alpha_0 - d/p_0 = \alpha_1 - d/p_1$, and $0 < q_0 = q_1 \leq \infty$, $b_0 = b_1 > 1/q$. More interestingly, we also deal with the extreme cases $p_0 = 1$ and/or $p_1 = \infty$. In particular, setting $p_1 = \infty$, we extend the Brézis-Wainger embedding (1.2) showing that

$$\text{Lip}_{p,q}^{(\alpha+d/p,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,q}^{(\alpha,-b-1+1/p)}(\mathbb{R}^d), \quad 1 < p < \infty,$$

and

$$\text{Lip}_{1,q}^{(k+d,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,q}^{(k,-b)}(\mathbb{R}^d), \quad k \in \mathbb{N}.$$

Next, our third goal is to give a characterization of the Lipschitz spaces via wavelets (Section 6). We show that the truncated square function

$$\left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\beta 2} |\lambda_m^{j,G}(f) \chi_{j,m}(\cdot)|^2 \right)^{1/2},$$

given in terms of wavelet coefficients and characteristic functions of dyadic cubes, can be used to develop a unified treatment of function spaces which includes Besov, Sobolev, Lebesgue and Lipschitz spaces; see Section 6. To be more precise, we prove that the family of norms given by

$$\left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\beta 2} |\lambda_m^{j,G}(f) \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}, \quad \beta \geq \alpha,$$

allows us to characterize Besov spaces ($\beta > \alpha$), Lipschitz spaces ($\beta = \alpha$), Lebesgue spaces ($\beta = \alpha = 0, q = \infty$ and $b = 0$), and Sobolev spaces ($\beta = \alpha, q = \infty$ and $b = 0$).

Concluding the introduction, we briefly present one of the novel ideas to investigate the structure of the Lipschitz spaces. Using the fact that Lipschitz spaces can be characterized as limiting interpolation spaces between classical Lebesgue and Sobolev spaces, i.e.,

$$\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d))_{(1,-b),q}, \quad (1.12)$$

we employ the machinery of limiting interpolation with $\theta = 1$ (see [13,28,29,35]) to transfer properties from $H_p^\alpha(\mathbb{R}^d)$ to $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$.

It is worth mentioning that a similar approach with $\theta = 0$ was applied in the study of Besov spaces with smoothness close to zero (cf. [11,22]):

$$\mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d))_{(0,b),q}, \quad (1.13)$$

where $\mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d)$ is the Besov space with the logarithmic smoothness b . However, there is a fundamental difference in applications of (1.12) and (1.13) due to the fact that Lipschitz and Besov spaces have a

completely different structure. For instance, a crucial step in the proofs of the wavelet description of the spaces $\mathbb{B}_{p,q}^{0,b}(\mathbb{R}^d)$ (cf. [11, Theorems 4.3 and 5.5]) is the fact that these spaces can be characterized via approximation processes. Such characterization fails to be true for Lipschitz spaces and this forces us to apply more sophisticated limiting interpolation arguments.

We collect the necessary background in interpolation theory in Section 2 and give some auxiliary interpolation results in Section 3. Some of them are interesting in themselves. For instance, we show that the scale of Lipschitz spaces with fixed integrability is stable under interpolation (see Lemma 3.6). Furthermore, our interpolation-based method also works with Lipschitz spaces formed by periodic functions.

Throughout the paper, we use the notation $F \lesssim G$ with $F, G \geq 0$ for the estimate $F \leq C G$, where C is a positive constant independent of the essential variables in F and G . If $F \lesssim G \lesssim F$, we write $F \asymp G$ and say that F is equivalent to G .

2. Preliminaries

2.1. Function spaces

Let $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on \mathbb{R}^d , and the space of tempered distributions on \mathbb{R}^d , respectively. The Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$\widehat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^d,$$

where $x \cdot \xi = \sum_{j=1}^d x_j \xi_j$, and the inverse Fourier transform is given by

$$f^\vee(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{ix \cdot \xi} dx.$$

These operators are extended to $\mathcal{S}'(\mathbb{R}^d)$ in the usual way.

Let $-\infty < \alpha < \infty$ and $1 < p < \infty$. The (fractional) Sobolev space (also called, Bessel potential space) $H_p^\alpha(\mathbb{R}^d)$ is the set of all $f \in L_p(\mathbb{R}^d)$ such that

$$\|f\|_{H_p^\alpha(\mathbb{R}^d)} = \|((1 + |x|^2)^{\alpha/2} \widehat{f})^\vee\|_{L_p(\mathbb{R}^d)} < \infty.$$

Note that $H_p^0(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and $H_p^k(\mathbb{R}^d) = W_p^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, the classical Sobolev space.

Let $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $-\infty < s, b < \infty$. The Besov space $B_{p,q}^{s,b}(\mathbb{R}^d)$ is formed by all $f \in \mathcal{S}'(\mathbb{R}^d)$ having a finite quasi-norm

$$\|f\|_{B_{p,q}^{s,b}(\mathbb{R}^d)} = \left(\sum_{j=0}^{\infty} (2^{js} (1+j)^b \|(\varphi_j \widehat{f})^\vee\|_{L_p(\mathbb{R}^d)})^q \right)^{1/q}. \quad (2.1)$$

Here, $(\varphi_j)_{j=0}^\infty$ is a smooth dyadic resolution of unity in \mathbb{R}^d given by $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\text{supp } \varphi_0 \subset \{x \in \mathbb{R}^d : |x| \leq 2\} \quad \text{and} \quad \varphi_0(x) = 1 \quad \text{if} \quad |x| \leq 1$$

and $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$. For further details on these spaces, we refer the reader to [26,27,31,42,46,50]. In particular, we remark that the definition of $B_{p,q}^{s,b}(\mathbb{R}^d)$ is independent (in the

sense of equivalent quasi-norms) of the particular choice of φ_0 . Note that if $b = 0$ in $B_{p,q}^{s,b}(\mathbb{R}^d)$ then we get the classical Besov spaces $B_{p,q}^s(\mathbb{R}^d)$. We also mention that the quasi-norm (2.1) can be used to introduce $B_{p,q}^{s,b}(\mathbb{R}^d)$ for $0 < p \leq \infty$.

For $\alpha > 0$ and $h \in \mathbb{R}^d$, we let

$$\Delta_h^\alpha f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x + (\alpha - j)h), \quad x \in \mathbb{R}^d,$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}$, $\binom{\alpha}{0} = 1$. We denote by $\omega_\alpha(f, t)_p$, $1 \leq p \leq \infty$, the *modulus of smoothness of fractional order α of $f \in L_p(\mathbb{R}^d)$* which is given by

$$\omega_\alpha(f, t)_p = \sup_{|h| \leq t} \|\Delta_h^\alpha f\|_{L_p(\mathbb{R}^d)}, \quad t > 0.$$

Clearly, if $\alpha \in \mathbb{N}$ then we recover the classical modulus of smoothness.

It is well known that Besov spaces with positive smoothness can be characterized through moduli of smoothness. See, e.g., [56, Section 2.5.12] and [38, Theorem 2.5]. Assume $0 < s < \alpha$. Then

$$\|f\|_{B_{p,q}^{s,b}(\mathbb{R}^d)} \asymp \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 (t^{-s}(1 - \log t)^b \omega_\alpha(f, t)_p)^q \frac{dt}{t} \right)^{1/q} \quad (2.2)$$

(if $q = \infty$, then the integral should be replaced by the appropriate supremum).

Let $\alpha > 0$, $1 \leq p \leq \infty$, $0 < q \leq \infty$, and $-\infty < b < \infty$. The *logarithmic Lipschitz space* $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ is the collection of all $f \in L_p(\mathbb{R}^d)$ such that

$$\|f\|_{\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \left(\int_0^1 (t^{-\alpha}(1 - \log t)^{-b} \omega_\alpha(f, t)_p)^q \frac{dt}{t} \right)^{1/q} < \infty \quad (2.3)$$

(with the usual modification if $q = \infty$). We shall assume that $b > 1/q$ ($b \geq 0$ if $q = \infty$). Otherwise, $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) = \{0\}$. Setting $\alpha = 1$ we obtain the spaces $\text{Lip}_{p,q}^{(1,-b)}(\mathbb{R}^d)$ studied in detail in [37]. See also the paper [40] by Janson for a thorough study of (generalized) Lipschitz spaces. The reader is advised to take care that the spaces $\text{Lip}_{\infty,q}^{(\alpha,-b)}(\mathbb{R}^d)$, $0 < \alpha < 1$, endowed with (2.3) differ from those considered by Haroske [37]. To be more precise, in our notation the Lipschitz-type spaces introduced in [37, Definition 2.26] coincide with $B_{\infty,q}^{\alpha,-b}(\mathbb{R}^d)$ (see (2.2)).

If $p = q = \infty$, $\alpha = 1$ and $b = 0$ in $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ then we obtain the classical Lipschitz spaces $\text{Lip}(\mathbb{R}^d)$ formed by all $f \in L_\infty(\mathbb{R}^d)$ such that

$$\|f\|_{\text{Lip}(\mathbb{R}^d)} = \|f\|_{L_\infty(\mathbb{R}^d)} + \sup_{0 < |x-y| < 1} \frac{|f(x) - f(y)|}{|x - y|}.$$

More generally, $\text{Lip}_{\infty,\infty}^{(k,0)}(\mathbb{R}^d) = \text{Lip}^k(\mathbb{R}^d)$ where $\text{Lip}^1(\mathbb{R}^d) = \text{Lip}(\mathbb{R}^d)$ and

$$\|f\|_{\text{Lip}^k(\mathbb{R}^d)} = \sum_{|\beta| \leq k-1} \|D^\beta f\|_{\text{Lip}(\mathbb{R}^d)}, \quad k \geq 2, \quad k \in \mathbb{N}.$$

See [22, Lemma 14.5].

If $p = 1, q = \infty, \alpha = 1$ and $b = 0$ then we recover $BV(\mathbb{R}^d)$, the space of integrable functions of bounded variation. More generally, we can deal with spaces of primitives of functions of bounded variation. That is, setting $\alpha = k \in \mathbb{N}$ we get $BV^{k-1}(\mathbb{R}^d)$ where

$$BV^0(\mathbb{R}^d) = BV(\mathbb{R}^d) = \text{Lip}_{1,\infty}^{(1,0)}(\mathbb{R}^d), \quad (2.4)$$

and

$$\|f\|_{BV^k(\mathbb{R}^d)} = \sum_{|\beta| \leq k} \|D^\beta f\|_{BV(\mathbb{R}^d)}, \quad k \geq 1.$$

See [22, Lemma 14.5].

Moreover, the space $H_p^\alpha(\mathbb{R}^d)$ with $\alpha > 0$ is a special case of Lipschitz spaces, since

$$\text{Lip}_{p,\infty}^{(\alpha,0)}(\mathbb{R}^d) = H_p^\alpha(\mathbb{R}^d), \quad 1 < p < \infty; \quad (2.5)$$

see, e.g., [62, Corollary 10].

Analogously, one can introduce the periodic counterparts $H_p^\alpha(\mathbb{T}^d)$, $B_{p,q}^{s,b}(\mathbb{T}^d)$, and $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{T}^d)$.

2.2. Limiting interpolation

Let (A_0, A_1) be an ordered couple of quasi-Banach spaces, that is, $A_1 \hookrightarrow A_0$. The *Peetre K -functional* is defined by

$$K(t, f) = K(t, f; A_0, A_1) = \inf_{f_1 \in A_1} (\|f - f_1\|_{A_0} + t\|f_1\|_{A_1}), \quad t > 0, \quad f \in A_0. \quad (2.6)$$

Let $0 < \theta < 1$, $-\infty < b < \infty$, and $0 < q \leq \infty$. The *logarithmic interpolation space* $(A_0, A_1)_{\theta,q;b}$ is the set formed by all $f \in A_0$ such that

$$\|f\|_{(A_0, A_1)_{\theta,q;b}} = \left(\int_0^\infty (t^{-\theta} (1 + |\log t|)^b K(t, f))^q \frac{dt}{t} \right)^{1/q} < \infty$$

(appropriately modified if $q = \infty$). For further details and properties, we refer to [36]. In particular, if $b = 0$ in $(A_0, A_1)_{\theta,q;b}$ then we obtain the classical real interpolation space $(A_0, A_1)_{\theta,q}$; see [4, 5, 55].

The interpolation properties of Besov spaces of generalized smoothness were investigated by Cobos and Fernández [12]. For later use, we record some interpolation formulas obtained in [12, Theorem 5.3]. Let $1 \leq p \leq \infty$, $0 < q_0, q_1, q \leq \infty$, $-\infty < s_0 \neq s_1 < \infty$, $-\infty < b_0, b_1, \alpha < \infty$ and $0 < \theta < 1$. Put $s = (1 - \theta)s_0 + \theta s_1$, and $b = (1 - \theta)b_0 + \theta b_1$. Then we have

$$(B_{p,q_0}^{s_0,b_0}(\mathbb{R}^d), B_{p,q_1}^{s_1,b_1}(\mathbb{R}^d))_{\theta,q;\alpha} = B_{p,q}^{s,b+\alpha}(\mathbb{R}^d) \quad (2.7)$$

and

$$(H_p^{s_0}(\mathbb{R}^d), H_p^{s_1}(\mathbb{R}^d))_{\theta,q;\alpha} = B_{p,q}^{s,\alpha}(\mathbb{R}^d). \quad (2.8)$$

In particular, if $b_0 = b_1 = \alpha = 0$ in (2.7) then

$$(B_{p,q_0}^{s_0}(\mathbb{R}^d), B_{p,q_1}^{s_1}(\mathbb{R}^d))_{\theta,q} = B_{p,q}^s(\mathbb{R}^d); \quad (2.9)$$

see also [55, Section 2.4.1, pages 181–184] and [4, Section 5, Theorem 4.17, page 343]. On the other hand, setting $s_0 = 0$ and $s_1 > 0$ in (2.8),

$$(L_p(\mathbb{R}^d), H_p^{s_1}(\mathbb{R}^d))_{\theta, q; \alpha} = B_{p, q}^{\theta s_1, \alpha}(\mathbb{R}^d). \quad (2.10)$$

For the extensions of (2.8) and (2.10) to Triebel-Lizorkin spaces of generalized smoothness, we refer the reader to [12].

The corresponding formulas for periodic Besov spaces also hold true.

Since $A_1 \hookrightarrow A_0$, it is not hard to check that $K(t, f) \asymp \|f\|_{A_0}$ for $t > 1$. Consequently, we have

$$\|f\|_{(A_0, A_1)_{\theta, q; b}} \asymp \left(\int_0^1 (t^{-\theta} (1 + |\log t|)^b K(t, f))^q \frac{dt}{t} \right)^{1/q}.$$

This fact together with the finer tuning given by logarithmic weights allows us to introduce *limiting interpolation spaces* with $\theta = 1$. Namely, the space $(A_0, A_1)_{(1, b), q}$ is the collection of all $f \in A_0$ for which

$$\|f\|_{(A_0, A_1)_{(1, b), q}} = \left(\int_0^1 (t^{-1} (1 + |\log t|)^b K(t, f))^q \frac{dt}{t} \right)^{1/q} < \infty. \quad (2.11)$$

See [28], [29], [35] and [13]. Note that this space becomes trivial if $b \geq -1/q$ ($b > 0$ if $q = \infty$). Then we shall assume that $b < -1/q$ ($b \leq 0$ if $q = \infty$).

Lipschitz spaces $\text{Lip}_{p, q}^{(\alpha, -b)}(\mathbb{R}^d)$, $1 < p < \infty$, can be characterized as limiting interpolation spaces between $L_p(\mathbb{R}^d)$ and $H_p^\alpha(\mathbb{R}^d)$. Indeed, if $1 < p < \infty$ then

$$K(t^\alpha, f; L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d)) \asymp t^\alpha \|f\|_{L_p(\mathbb{R}^d)} + \omega_\alpha(f, t)_p, \quad 0 < t < 1 \quad (2.12)$$

(see [62, (4.2)]), which yields that

$$(L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d))_{(1, -b), q} = \text{Lip}_{p, q}^{(\alpha, -b)}(\mathbb{R}^d). \quad (2.13)$$

The corresponding formula for periodic spaces also holds true (see [63, (21)]). Here it is important to mention that (2.12) and (2.13) can be extended to cover the extreme cases $p = 1, \infty$. This can be done with the help of the Sobolev-type spaces $\mathcal{H}_p^\alpha(\mathbb{R}^d)$, $\alpha > 0$, $1 \leq p \leq \infty$, defined by

$$\mathcal{H}_p^\alpha(\mathbb{R}^d) = \left\{ f \in L_p(\mathbb{R}^d) : \|f\|_{\mathcal{H}_p^\alpha(\mathbb{R}^d)} = \|f\|_{L_p(\mathbb{R}^d)} + \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha f\|_{L_p(\mathbb{R}^d)} < \infty \right\}$$

where, $D_\zeta^\alpha f(x) = ((i\xi \cdot \zeta)^\alpha \widehat{f}(\xi))^\vee(x)$. Note that $\mathcal{H}_p^\alpha(\mathbb{R}^d) = H_p^\alpha(\mathbb{R}^d)$ if $1 < p < \infty$. It was shown in [44, Property 13] that

$$K(t^\alpha, f; L_p(\mathbb{R}^d), \mathcal{H}_p^\alpha(\mathbb{R}^d)) \asymp t^\alpha \|f\|_{L_p(\mathbb{R}^d)} + \omega_\alpha(f, t)_p, \quad 1 \leq p \leq \infty,$$

and thus

$$(L_p(\mathbb{R}^d), \mathcal{H}_p^\alpha(\mathbb{R}^d))_{(1, -b), q} = \text{Lip}_{p, q}^{(\alpha, -b)}(\mathbb{R}^d), \quad 1 \leq p \leq \infty. \quad (2.14)$$

The periodic counterpart of the latter formula also holds true.

Furthermore, it is well known that the corresponding results for the classical Sobolev spaces $W_p^k(\mathbb{R}^d)$ also hold true if $p = 1, \infty$ (see [4, Chapter 5, Theorem 4.12]). More precisely, if $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ then

$$K(t^k, f; L_p(\mathbb{R}^d), W_p^k(\mathbb{R}^d)) \asymp t^k \|f\|_{L_p(\mathbb{R}^d)} + \omega_k(f, t)_p, \quad 0 < t < 1,$$

and so

$$(L_p(\mathbb{R}^d), W_p^k(\mathbb{R}^d))_{(1, -b), q} = \text{Lip}_{p, q}^{(k, -b)}(\mathbb{R}^d). \quad (2.15)$$

The latter is the limiting version of the well-known interpolation formula for Besov spaces

$$(L_p(\mathbb{R}^d), W_p^k(\mathbb{R}^d))_{\theta, q; -b} = B_{p, q}^{\theta k, -b}(\mathbb{R}^d). \quad (2.16)$$

The following reiteration formulas for classical and limiting interpolation methods will be useful later. Let $0 < \theta < 1$, $0 < p, q, q_0, q_1 \leq \infty$, $b < -1/q$, and $b_0 + 1/q_0 < b_1 + 1/q_1 < 0$. Then

$$(A_0, A_1)_{\theta, q; b+1/\min\{p, q\}} \hookrightarrow (A_0, (A_0, A_1)_{\theta, p})_{(1, b), q} \hookrightarrow (A_0, A_1)_{\theta, q; b+1/\max\{p, q\}}; \quad (2.17)$$

$$(A_0, (A_0, A_1)_{(1, b), q})_{\theta, p} = (A_0, A_1)_{\theta, p; \theta(b+1/q)}; \quad (2.18)$$

$$((A_0, A_1)_{\theta, p}, A_1)_{(1, b), q} = (A_0, A_1)_{(1, b), q}; \quad (2.19)$$

$$((A_0, A_1)_{(1, b_0), q_0}, (A_0, A_1)_{(1, b_1), q_1})_{\theta, p} = (A_0, A_1)_{(1, (1-\theta)(b_0+1/q_0)+\theta(b_1+1/q_1)-1/p), p}. \quad (2.20)$$

See [28, Theorem 4.7*+], [29, Theorems 7.1(v), 7.4* and Corollary 7.11] and [9, Lemma 2.5(a)].

3. Interpolation lemmas

In this section we show some interpolation lemmas that will be useful in forthcoming considerations. Our first result concerns the limiting interpolation of vector-valued sequence spaces $\ell_p(A_j)$. Before we state it, let us recall some notation. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 < p \leq \infty$. If $(\lambda_j)_{j \in \mathbb{N}_0}$ is a sequence of positive numbers and $(A_j)_{j \in \mathbb{N}_0}$ is a sequence of quasi-Banach spaces, by $\ell_p(\lambda_j A_j)$ we mean the space formed by all vector-valued sequences $a = (a_j)_{j \in \mathbb{N}_0}$ with $a_j \in A_j$ endowed with the quasi-norm

$$\|a\|_{\ell_p(\lambda_j A_j)} = \left(\sum_{j=0}^{\infty} \lambda_j^p \|a_j\|_{A_j}^p \right)^{1/p}$$

(where the sum should be replaced by the sup if $p = \infty$).

Working with vector-valued sequence spaces, it is convenient to introduce the so called K_p -functional (cf. [5, page 75]). For $0 < p \leq \infty$, the K_p -functional is defined by

$$K_p(t, f) = K_p(t, f; A_0, A_1) = \inf_{f_1 \in A_1} (\|f - f_1\|_{A_0}^p + t^p \|f_1\|_{A_1}^p)^{1/p}.$$

Note that if $p = 1$ then we obtain the usual K -functional (2.6). In general, we have

$$K_p(t, f) \asymp K(t, f) \quad (3.1)$$

where the equivalence constants only depend on p .

Lemma 3.1. Let $0 < p, q \leq \infty, b < -1/q$ ($b \leq 0$ if $q = \infty$) and let $(A_j)_{j \in \mathbb{N}_0}, (B_j)_{j \in \mathbb{N}_0}$ be sequences of quasi-Banach spaces such that $B_j \hookrightarrow A_j$ uniformly with respect to j . Then

$$\ell_{\min\{p,q\}}((A_j, B_j)_{(1,b),q}) \hookrightarrow (\ell_p(A_j), \ell_p(B_j))_{(1,b),q} \hookrightarrow \ell_{\max\{p,q\}}((A_j, B_j)_{(1,b),q}). \quad (3.2)$$

Remark 3.2. The assumption $\sup_{j \in \mathbb{N}_0} \sup_{\|a\|_{B_j} \leq 1} \|a\|_{A_j} < \infty$ ensures that $\ell_p(B_j) \hookrightarrow \ell_p(A_j)$.

Proof of Lemma 3.1. Let $a = (a_j)_{j \in \mathbb{N}_0} \in \ell_p(A_j)$. Assume, for convenience, $p < \infty$, the case $p = \infty$ can be done similarly. Elementary computations lead to

$$K_p(t, a; \ell_p(A_j), \ell_p(B_j)) = \left(\sum_{j=0}^{\infty} K_p(t, a_j; A_j, B_j)^p \right)^{1/p}.$$

Then, by (2.11) and (3.1),

$$\|a\|_{(\ell_p(A_j), \ell_p(B_j))_{(1,b),q}} \asymp \left(\int_0^1 \left(\sum_{j=0}^{\infty} (t^{-1}(1 - \log t)^b K_p(t, a_j; A_j, B_j))^p \right)^{q/p} \frac{dt}{t} \right)^{1/q}.$$

Suppose now that $q \geq p$. Then, applying Minkowski's inequality, we have

$$\begin{aligned} \|a\|_{(\ell_p(A_j), \ell_p(B_j))_{(1,b),q}} &\lesssim \left(\sum_{j=0}^{\infty} \left(\int_0^1 t^{-q}(1 - \log t)^{bq} K_p(t, a_j; A_j, B_j)^q \frac{dt}{t} \right)^{p/q} \right)^{1/p} \\ &\asymp \left(\sum_{j=0}^{\infty} \|a_j\|_{(A_j, B_j)_{(1,b),q}}^p \right)^{1/p} = \|a\|_{\ell_p((A_j, B_j)_{(1,b),q})}. \end{aligned}$$

On the other hand, if $q < p$ then

$$\begin{aligned} \|a\|_{(\ell_p(A_j), \ell_p(B_j))_{(1,b),q}} &\lesssim \left(\int_0^1 \sum_{j=0}^{\infty} (t^{-1}(1 - \log t)^b K_p(t, a_j; A_j, B_j))^q \frac{dt}{t} \right)^{1/q} \\ &\asymp \left(\sum_{j=0}^{\infty} \|a_j\|_{(A_j, B_j)_{(1,b),q}}^q \right)^{1/q} = \|a\|_{\ell_q((A_j, B_j)_{(1,b),q})}. \end{aligned}$$

This finishes the proof of the left-hand side embedding in (3.2). The proof of the right-hand side embedding is similar. Further details are left to the reader. \square

As usual, given $\lambda > 0$, we denote by λA the space A equipped with the quasi-norm

$$\|a\|_{\lambda A} = \lambda \|a\|_A, \quad a \in A.$$

Lemma 3.3. Let $\lambda > 0$ and let (A_0, A_1) be a couple of quasi-Banach spaces with $A_1 \hookrightarrow A_0$. If $0 < q \leq \infty$ and $b < -1/q$ ($b \leq 0$ if $q = \infty$) then

$$(\lambda A_0, \lambda A_1)_{(1,b),q} = \lambda(A_0, A_1)_{(1,b),q} \quad (3.3)$$

with equivalence constants which are independent of λ .

Proof. Clearly, $K(t, a; \lambda A_0, \lambda A_1) = \lambda K(t, a; A_0, A_1)$. Then (3.3) follows. \square

Lemma 3.4. Let $\lambda \geq 1$ and let A be a quasi-Banach space. If $0 < q \leq \infty$ and $b < -1/q$ ($b \leq 0$ if $q = \infty$) then

$$(A, \lambda A)_{(1,b),q} = \lambda(1 + \log \lambda)^{b+1/q} A \quad (3.4)$$

with equivalence constants which are independent of λ .

Proof. Since $K(t, a; A, \lambda A) \asymp \min\{1, t\lambda\} \|a\|_A$, we obtain

$$\|a\|_{(A, \lambda A)_{(1,b),q}} = \left(\int_0^1 (t^{-1}(1 - \log t)^b \min\{1, t\lambda\})^q \frac{dt}{t} \right)^{1/q} \|a\|_A.$$

Assume $q < \infty$. We have

$$\begin{aligned} & \left(\int_0^1 (t^{-1}(1 - \log t)^b \min\{1, t\lambda\})^q \frac{dt}{t} \right)^{1/q} \\ & \asymp \left(\int_0^{1/\lambda} (1 - \log t)^{bq} \frac{dt}{t} \right)^{1/q} \lambda + \left(\int_{1/\lambda}^1 t^{-q}(1 - \log t)^{bq} \frac{dt}{t} \right)^{1/q} \\ & \asymp \lambda(1 + \log \lambda)^{b+1/q} \end{aligned}$$

because $b + 1/q < 0$. This gives the desired equivalence (3.4).

The case $q = \infty$ is easier and we omit further details. \square

Before going further, we recall the definitions of some vector-valued spaces that we shall use in the sequel.

Let A be a Banach space. Let $-\infty < \alpha < \infty$ and $0 < r \leq \infty$. We denote by $\ell_r^\alpha(\mathbb{N}_0; A) = \ell_r^\alpha(A)$ the space formed by all sequences $x = (x_j)_{j \in \mathbb{N}_0}$ with $x_j \in A$ such that

$$\|x\|_{\ell_r^\alpha(A)} = \left(\sum_{j=0}^{\infty} 2^{j\alpha r} \|x_j\|_A^r \right)^{1/r} < \infty \quad (3.5)$$

(with the usual modification if $r = \infty$).

If $1 \leq p \leq \infty$ then $L_p(\mathbb{R}^d; A)$ is the usual vector-valued Lebesgue space in the sense of the Bochner integral, that is, the set formed by all strongly measurable functions $f : \mathbb{R}^d \rightarrow A$ for which

$$\|f\|_{L_p(\mathbb{R}^d; A)} = \left(\int_{\mathbb{R}^d} \|f(x)\|_A^p dx \right)^{1/p} < \infty$$

(with the usual modification if $p = \infty$).

Our next result provides a characterization of the limiting interpolation space with $\theta = 1$ with respect to the couple formed by the vector-valued Lebesgue spaces $(L_p(\mathbb{R}^d; \ell_r(A)), L_p(\mathbb{R}^d; \ell_r^\alpha(A)))$.

Lemma 3.5. *Let $\alpha > 0, 1 \leq p, r \leq \infty, 0 < q \leq \infty$, and $b > 1/q$ ($b \geq 0$ if $q = \infty$). Let A be a Banach space. Then*

$$\|(f_j)\|_{(L_p(\mathbb{R}^d; \ell_r(A)), L_p(\mathbb{R}^d; \ell_r^\alpha(A)))_{(1, -b), q}} \asymp \left(\sum_{k=0}^{\infty} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k 2^{j\alpha r} \|f_j(\cdot)\|_A^r \right)^{1/r} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}. \quad (3.6)$$

The corresponding result for periodic functions also holds true.

Proof. We start by computing the K -functional for the couple $(L_p(\mathbb{R}^d; \ell_r(A)), L_p(\mathbb{R}^d; \ell_r^\alpha(A)))$. Let $k \in \mathbb{N}_0$. We claim that

$$\begin{aligned} K(2^{-k\alpha}, (f_j); L_p(\mathbb{R}^d; \ell_r(A)), L_p(\mathbb{R}^d; \ell_r^\alpha(A))) \\ \asymp \left\| \left(\sum_{j=k}^{\infty} \|f_j(\cdot)\|_A^r \right)^{1/r} \right\|_{L_p(\mathbb{R}^d)} + 2^{-k\alpha} \left\| \left(\sum_{j=0}^k 2^{j\alpha r} \|f_j(\cdot)\|_A^r \right)^{1/r} \right\|_{L_p(\mathbb{R}^d)}. \end{aligned} \quad (3.7)$$

Indeed, straightforward computations show that

$$K(t, f; L_p(\mathbb{R}^d; A_0), L_p(\mathbb{R}^d; A_1)) \asymp \left(\int_{\mathbb{R}^d} K(t, f(x); A_0, A_1)^p dx \right)^{1/p}$$

for any Banach couple (A_0, A_1) and

$$K(t, (x_j); \ell_r(A), \ell_r^\alpha(A)) \asymp \left(\sum_{j=0}^{\infty} (\min\{1, t2^{j\alpha}\} \|x_j\|_A)^r \right)^{1/r},$$

which implies (3.7). It follows from (3.7) that

$$\begin{aligned} \|(f_j)\|_{(L_p(\mathbb{R}^d; \ell_r(A)), L_p(\mathbb{R}^d; \ell_r^\alpha(A)))_{(1, -b), q}} \\ \asymp \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} K(2^{-k\alpha}, (f_j); L_p(\mathbb{R}^d; \ell_r(A)), L_p(\mathbb{R}^d; \ell_r^\alpha(A)))^q \right)^{1/q} \\ \asymp \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \left\| \left(\sum_{j=k}^{\infty} \|f_j(\cdot)\|_A^r \right)^{1/r} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} \\ + \left(\sum_{k=0}^{\infty} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k 2^{j\alpha r} \|f_j(\cdot)\|_A^r \right)^{1/r} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} \\ = I + II. \end{aligned}$$

Hence, to get (3.6) it will be enough to show that $I \lesssim II$. Let us distinguish two possible cases.

CASE 1: Assume $p \geq r$. For $k \in \mathbb{N}_0$, we apply Minkowski's inequality to obtain

$$\left\| \left(\sum_{j=k}^{\infty} \|f_j(\cdot)\|_A^r \right)^{1/r} \right\|_{L_p(\mathbb{R}^d)} \leq \left(\sum_{j=k}^{\infty} \|f_j\|_{L_p(\mathbb{R}^d; A)}^r \right)^{1/r}$$

and so,

$$I \leq \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \left(\sum_{j=k}^{\infty} \|f_j\|_{L_p(\mathbb{R}^d; A)}^r \right)^{q/r} \right)^{1/q}. \quad (3.8)$$

If $q \geq r$ then we apply Hardy's inequality to get

$$I \lesssim \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \|f_k\|_{L_p(\mathbb{R}^d; A)}^q \right)^{1/q} \leq II.$$

Assume now that $q < r$. Then, by (3.8) and changing the order of summation, we have

$$\begin{aligned} I &\leq \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \sum_{j=k}^{\infty} \|f_j\|_{L_p(\mathbb{R}^d; A)}^q \right)^{1/q} \\ &\asymp \left(\sum_{j=0}^{\infty} 2^{j\alpha q} (1+j)^{-bq} \|f_j\|_{L_p(\mathbb{R}^d; A)}^q \right)^{1/q} \leq II. \end{aligned}$$

CASE 2: Assume $p < r$. Then

$$I \leq \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \left(\sum_{j=k}^{\infty} \|f_j\|_{L_p(\mathbb{R}^d; A)}^p \right)^{q/p} \right)^{1/q}$$

Let $q \geq p$. Invoking Hardy's inequality,

$$I \lesssim \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \|f_k\|_{L_p(\mathbb{R}^d; A)}^q \right)^{1/q} \leq II.$$

On the other hand, if $q < p$ then

$$I \leq \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \sum_{j=k}^{\infty} \|f_j\|_{L_p(\mathbb{R}^d; A)}^q \right)^{1/q} \lesssim II. \quad \square$$

The following result, which is interesting by its own sake, shows that the scale of Lipschitz spaces is closed under real interpolation. Namely, we obtain the following

Lemma 3.6. *Let $\alpha > 0, 1 \leq p \leq \infty, 0 < q, q_0, q_1 \leq \infty, 0 < \theta < 1$, and $b_0 - 1/q_0 > b_1 - 1/q_1 > 0$. Then*

$$(\text{Lip}_{p, q_0}^{(\alpha, -b_0)}(\mathbb{R}^d), \text{Lip}_{p, q_1}^{(\alpha, -b_1)}(\mathbb{R}^d))_{\theta, q} = \text{Lip}_{p, q}^{(\alpha, -b)}(\mathbb{R}^d)$$

where $b - 1/q = (1 - \theta)(b_0 - 1/q_0) + \theta(b_1 - 1/q_1)$.

The corresponding result for periodic spaces also holds true.

Proof. By (2.14),

$$\text{Lip}_{p,q_i}^{(\alpha,-b_i)}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), \mathcal{H}_p^\alpha(\mathbb{R}^d))_{(1,-b_i),q_i}, \quad i = 0, 1.$$

Applying now (2.20), we have

$$\begin{aligned} & (\text{Lip}_{p,q_0}^{(\alpha,-b_0)}(\mathbb{R}^d), \text{Lip}_{p,q_1}^{(\alpha,-b_1)}(\mathbb{R}^d))_{\theta,q} \\ &= ((L_p(\mathbb{R}^d), \mathcal{H}_p^\alpha(\mathbb{R}^d))_{(1,-b_0),q_0}, (L_p(\mathbb{R}^d), \mathcal{H}_p^\alpha(\mathbb{R}^d))_{(1,-b_1),q_1})_{\theta,q} \\ &= (L_p(\mathbb{R}^d), \mathcal{H}_p^\alpha(\mathbb{R}^d))_{(1,-((1-\theta)(b_0-1/q_0)+\theta(b_1-1/q_1)+1/q)),q} \\ &= \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d), \end{aligned}$$

where we have used (2.14) in the last step. \square

4. Interrelations between Lipschitz and Besov spaces

The goal of this section is to show that Lipschitz spaces are closely related to Besov spaces. Specifically, we are interested in the relationships between the spaces $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ and $B_{p_0,r}^{\alpha_0,-c}(\mathbb{R}^d)$ with constant differential dimension, that is, $\alpha - d/p = \alpha_0 - d/p_0$. Firstly, we shall study the case $p = p_0$ (or equivalently, $\alpha = \alpha_0$), i.e., embeddings between $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ and $B_{p,r}^{\alpha,-c}(\mathbb{R}^d)$ (see Theorem 4.1). Secondly, we shall deal with Jawerth-Franke embeddings, that is, embeddings between $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ and $B_{p_0,r}^{\alpha_0,-c}(\mathbb{R}^d)$ with $\alpha - d/p = \alpha_0 - d/p_0$, $p \neq p_0$ (see Theorem 4.4).

4.1. Embeddings between Lipschitz and Besov spaces with fixed integrability

It is well known that

$$B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d) \hookrightarrow H_p^\alpha(\mathbb{R}^d) \hookrightarrow B_{p,\max\{2,p\}}^\alpha(\mathbb{R}^d), \quad -\infty < \alpha < \infty, \quad 1 < p < \infty; \quad (4.1)$$

see, e.g., [56, Section 2.3.2]. Note that these embeddings can be rewritten in terms of Lipschitz spaces as

$$B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,\infty}^{(\alpha,0)}(\mathbb{R}^d) \hookrightarrow B_{p,\max\{2,p\}}^\alpha(\mathbb{R}^d), \quad \alpha > 0 \quad (4.2)$$

(see (2.5)).

Our next result establishes the counterparts of (4.2) for the Lipschitz spaces $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$, $b > 1/q$.

Theorem 4.1. *Let $\alpha > 0$, $1 < p < \infty$, $0 < q \leq \infty$, and $b > 1/q$. Then we have*

$$B_{p,q}^{\alpha,-b+1/\min\{2,p,q\}}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p,q}^{\alpha,-b+1/\max\{2,p,q\}}(\mathbb{R}^d) \quad (4.3)$$

and

$$B_{p,\min\{2,p,q\}}^{\alpha,-b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p,\max\{2,p,q\}}^{\alpha,-b+1/q}(\mathbb{R}^d). \quad (4.4)$$

In particular,

$$\text{Lip}_{2,2}^{(\alpha,-b)}(\mathbb{R}^d) = B_{2,2}^{\alpha,-b+1/2}(\mathbb{R}^d), \quad b > 1/2. \quad (4.5)$$

The corresponding embeddings for periodic spaces also hold true.

Remark 4.2. (i) In fact one can show that (4.3), (4.4), and (4.5) are optimal in the following sense: Under conditions of Theorem 4.1, if $\frac{2d}{d+1} < p < \infty$, $0 < r \leq \infty$, and $-\infty < \xi < \infty$, then

$$\begin{aligned} B_{p,q}^{\alpha,-b+\xi}(\mathbb{R}^d) &\hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \iff \xi \geq 1/\min\{2,p,q\}, \\ \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) &\hookrightarrow B_{p,q}^{\alpha,-b+\xi}(\mathbb{R}^d) \iff \xi \leq 1/\max\{2,p,q\}, \\ B_{p,r}^{\alpha,-b+1/q}(\mathbb{R}^d) &\hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \iff r \leq \min\{2,p,q\}, \\ \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) &\hookrightarrow B_{p,r}^{\alpha,-b+1/q}(\mathbb{R}^d) \iff r \geq \max\{2,p,q\}, \\ \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) &= B_{p,q}^{\alpha,\xi}(\mathbb{R}^d) \iff p = q = 2 \quad \text{and} \quad \xi = -b + 1/2. \end{aligned}$$

Proofs of the “only if” part in these assertions are rather technical (based on the characterizations of Lipschitz spaces for two important classes of functions: trigonometric series with lacunary Fourier coefficients and functions with monotone-type Fourier transforms) and will be given elsewhere.

(ii) The left-hand side embeddings of (4.3) and (4.4) coincide if $q = \min\{2,p,q\}$. However, in general, these embeddings are independent of each other. For instance, assume that either $q > 2$ or $q > p$. Then we have

$$B_{p,q}^{\alpha,-b+1/\min\{2,p\}}(\mathbb{R}^d) \not\hookrightarrow B_{p,\min\{2,p\}}^{\alpha,-b+1/q}(\mathbb{R}^d)$$

and

$$B_{p,\min\{2,p\}}^{\alpha,-b+1/q}(\mathbb{R}^d) \not\hookrightarrow B_{p,q}^{\alpha,-b+1/\min\{2,p\}}(\mathbb{R}^d);$$

see [46, Theorem 1] and [22, Proposition 6.1]. The corresponding comment also applies to the right-hand side embeddings of (4.3) and (4.4).

The special case $\alpha = 1$ in (4.3) was already shown in [10, Theorem 5.2]. On the other hand, (4.4) with $\alpha = 1$ improves the following chain of embeddings given in [37, (7.59) and (7.61)]

$$B_{p,\min\{1,q\}}^{1,-b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(1,-b)}(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^{1,-b+1/q}(\mathbb{R}^d)$$

because

$$B_{p,\min\{1,q\}}^{1,-b+1/q}(\mathbb{R}^d) \hookrightarrow B_{p,\min\{2,p,q\}}^{1,-b+1/q}(\mathbb{R}^d), \quad B_{p,\min\{1,q\}}^{1,-b+1/q}(\mathbb{R}^d) \neq B_{p,\min\{2,p,q\}}^{1,-b+1/q}(\mathbb{R}^d) \quad \text{if } q > 1,$$

and

$$B_{p,\max\{2,p,q\}}^{1,-b+1/q}(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^{1,-b+1/q}(\mathbb{R}^d), \quad B_{p,\max\{2,p,q\}}^{1,-b+1/q}(\mathbb{R}^d) \neq B_{p,\infty}^{1,-b+1/q}(\mathbb{R}^d) \quad \text{if } q < \infty.$$

(iii) If $q = \infty$ in (4.4) then

$$B_{p,\min\{2,p\}}^{\alpha,-b}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,\infty}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^{\alpha,-b}(\mathbb{R}^d), \quad b > 0. \quad (4.6)$$

Comparing the right-hand side embeddings of (4.2) and (4.6), one can observe the significant role played by logarithmic smoothness in the fine indices of Besov spaces. As noticed in part (i), the embedding $\text{Lip}_{p,\infty}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^{\alpha,-b}(\mathbb{R}^d)$ is optimal in the sense that one cannot replace the target space $B_{p,\infty}^{\alpha,-b}(\mathbb{R}^d)$ by $B_{p,r}^{\alpha,-b}(\mathbb{R}^d)$, $r < \infty$.

(iv) In the limiting cases $p = 1$ and $p = \infty$ analogues of (4.3) and (4.4) read as follows:

$$B_{p,q}^{\alpha, -b+1/\min\{1,q\}}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha, -b)}(\mathbb{R}^d) \hookrightarrow B_{p,q}^{\alpha, -b}(\mathbb{R}^d)$$

and

$$B_{p,\min\{1,q\}}^{\alpha, -b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha, -b)}(\mathbb{R}^d) \hookrightarrow B_{p,\infty}^{\alpha, -b+1/q}(\mathbb{R}^d).$$

See [37, Corollary 7.20] for the case $\alpha = 1$.

Proof of Theorem 4.1. Applying the interpolation property of the limiting interpolation method with $\theta = 1$ (see (2.11)) to the embeddings (4.1), we obtain

$$\begin{aligned} (B_{p,\min\{2,p\}}^0(\mathbb{R}^d), B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} &\hookrightarrow (L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d))_{(1,-b),q} \\ &\hookrightarrow (B_{p,\max\{2,p\}}^0(\mathbb{R}^d), B_{p,\max\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} \end{aligned}$$

and then, by (2.13),

$$\begin{aligned} (B_{p,\min\{2,p\}}^0(\mathbb{R}^d), B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} &\hookrightarrow \text{Lip}_{p,q}^{(\alpha, -b)}(\mathbb{R}^d) \\ &\hookrightarrow (B_{p,\max\{2,p\}}^0(\mathbb{R}^d), B_{p,\max\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q}. \end{aligned} \quad (4.7)$$

Next we show that

$$B_{p,q}^{\alpha, -b+1/\min\{2,p,q\}}(\mathbb{R}^d) + B_{p,\min\{2,p,q\}}^{\alpha, -b+1/q}(\mathbb{R}^d) \hookrightarrow (B_{p,\min\{2,p\}}^0(\mathbb{R}^d), B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q}. \quad (4.8)$$

Let α_0 be such that $\alpha_0 > \alpha$ and set $\theta = \alpha/\alpha_0$. By (2.9),

$$(B_{p,r}^0(\mathbb{R}^d), B_{p,r}^{\alpha_0}(\mathbb{R}^d))_{\theta,r} = B_{p,r}^\alpha(\mathbb{R}^d), \quad 0 < r \leq \infty. \quad (4.9)$$

Putting $r = \min\{2,p\}$, it follows from (2.17) and (2.7) that

$$\begin{aligned} (B_{p,\min\{2,p\}}^0(\mathbb{R}^d), B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} \\ \hookrightarrow (B_{p,\min\{2,p\}}^0(\mathbb{R}^d), B_{p,\min\{2,p\}}^{\alpha_0}(\mathbb{R}^d))_{\theta,q; -b+1/\min\{2,p,q\}} \\ = B_{p,q}^{\alpha, -b+1/\min\{2,p,q\}}(\mathbb{R}^d). \end{aligned}$$

On the other hand, by (3.2) and (3.4),

$$\begin{aligned} (\ell_{\min\{2,p\}}(L_p(\mathbb{R}^d)), \ell_{\min\{2,p\}}(2^{j\alpha}L_p(\mathbb{R}^d)))_{(1,-b),q} \\ \hookrightarrow \ell_{\min\{2,p,q\}}((L_p(\mathbb{R}^d), 2^{j\alpha}L_p(\mathbb{R}^d)))_{(1,-b),q} \\ = \ell_{\min\{2,p,q\}}(2^{j\alpha}(1+j)^{-b+1/q}L_p(\mathbb{R}^d)). \end{aligned}$$

Now, the embedding

$$(B_{p,\min\{2,p\}}^0(\mathbb{R}^d), B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} \hookrightarrow B_{p,\min\{2,p,q\}}^{\alpha, -b+1/q}(\mathbb{R}^d)$$

is an immediate consequence of the retract theorem for interpolation methods (see [55, Sects. 1.2.4, 2.4.1]) since $B_{p,\min\{2,p\}}^0(\mathbb{R}^d)$, $B_{p,\min\{2,p\}}^\alpha(\mathbb{R}^d)$ and $B_{p,\min\{2,p,q\}}^{\alpha, -b+1/q}(\mathbb{R}^d)$ are retracts of the vector-valued sequence spaces $\ell_{\min\{2,p\}}(L_p(\mathbb{R}^d))$, $\ell_{\min\{2,p\}}(2^{j\alpha}L_p(\mathbb{R}^d))$ and $\ell_{\min\{2,p,q\}}(2^{j\alpha}(1+j)^{-b+1/q}L_p(\mathbb{R}^d))$, respectively. This completes the proof of (4.8).

We claim that

$$(B_{p,\max\{2,p\}}^0(\mathbb{R}^d), B_{p,\max\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} \hookrightarrow B_{p,q}^{\alpha,-b+1/\max\{2,p,q\}}(\mathbb{R}^d) \cap B_{p,\max\{2,p,q\}}^{\alpha,-b+1/q}(\mathbb{R}^d). \quad (4.10)$$

Indeed, applying (4.9) with $r = \max\{2, p\}$, (2.17) and (2.7), we get

$$\begin{aligned} & (B_{p,\max\{2,p\}}^0(\mathbb{R}^d), B_{p,\max\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} \\ & \hookrightarrow (B_{p,\max\{2,p\}}^0(\mathbb{R}^d), B_{p,\max\{2,p\}}^{\alpha_0}(\mathbb{R}^d))_{\theta,q;-b+1/\max\{2,p,q\}} \\ & = B_{p,q}^{\alpha,-b+1/\max\{2,p,q\}}(\mathbb{R}^d). \end{aligned}$$

Furthermore, in light of (3.2) and (3.4),

$$\begin{aligned} & (\ell_{\max\{2,p\}}(L_p(\mathbb{R}^d)), \ell_{\max\{2,p\}}(2^{j\alpha} L_p(\mathbb{R}^d)))_{(1,-b),q} \\ & \hookrightarrow \ell_{\max\{2,p,q\}}((L_p(\mathbb{R}^d), 2^{j\alpha} L_p(\mathbb{R}^d))_{(1,-b),q}) \\ & = \ell_{\max\{2,p,q\}}(2^{j\alpha}(1+j)^{-b+1/q} L_p(\mathbb{R}^d)) \end{aligned}$$

and so, the retraction method for interpolation allows us to derive

$$(B_{p,\max\{2,p\}}^0(\mathbb{R}^d), B_{p,\max\{2,p\}}^\alpha(\mathbb{R}^d))_{(1,-b),q} \hookrightarrow B_{p,\max\{2,p,q\}}^{\alpha,-b+1/q}(\mathbb{R}^d).$$

Combining (4.7), (4.8) and (4.10) we arrive at the desired embeddings (4.3) and (4.4). \square

4.2. Jawerth-Franke type embeddings for Lipschitz spaces

We start by recalling the classical embeddings of Jawerth-Franke between Besov spaces and Sobolev spaces. See [41,33] (cf. also [47] and [61]).

Theorem 4.3 (Embeddings of Jawerth-Franke for Sobolev spaces). *Let $1 \leq p_0 < p < p_1 \leq \infty$ and $-\infty < \alpha < \infty$. Then*

$$B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d) \hookrightarrow H_p^\alpha(\mathbb{R}^d) \hookrightarrow B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d). \quad (4.11)$$

The periodic counterparts also hold true.

In view of (2.5), the embeddings (4.11) can be rewritten as follows

$$B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,\infty}^{(\alpha,0)}(\mathbb{R}^d) \hookrightarrow B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d), \quad \alpha > 0. \quad (4.12)$$

The goal of this section is to extend (4.12) to the full scale of the logarithmic Lipschitz spaces $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$. In the first attempt, one can combine (4.3), (4.4) and (5.4) below to get

$$B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/\min\{2,p,q\}}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/\max\{2,p,q\}}(\mathbb{R}^d)$$

and

$$B_{p_0,\min\{2,p,q\}}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,\max\{2,p,q\}}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d)$$

for $\alpha > 0, 1 \leq p_0 < p < p_1 \leq \infty, 0 < q \leq \infty$, and $b > 1/q$. However, our next result shows that these embeddings can be strengthened.

Theorem 4.4 (Embeddings of Jawerth-Franke type for Lipschitz spaces). Let $\alpha > 0$, $1 \leq p_0 < p < p_1 \leq \infty$, $0 < q \leq \infty$, and $b > 1/q$. Then we have

$$B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/\min\{p,q\}}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/\max\{p,q\}}(\mathbb{R}^d) \quad (4.13)$$

and

$$B_{p_0,\min\{p,q\}}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,\max\{p,q\}}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d). \quad (4.14)$$

The corresponding embeddings for periodic spaces also hold true.

Remark 4.5. (i) In fact one can show that (4.13) and (4.14) are optimal in the following sense: Under conditions of Theorem 4.4, if $\frac{2d}{d+1} < p_0 < \infty$, $0 < r \leq \infty$, and $-\infty < \xi < \infty$, then

$$\begin{aligned} B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+\xi}(\mathbb{R}^d) &\hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \iff \xi \geq 1/\min\{p,q\}, \\ \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) &\hookrightarrow B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+\xi}(\mathbb{R}^d) \iff \xi \leq 1/\max\{p,q\}, \\ B_{p_0,r}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) &\hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \iff r \leq \min\{p,q\}, \\ \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) &\hookrightarrow B_{p_1,r}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d) \iff r \geq \max\{p,q\}. \end{aligned}$$

Proofs of these results are omitted and will be given elsewhere.

(ii) Note that the left-hand side embeddings (respectively, right-hand side embeddings) of (4.13) and (4.14) coincide if $q \leq p$ (respectively, $q \geq p$). However, in general, these embeddings are not comparable. For instance, we give the left-hand side embeddings of (4.13) and (4.14) with $q > p$,

$$B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/p}(\mathbb{R}^d) + B_{p_0,p}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d),$$

and we observe that

$$B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/p}(\mathbb{R}^d) \not\subseteq B_{p_0,p}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d)$$

and

$$B_{p_0,p}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) \not\subseteq B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/p}(\mathbb{R}^d),$$

see [46, Theorem 1] and [22, Remark 6.4]. The same comment applies to the right-hand side embeddings of (4.13) and (4.14) with $q < p$. More precisely, we have

$$\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/p}(\mathbb{R}^d) \cap B_{p_1,p}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d),$$

and

$$\begin{aligned} B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/p}(\mathbb{R}^d) &\not\subseteq B_{p_1,p}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d), \\ B_{p_1,p}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d) &\not\subseteq B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/p}(\mathbb{R}^d). \end{aligned}$$

Proof of Theorem 4.4. Since

$$B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \hookrightarrow B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d)$$

and

$$B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d) \hookrightarrow H_p^\alpha(\mathbb{R}^d) \hookrightarrow B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d)$$

(see (4.11)), we can apply the limiting interpolation method with $\theta = 1$ (see (2.11)) to obtain

$$\begin{aligned} (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d))_{(1,-b),q} &\hookrightarrow (L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d))_{(1,-b),q} \\ &\hookrightarrow (B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d))_{(1,-b),q}. \end{aligned} \quad (4.15)$$

According to (2.13), we have

$$(L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d))_{(1,-b),q} = \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d). \quad (4.16)$$

Next, we show that

$$\begin{aligned} B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/\min\{p,q\}}(\mathbb{R}^d) + B_{p_0,\min\{p,q\}}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) \\ \hookrightarrow (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d))_{(1,-b),q}. \end{aligned} \quad (4.17)$$

Let $\alpha_0 > \alpha + d(1/p_0 - 1/p)$ and $\theta_0 = \alpha/(\alpha_0 - d/p_0 + d/p)$. By (2.9),

$$B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d) = (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), B_{p_0,p}^{\alpha_0}(\mathbb{R}^d))_{\theta_0,p}.$$

Therefore, by the left-hand side embedding in (2.17) and (2.7), we get

$$\begin{aligned} (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d))_{(1,-b),q} \\ = (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), B_{p_0,p}^{\alpha_0}(\mathbb{R}^d))_{\theta_0,p})_{(1,-b),q} \\ \hookleftarrow (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), B_{p_0,p}^{\alpha_0}(\mathbb{R}^d))_{\theta_0,q;-b+1/\min\{p,q\}} \\ = B_{p_0,q}^{\alpha+d(1/p_0-1/p),-b+1/\min\{p,q\}}(\mathbb{R}^d). \end{aligned} \quad (4.18)$$

We claim that

$$B_{p_0,\min\{p,q\}}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d) \hookrightarrow (B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d))_{(1,-b),q}. \quad (4.19)$$

To get this we will apply the retraction property of Besov spaces (see [55, Sections 1.2.4 and 2.4.1]). It follows from the left-hand side embedding in (3.2), (3.3) and (3.4) that

$$\begin{aligned} (\ell_p(2^{j(d/p_0-d/p)}L_{p_0}(\mathbb{R}^d)), \ell_p(2^{j(\alpha+d/p_0-d/p)}L_{p_0}(\mathbb{R}^d)))_{(1,-b),q} \\ \hookleftarrow \ell_{\min\{p,q\}}((2^{j(d/p_0-d/p)}L_{p_0}(\mathbb{R}^d), 2^{j(\alpha+d/p_0-d/p)}L_{p_0}(\mathbb{R}^d)))_{(1,-b),q} \\ = \ell_{\min\{p,q\}}(2^{j(d/p_0-d/p)}L_{p_0}(\mathbb{R}^d), 2^{j\alpha}L_{p_0}(\mathbb{R}^d))_{(1,-b),q} \\ = \ell_{\min\{p,q\}}(2^{j(\alpha+d/p_0-d/p)}(1+j)^{-b+1/q}L_{p_0}(\mathbb{R}^d)). \end{aligned}$$

Now the embedding (4.19) follows from the fact that

$$B_{p_0,\min\{p,q\}}^{\alpha+d(1/p_0-1/p),-b+1/q}(\mathbb{R}^d), \quad B_{p_0,p}^{d(1/p_0-1/p)}(\mathbb{R}^d), \quad \text{and} \quad B_{p_0,p}^{\alpha+d(1/p_0-1/p)}(\mathbb{R}^d)$$

are retracts of $\ell_{\min\{p,q\}}(2^{j(\alpha+d/p_0-d/p)}(1+j)^{-b+1/q}L_{p_0}(\mathbb{R}^d))$, $\ell_p(2^{j(d/p_0-d/p)}L_{p_0}(\mathbb{R}^d))$, and $\ell_p(2^{j(\alpha+d/p_0-d/p)}L_{p_0}(\mathbb{R}^d))$, respectively.

As a byproduct of (4.18) and (4.19), we arrive at (4.17).

Next we will prove that the following embedding holds

$$\begin{aligned} (B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d))_{(1,-b),q} \\ \hookrightarrow B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/\max\{p,q\}}(\mathbb{R}^d) \cap B_{p_1,\max\{p,q\}}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d). \end{aligned} \quad (4.20)$$

Let $\alpha_1 > \alpha + d(1/p_1 - 1/p)$ and $\theta_1 = \alpha/(\alpha_1 - d/p_1 + d/p)$. According to (2.9),

$$B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d) = (B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), B_{p_1,p}^{\alpha_1}(\mathbb{R}^d))_{\theta_1,p}.$$

Therefore, invoking the right-hand side embedding in (2.17) together with (2.7),

$$\begin{aligned} (B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d))_{(1,-b),q} \\ = (B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), (B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), B_{p_1,p}^{\alpha_1}(\mathbb{R}^d))_{\theta_1,p})_{(1,-b),q} \\ \hookrightarrow (B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), B_{p_1,p}^{\alpha_1}(\mathbb{R}^d))_{\theta_1,q;-b+1/\max\{p,q\}} \\ = B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/\max\{p,q\}}(\mathbb{R}^d). \end{aligned}$$

On the other hand, using the right-hand side embedding in (3.2), (3.3) and (3.4),

$$\begin{aligned} (\ell_p(2^{j(d/p_1-d/p)}L_{p_1}(\mathbb{R}^d)), \ell_p(2^{j(\alpha+d/p_1-d/p)}L_{p_1}(\mathbb{R}^d)))_{(1,-b),q} \\ \hookrightarrow \ell_{\max\{p,q\}}((2^{j(d/p_1-d/p)}L_{p_1}(\mathbb{R}^d), 2^{j(\alpha+d/p_1-d/p)}L_{p_1}(\mathbb{R}^d)))_{(1,-b),q} \\ = \ell_{\max\{p,q\}}(2^{j(d/p_1-d/p)}(L_{p_1}(\mathbb{R}^d), 2^{j\alpha}L_{p_1}(\mathbb{R}^d)))_{(1,-b),q} \\ = \ell_{\max\{p,q\}}(2^{j(\alpha+d/p_1-d/p)}(1+j)^{-b+1/q}L_{p_1}(\mathbb{R}^d)). \end{aligned}$$

Consequently, the retraction method allows us to get

$$(B_{p_1,p}^{d(1/p_1-1/p)}(\mathbb{R}^d), B_{p_1,p}^{\alpha+d(1/p_1-1/p)}(\mathbb{R}^d))_{(1,-b),q} \hookrightarrow B_{p_1,\max\{p,q\}}^{\alpha+d(1/p_1-1/p),-b+1/q}(\mathbb{R}^d).$$

This completes the proof of (4.20).

Combining (4.15), (4.16), (4.17) and (4.20), we derive (4.13) and (4.14).

The same methodology can be applied to deal with periodic function spaces. Further details are left to the reader. \square

5. Embeddings between Lipschitz spaces

5.1. Embeddings with constant integrability

In this section we investigate embeddings between the Lipschitz spaces $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ with fixed integrability p . Namely, we obtain the following

Theorem 5.1. *Let $1 < p < \infty$, $\alpha_i > 0$, $0 < q_i \leq \infty$, and $b_i > 1/q_i$, $i = 0, 1$. Assume that one of the following conditions is satisfied*

- (i) $\alpha_0 > \alpha_1$,
- (ii) $\alpha_0 = \alpha_1$, $b_1 - \frac{1}{q_1} > b_0 - \frac{1}{q_0}$,
- (iii) $\alpha_0 = \alpha_1$, $b_1 - \frac{1}{q_1} = b_0 - \frac{1}{q_0}$, $q_0 \leq q_1$.

Then

$$\text{Lip}_{p,q_0}^{(\alpha_0,-b_0)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q_1}^{(\alpha_1,-b_1)}(\mathbb{R}^d). \quad (5.1)$$

The corresponding embeddings for periodic spaces also hold true.

Remark 5.2. It can be shown that, in fact, if (5.1) holds, then necessarily one of the conditions (i), (ii), (iii) follows.

Proof of Theorem 5.1. (i): We will make use of the following well-known embeddings for Besov spaces

$$B_{p,q_0}^{\alpha_0,\xi_0}(\mathbb{R}^d) \hookrightarrow B_{p,q_1}^{\alpha_1,\xi_1}(\mathbb{R}^d) \quad (5.2)$$

for $-\infty < \alpha_1 < \alpha_0 < \infty$, $-\infty < \xi_0, \xi_1 < \infty$, and $0 < q_0, q_1 \leq \infty$. See [56, Section 2.3.2, Proposition 2], [50, Proposition 1.9(ii)] and [7, Proposition 5.3].

In light of (4.3) and (5.2), we have

$$\begin{aligned} \text{Lip}_{p,q_0}^{(\alpha_0,-b_0)}(\mathbb{R}^d) &\hookrightarrow B_{p,q_0}^{\alpha_0,-b_0+1/\max\{2,p,q_0\}}(\mathbb{R}^d) \hookrightarrow B_{p,q_1}^{\alpha_1,-b_1+1/\min\{2,p,q_1\}}(\mathbb{R}^d) \\ &\hookrightarrow \text{Lip}_{p,q_1}^{(\alpha_1,-b_1)}(\mathbb{R}^d). \end{aligned}$$

(ii), (iii): Let $\alpha = \alpha_0 = \alpha_1$. Firstly, we assume $q_1 < q_0$ and $b_1 - \frac{1}{q_1} > b_0 - \frac{1}{q_0}$. Applying Hölder's inequality,

$$\begin{aligned} &\left(\int_0^1 (t^{-\alpha}(1-\log t)^{-b_1} \omega_\alpha(f,t)_p)^{q_1} \frac{dt}{t} \right)^{1/q_1} \\ &\leq \left(\int_0^1 (t^{-\alpha}(1-\log t)^{-b_0} \omega_\alpha(f,t)_p)^{q_0} \frac{dt}{t} \right)^{1/q_0} \left(\int_0^1 (1-\log t)^{(b_0-b_1)(\frac{1}{q_1}-\frac{1}{q_0})-1} \frac{dt}{t} \right)^{\frac{1}{q_1}-\frac{1}{q_0}} \\ &\lesssim \left(\int_0^1 (t^{-\alpha}(1-\log t)^{-b_0} \omega_\alpha(f,t)_p)^{q_0} \frac{dt}{t} \right)^{1/q_0}. \end{aligned}$$

Secondly, suppose that $q_0 \leq q_1$ and $b_1 - \frac{1}{q_1} \geq b_0 - \frac{1}{q_0}$. We have

$$\left(\int_0^1 (t^{-\alpha}(1-\log t)^{-b_1} \omega_\alpha(f,t)_p)^{q_1} \frac{dt}{t} \right)^{1/q_1} \asymp I + II,$$

where

$$I = \left(\int_0^{1/2} (t^{-\alpha}(1-\log t)^{-b_1} \omega_\alpha(f,t)_p)^{q_1} \frac{dt}{t} \right)^{1/q_1}, \quad II = \left(\int_{1/2}^1 (t^{-\alpha}(1-\log t)^{-b_1} \omega_\alpha(f,t)_p)^{q_1} \frac{dt}{t} \right)^{1/q_1}.$$

Since $\omega_\alpha(f,t)_p \lesssim \|f\|_{L_p(\mathbb{R}^d)}$, we get $II \lesssim \|f\|_{L_p(\mathbb{R}^d)}$. On the other hand, to estimate I , we use monotonicity properties (noting that $\omega_\alpha(f,t)_p/t^\alpha$ is equivalent to a decreasing function, see, e.g., [44]),

$$\begin{aligned}
\left(\int_0^{1/2} (t^{-\alpha} (1 - \log t)^{-b_1} \omega_\alpha(f, t)_p)^{q_1} \frac{dt}{t} \right)^{1/q_1} &\lesssim \left(\sum_{n=0}^{\infty} (2^{2^{n+1}\alpha} 2^{n(-b_1+1/q_1)} \omega_\alpha(f, 2^{-2^{n+1}})_p)^{q_1} \right)^{1/q_1} \\
&\leq \left(\sum_{n=0}^{\infty} (2^{2^{n+1}\alpha} 2^{n(-b_0+1/q_0)} \omega_\alpha(f, 2^{-2^{n+1}})_p)^{q_0} \right)^{1/q_0} \\
&\lesssim \left(\int_0^1 (t^{-\alpha} (1 - \log t)^{-b_0} \omega_\alpha(f, t)_p)^{q_0} \frac{dt}{t} \right)^{1/q_0}.
\end{aligned}$$

Therefore, $I + II \lesssim \|f\|_{\text{Lip}_{p,q_0}^{(\alpha,-b_0)}(\mathbb{R}^d)}$. \square

5.2. Embeddings with constant differential dimension

Let $1 < p_0 < p_1 < \infty$, $0 < q \leq \infty$ and $-\infty < \alpha_1 < \alpha_0 < \infty$. Further, we assume that

$$\alpha_0 - \frac{d}{p_0} = \alpha_1 - \frac{d}{p_1}. \quad (5.3)$$

Then the classical Sobolev embeddings assert that

$$B_{p_0,q}^{\alpha_0,b}(\mathbb{R}^d) \hookrightarrow B_{p_1,q}^{\alpha_1,b}(\mathbb{R}^d), \quad -\infty < b < \infty, \quad (5.4)$$

and

$$H_{p_0}^{\alpha_0}(\mathbb{R}^d) \hookrightarrow H_{p_1}^{\alpha_1}(\mathbb{R}^d). \quad (5.5)$$

We remark that (5.4) also holds true for $1 \leq p_0 < p_1 \leq \infty$ satisfying (5.3), and (5.5) can be extended to the broader scale of Triebel-Lizorkin spaces. Furthermore, both embeddings can be given in the more general setting of quasi-Banach spaces. For further details, we refer to [56, Section 2.7.1], [50, Proposition 1.9], [7, Proposition 5.3] and the references within. We note that since $H_p^\alpha(\mathbb{R}^d) = \text{Lip}_{p,\infty}^{(\alpha,0)}(\mathbb{R}^d)$, $\alpha > 0$, (5.5) can be rewritten in terms of Lipschitz spaces as

$$\text{Lip}_{p_0,\infty}^{(\alpha_0,0)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p_1,\infty}^{(\alpha_1,0)}(\mathbb{R}^d), \quad 0 < \alpha_1 < \alpha_0 < \infty.$$

Next we extend this embedding to the full range of parameters. More precisely, we obtain the following Sobolev-type embedding.

Theorem 5.3. *Let $1 < p_0 < p_1 < \infty$, $0 < \alpha_1 < \alpha_0 < \infty$ with $\alpha_0 - d/p_0 = \alpha_1 - d/p_1$. Let $0 < q \leq \infty$ and $b > 1/q$. Then*

$$\text{Lip}_{p_0,q}^{(\alpha_0,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p_1,q}^{(\alpha_1,-b)}(\mathbb{R}^d).$$

The corresponding result for periodic spaces also holds true.

Remark 5.4. The corresponding results in the endpoint cases $p_0 = 1$ and/or $p_1 = \infty$ are more delicate and will be stated in Theorems 5.5 and 5.7 below.

Proof of Theorem 5.3. We choose θ such that

$$1 - \frac{\alpha_1}{\alpha_0} < \theta < \min \left\{ 1, \frac{d}{\alpha_0 p_0} \right\}$$

and let $\lambda = \theta\alpha_0 - d\left(\frac{1}{p_0} - \frac{1}{p_1}\right)$. According to (5.4) and (5.5), we have

$$B_{p_0,q}^{\theta\alpha_0}(\mathbb{R}^d) \hookrightarrow B_{p_1,q}^{\lambda}(\mathbb{R}^d) \quad \text{and} \quad H_{p_0}^{\alpha_0}(\mathbb{R}^d) \hookrightarrow H_{p_1}^{\alpha_1}(\mathbb{R}^d).$$

Then, by the interpolation property,

$$(B_{p_0,q}^{\theta\alpha_0}(\mathbb{R}^d), H_{p_0}^{\alpha_0}(\mathbb{R}^d))_{(1,-b),q} \hookrightarrow (B_{p_1,q}^{\lambda}(\mathbb{R}^d), H_{p_1}^{\alpha_1}(\mathbb{R}^d))_{(1,-b),q}. \quad (5.6)$$

Next we identify these interpolation spaces. Since $B_{p_0,q}^{\theta\alpha_0}(\mathbb{R}^d) = (L_{p_0}, H_{p_0}^{\alpha_0}(\mathbb{R}^d))_{\theta,q}$ (see (2.10)), it follows from (2.19) and (2.13) that

$$(B_{p_0,q}^{\theta\alpha_0}(\mathbb{R}^d), H_{p_0}^{\alpha_0}(\mathbb{R}^d))_{(1,-b),q} = (L_{p_0}(\mathbb{R}^d), H_{p_0}^{\alpha_0}(\mathbb{R}^d))_{(1,-b),q} = \text{Lip}_{p_0,q}^{(\alpha_0,-b)}(\mathbb{R}^d). \quad (5.7)$$

On the other hand, setting $\eta = 1 - (1 - \theta)\alpha_0/\alpha_1 \in (0, 1)$ then

$$B_{p_1,q}^{\lambda}(\mathbb{R}^d) = (L_{p_1}(\mathbb{R}^d), H_{p_1}^{\alpha_1}(\mathbb{R}^d))_{\eta,q}$$

and applying again (2.19) and (2.13), we get

$$(B_{p_1,q}^{\lambda}(\mathbb{R}^d), H_{p_1}^{\alpha_1}(\mathbb{R}^d))_{(1,-b),q} = \text{Lip}_{p_1,q}^{(\alpha_1,-b)}(\mathbb{R}^d). \quad (5.8)$$

Plugging (5.7) and (5.8) into (5.6), we arrive at $\text{Lip}_{p_0,q}^{(\alpha_0,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p_1,q}^{(\alpha_1,-b)}(\mathbb{R}^d)$. \square

Next we turn our attention to the counterpart of Theorem 5.3 in the limiting case $p_0 = 1$.

Theorem 5.5. *Let $1 < p < \infty$, $0 < q \leq \infty$, and $b > 1/q$ ($b \geq 0$ if $q = \infty$).*

(i) *Let $d \in \mathbb{N}$, $0 < \alpha_1 < \alpha_0 < \infty$ with $\alpha_0 - d = \alpha_1 - d/p$. Then*

$$\text{Lip}_{1,q}^{(\alpha_0,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha_1,-b-1/p)}(\mathbb{R}^d). \quad (5.9)$$

(ii) *Let $d \geq 2$, $k \in \mathbb{N}$, $k \geq 2$, $0 < \alpha_1 \leq k - 1$ and $k - d = \alpha_1 - d/p$. Then*

$$\text{Lip}_{1,q}^{(k,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha_1,-b)}(\mathbb{R}^d). \quad (5.10)$$

The corresponding results for periodic spaces also hold true.

Remark 5.6. (i) One can observe that the target spaces in (5.9) involve the additional logarithmic smoothness of order $1/p$. Such a phenomenon does not arise in the non-limiting case given in Theorem 5.3. Furthermore, it can be seen that, in general, the embedding (5.9) is optimal. However, (5.10) shows that under additional restrictions (5.9) can be sharpened. Note that the assumptions given in the second part of the theorem imply $\alpha_1 > k - d$.

(ii) The case $\alpha_0 = 1$, $q = \infty$ and $b = 0$ in (5.9) is of special interest. Namely, the following embedding holds, cf. (2.4),

$$\text{BV}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,\infty}^{(1-d+d/p,-1/p)}(\mathbb{R}), \quad 1 < p < \frac{d}{d-1} \quad (1 < p < \infty \quad \text{if} \quad d = 1). \quad (5.11)$$

In general, this result is optimal. This shows a striking difference between $\text{BV}(\mathbb{R})$ and Sobolev spaces $H_p^{1/p}(\mathbb{R})$, $p > 1$. To be more precise, by (5.5),

$$H_{p_0}^{1/p_0}(\mathbb{R}) = \text{Lip}_{p_0, \infty}^{(1/p_0, 0)}(\mathbb{R}) \hookrightarrow H_{p_1}^{1/p_1}(\mathbb{R}) = \text{Lip}_{p_1, \infty}^{(1/p_1, 0)}(\mathbb{R}), \quad 1 < p_0 < p_1 < \infty.$$

However, the latter fails to be true if $p_0 = 1$ (that is, working with $\text{BV}(\mathbb{R}) = \text{Lip}_{1, \infty}^{(1, 0)}(\mathbb{R})$) and the best possible embedding result involves additional logarithmic smoothness (see (5.11)). Note that $\text{Lip}_{p, \infty}^{(1/p, 0)}(\mathbb{R}) \subsetneq \text{Lip}_{p, \infty}^{(1/p, -1/p)}(\mathbb{R})$ (see Theorem 5.1).

Proof of Theorem 5.5. (i): Let $q = \infty$ and $b \geq 0$. Since $\omega_\alpha(f, t)_1 \lesssim \omega_{\alpha_0}(f, t)_1$, $\alpha > \alpha_0$, we have $\text{Lip}_{1, \infty}^{(\alpha_0, -b)}(\mathbb{R}^d) \hookrightarrow B_{1, \infty}^{\alpha_0, -b}(\mathbb{R}^d)$. On the other hand, by (4.13), $B_{1, \infty}^{\alpha_0, -b}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p, \infty}^{(\alpha_1, -b-1/p)}(\mathbb{R}^d)$. Therefore,

$$\text{Lip}_{1, \infty}^{(\alpha_0, -b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p, \infty}^{(\alpha_1, -b-1/p)}(\mathbb{R}^d), \quad b \geq 0. \quad (5.12)$$

Suppose now $q < \infty$ and $b - 1/q > 0$. Let $0 < b_1 < b - 1/q < b_0$. Then there exists $\theta \in (0, 1)$ such that $b - 1/q = (1 - \theta)b_0 + \theta b_1$. Further, it follows from (5.12) that

$$\text{Lip}_{1, \infty}^{(\alpha_0, -b_i)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p, \infty}^{(\alpha_1, -b_i-1/p)}(\mathbb{R}^d), \quad i = 0, 1,$$

and so, by the interpolation property,

$$(\text{Lip}_{1, \infty}^{(\alpha_0, -b_0)}(\mathbb{R}^d), \text{Lip}_{1, \infty}^{(\alpha_0, -b_1)}(\mathbb{R}^d))_{\theta, q} \hookrightarrow (\text{Lip}_{p, \infty}^{(\alpha_1, -b_0-1/p)}(\mathbb{R}^d), \text{Lip}_{p, \infty}^{(\alpha_1, -b_1-1/p)}(\mathbb{R}^d))_{\theta, q}. \quad (5.13)$$

According to Lemma 3.6,

$$(\text{Lip}_{1, \infty}^{(\alpha_0, -b_0)}(\mathbb{R}^d), \text{Lip}_{1, \infty}^{(\alpha_0, -b_1)}(\mathbb{R}^d))_{\theta, q} = \text{Lip}_{1, q}^{(\alpha_0, -b)}(\mathbb{R}^d)$$

and

$$(\text{Lip}_{p, \infty}^{(\alpha_1, -b_0-1/p)}(\mathbb{R}^d), \text{Lip}_{p, \infty}^{(\alpha_1, -b_1-1/p)}(\mathbb{R}^d))_{\theta, q} = \text{Lip}_{p, q}^{(\alpha_1, -b-1/p)}(\mathbb{R}^d).$$

Inserting these formulas into (5.13), we arrive at

$$\text{Lip}_{1, q}^{(\alpha_0, -b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p, q}^{(\alpha_1, -b-1/p)}(\mathbb{R}^d).$$

(ii): By the classical Sobolev theorem and (5.5),

$$W_1^k(\mathbb{R}^d) \hookrightarrow W_{\frac{d}{d-1}}^{k-1}(\mathbb{R}^d) = H_{\frac{d}{d-1}}^{k-1}(\mathbb{R}^d) \hookrightarrow H_p^{\alpha_1}(\mathbb{R}^d).$$

Further, we will make use of the well-known embedding

$$B_{1, p}^{k-\alpha_1}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d),$$

cf. [52] or [57, Remark 11.8] for further details on the history. Applying now limiting interpolation and (2.13), we get

$$(B_{1, p}^{k-\alpha_1}(\mathbb{R}^d), W_1^k(\mathbb{R}^d))_{(1, -b), q} \hookrightarrow (L_p(\mathbb{R}^d), H_p^{\alpha_1}(\mathbb{R}^d))_{(1, -b), q} = \text{Lip}_{p, q}^{(\alpha_1, -b)}(\mathbb{R}^d).$$

It remains to compute the domain space $(B_{1, p}^{k-\alpha_1}(\mathbb{R}^d), W_1^k(\mathbb{R}^d))_{(1, -b), q}$. It follows from (2.16), (2.19) and (2.15) that

$$\begin{aligned} (B_{1,p}^{k-\alpha_1}(\mathbb{R}^d), W_1^k(\mathbb{R}^d))_{(1,-b),q} &= ((L_1(\mathbb{R}^d), W_1^k(\mathbb{R}^d))_{\frac{k-\alpha_1}{k},p}, W_1^k(\mathbb{R}^d))_{(1,-b),q} \\ &= (L_1(\mathbb{R}^d), W_1^k(\mathbb{R}^d))_{(1,-b),q} = \text{Lip}_{1,q}^{(k,-b)}(\mathbb{R}^d). \end{aligned}$$

This completes the proof of (5.10). \square

5.3. Brézis-Wainger embeddings

The Brézis-Wainger embedding [6] asserts that

$$H_p^{1+d/p}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(1,-1+1/p)}(\mathbb{R}^d), \quad 1 < p < \infty. \quad (5.14)$$

Note that this embedding can be rewritten as

$$\text{Lip}_{p,\infty}^{(1+d/p,0)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(1,-1+1/p)}(\mathbb{R}^d), \quad (5.15)$$

which corresponds to a special case of the limiting version of Theorem 5.3 with $p_1 = \infty$.

The goal of this section is to provide the counterpart of Theorem 5.3 in the limiting case $p_1 = \infty$, or equivalently, to extend (5.14) and (5.15) to the full range of parameters.

Theorem 5.7. *Let $b > 1/q$ ($b \geq 0$ if $q = \infty$).*

(i) *Let $1 < p < \infty$ and $0 < \alpha < \infty$. Then*

$$\text{Lip}_{p,q}^{(\alpha+d/p,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,q}^{(\alpha,-b-1+1/p)}(\mathbb{R}^d). \quad (5.16)$$

(ii) *Let $k \in \mathbb{N}$. Then*

$$\text{Lip}_{1,q}^{(k+d,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,q}^{(k,-b)}(\mathbb{R}^d). \quad (5.17)$$

The corresponding results for periodic spaces also hold true.

Remark 5.8. (i) It can be shown that the embedding (5.16) is optimal in the sense that the shift $-1 + 1/p$ given in the logarithmic smoothness of the target space cannot be improved.

(ii) Setting $q = \infty$ and $b = 0$ in (5.17), we obtain

$$\text{BV}^{k+d-1}(\mathbb{R}^d) \hookrightarrow \text{Lip}^k(\mathbb{R}^d), \quad k \in \mathbb{N}.$$

Proof of Theorem 5.7. (i): We shall divide the proof into several steps.

STEP 1: In virtue of Marchaud's inequality,

$$t^{-\alpha} \omega_{\alpha}(f, t)_{\infty} \lesssim \int_t^{\infty} \frac{\omega_{\alpha+d/p}(f, u)_{\infty}}{u^{\alpha}} \frac{du}{u},$$

see, e.g., [44], we can apply Hölder's inequality to obtain

$$t^{-\alpha} \omega_{\alpha}(f, t)_{\infty} \lesssim (1 - \log t)^{b-1/p} \|f\|_{B_{\infty,p}^{\alpha,-b+1}(\mathbb{R}^d)}, \quad b > 1/p,$$

or equivalently,

$$B_{\infty,p}^{\alpha,-b+1}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(\alpha,-b+1/p)}(\mathbb{R}^d), \quad b > 1/p. \quad (5.18)$$

STEP 2: We show (5.16) with $q = \infty$ and $b = 0$, that is,

$$H_p^{\alpha+d/p}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(\alpha,-1+1/p)}(\mathbb{R}^d).$$

Indeed, by (4.11) and (5.18), we derive

$$H_p^{\alpha+d/p}(\mathbb{R}^d) \hookrightarrow B_{\infty,p}^{\alpha}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(\alpha,-1+1/p)}(\mathbb{R}^d).$$

STEP 3: We make the following assertion

$$\text{Lip}_{p,1}^{(\alpha+d/p,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(\alpha,-b+1/p)}(\mathbb{R}^d), \quad b > 1. \quad (5.19)$$

Indeed, using (4.14), we have

$$\text{Lip}_{p,1}^{(\alpha+d/p,-b)}(\mathbb{R}^d) \hookrightarrow B_{\infty,p}^{\alpha,-b+1}(\mathbb{R}^d). \quad (5.20)$$

Then (5.19) follows from (5.18) and (5.20).

Let $0 < q \leq \infty$ and $b > 1/q$. We choose b_0 and b_1 satisfying $1 < b_1 < 1 + b - 1/q < b_0$ and let $\theta \in (0, 1)$ such that $1 + b - 1/q = (1 - \theta)b_1 + \theta b_0$. According to (5.19), we have

$$\text{Lip}_{p,1}^{(\alpha+d/p,-b_i)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(\alpha,-b_i+1/p)}(\mathbb{R}^d), \quad i = 0, 1,$$

and thus

$$(\text{Lip}_{p,1}^{(\alpha+d/p,-b_0)}(\mathbb{R}^d), \text{Lip}_{p,1}^{(\alpha+d/p,-b_1)}(\mathbb{R}^d))_{\theta,q} \hookrightarrow (\text{Lip}_{\infty,\infty}^{(\alpha,-b_0+1/p)}(\mathbb{R}^d), \text{Lip}_{\infty,\infty}^{(\alpha,-b_1+1/p)}(\mathbb{R}^d))_{\theta,q}. \quad (5.21)$$

By Lemma 3.6,

$$(\text{Lip}_{p,1}^{(\alpha+d/p,-b_0)}(\mathbb{R}^d), \text{Lip}_{p,1}^{(\alpha+d/p,-b_1)}(\mathbb{R}^d))_{\theta,q} = \text{Lip}_{p,q}^{(\alpha+d/p,-b)}(\mathbb{R}^d)$$

and

$$(\text{Lip}_{\infty,\infty}^{(\alpha,-b_0+1/p)}(\mathbb{R}^d), \text{Lip}_{\infty,\infty}^{(\alpha,-b_1+1/p)}(\mathbb{R}^d))_{\theta,q} = \text{Lip}_{\infty,q}^{(\alpha,-b-1+1/p)}(\mathbb{R}^d).$$

Inserting these formulas into (5.21) we achieve (5.16).

(ii): It follows from the trivial embedding $W_1^{k+d}(\mathbb{R}^d) \hookrightarrow W_{\infty}^k(\mathbb{R}^d)$ and the well-known result $B_{1,1}^d(\mathbb{R}^d) \hookrightarrow L_{\infty}(\mathbb{R}^d)$ (see [56, Theorem 2.8.3]) that

$$(B_{1,1}^d(\mathbb{R}^d), W_1^{k+d}(\mathbb{R}^d))_{(1,-b),q} \hookrightarrow (L_{\infty}(\mathbb{R}^d), W_{\infty}^k(\mathbb{R}^d))_{(1,-b),q} = \text{Lip}_{\infty,q}^{(k,-b)}(\mathbb{R}^d)$$

where we have also used (2.15). To find the space $(B_{1,1}^d(\mathbb{R}^d), W_1^{k+d}(\mathbb{R}^d))_{(1,-b),q}$, we make use of (2.16), (2.19) and (2.15). There holds

$$\begin{aligned} (B_{1,1}^d(\mathbb{R}^d), W_1^{k+d}(\mathbb{R}^d))_{(1,-b),q} &= ((L_1(\mathbb{R}^d), W_1^{k+d}(\mathbb{R}^d))_{\frac{d}{k+d},1}, W_1^{k+d}(\mathbb{R}^d))_{(1,-b),q} \\ &= (L_1(\mathbb{R}^d), W_1^{k+d}(\mathbb{R}^d))_{(1,-b),q} = \text{Lip}_{1,q}^{(k+d,-b)}(\mathbb{R}^d). \end{aligned}$$

Therefore, $\text{Lip}_{1,q}^{(k+d,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,q}^{(k,-b)}(\mathbb{R}^d)$. \square

5.4. Embeddings into Lip

The study of embeddings of smooth function spaces into the Lipschitz class Lip has a long history. In particular, it plays a key role in the computation of continuity envelopes of function spaces as can be seen in [57, Chapters 12 and 14] and [37, Chapter 9]. Here we shall only mention that if $1 \leq p \leq \infty$ then

$$B_{p,q}^{1+d/p}(\mathbb{R}^d) \hookrightarrow \text{Lip}(\mathbb{R}^d) \iff 0 < q \leq 1; \quad (5.22)$$

for further extensions of this result, the reader is referred to [7, Proposition 3.2]. Consequently, if $1 < p < \infty$ then

$$H_p^{1+d/p}(\mathbb{R}^d) \text{ is not continuously embedded into } \text{Lip}(\mathbb{R}^d). \quad (5.23)$$

Note that one can circumvent this obstruction using the Brézis-Wainger inequality [6], which asserts that $H_p^{1+d/p}(\mathbb{R}^d)$ is formed by almost Lipschitz-continuous functions. More precisely, the following embedding holds true

$$H_p^{1+d/p}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{\infty,\infty}^{(1,-1/p')}(\mathbb{R}^d), \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover, this result is optimal within the scale of the spaces $\text{Lip}_{\infty,\infty}^{(1,-b)}(\mathbb{R}^d)$. For further details, as well as generalizations to Besov and Triebel-Lizorkin spaces, we refer to [24, Theorem 2.1], [57, Theorem 11.4] and [37, Propositions 7.14 and 7.15] and the references therein.

Our next result gives a full characterization of the embeddings from $\text{Lip}_{p,q}^{(\alpha,-b)}$ into Lip.

Theorem 5.9. *Let $\alpha > 0$, $1 < p < \infty$, $0 < q \leq \infty$, and $b > 1/q$. Then*

$$\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}(\mathbb{R}^d) \iff \alpha > 1 + \frac{d}{p}.$$

Proof. Assume $\alpha > d/p + 1$. Let $p < p_1 < \infty$. Then according to (4.13) and (5.22), we have

$$\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{p_1,q}^{\alpha+d(1/p_1-1/p),-b+1/\max\{p,q\}}(\mathbb{R}^d) \hookrightarrow B_{p_1,1}^{1+d/p_1}(\mathbb{R}^d) \hookrightarrow \text{Lip}(\mathbb{R}^d),$$

where the second embedding follows from (5.2).

The converse statement will be shown by contradiction. Suppose that

$$\text{Lip}_{p,q}^{(d/p+1,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}(\mathbb{R}^d). \quad (5.24)$$

It is an immediate consequence of (2.5) that

$$H_p^\alpha(\mathbb{R}^d) \hookrightarrow \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d). \quad (5.25)$$

In particular, $H_p^{d/p+1}(\mathbb{R}^d) \hookrightarrow \text{Lip}(\mathbb{R}^d)$, which is not true because of (5.23). Hence, (5.24) does not hold. It follows now from Theorem 5.1 that if $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}(\mathbb{R}^d)$ then $\alpha > d/p + 1$. \square

5.5. Embeddings into BV

Some technical problems of the space of functions of bounded variation can be overcome using its relationships with the scale of Besov spaces. See [14,34,48]. In particular, the following embeddings hold

$$B_{1,1}^1(\mathbb{R}^d) \hookrightarrow \text{BV}(\mathbb{R}^d) \hookrightarrow B_{1,\infty}^1(\mathbb{R}^d). \quad (5.26)$$

The objective of this section is to characterize embeddings of Lipschitz spaces into $\text{BV}(\mathbb{R}^d) = \text{Lip}_{1,\infty}^{(1,0)}(\mathbb{R}^d)$. This will complement those embeddings given in (5.11).

Theorem 5.10. *Let $\alpha > 0, 0 < q \leq \infty$ and $b > 1/q$. Then*

$$\text{Lip}_{1,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow \text{BV}(\mathbb{R}^d) \iff \alpha > 1.$$

The corresponding result for periodic spaces also holds true.

Proof. We claim that

$$\text{Lip}_{1,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{1,1}^1(\mathbb{R}^d), \quad \alpha > 1. \quad (5.27)$$

Indeed, this follows from the trivial embeddings

$$\text{Lip}_{1,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{1,\infty}^{(\alpha,-b)}(\mathbb{R}^d), \quad \alpha > 0, \quad 0 < q < \infty, \quad b > 1/q,$$

and

$$\text{Lip}_{1,\infty}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow B_{1,1}^1(\mathbb{R}^d).$$

Combining (5.27) and (5.26), we arrive at $\text{Lip}_{1,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow \text{BV}(\mathbb{R}^d)$.

Let us prove that the condition $\alpha > 1$ is necessary. We shall proceed by contradiction, that is, assume that there exists $\alpha \leq 1$ such that

$$\text{Lip}_{1,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow \text{BV}(\mathbb{R}^d). \quad (5.28)$$

We observe that it is enough to disprove (5.28) with $\alpha = 1$ because

$$\text{Lip}_{1,\infty}^{(1,-b)}(\mathbb{R}^d) \hookrightarrow \text{Lip}_{1,q}^{(\alpha,-b)}(\mathbb{R}^d), \quad \alpha < 1.$$

This embedding is an immediate consequence of the Marchaud inequality for moduli of smoothness

$$\omega_\alpha(f, t)_1 \lesssim t^\alpha \int_t^\infty \frac{\omega_1(f, u)_1}{u^\alpha} \frac{du}{u},$$

see, e.g., [44]. Assume that (5.28) holds with $\alpha = 1$. For $\theta \in (0, 1)$, we have

$$(L_1(\mathbb{R}^d), \text{Lip}_{1,q}^{(1,-b)}(\mathbb{R}^d))_{\theta,q} \hookrightarrow (L_1(\mathbb{R}^d), \text{BV}(\mathbb{R}^d))_{\theta,q}. \quad (5.29)$$

Next we compute these interpolation spaces. Concerning the target space, we have

$$(L_1(\mathbb{R}^d), \text{BV}(\mathbb{R}^d))_{\theta,q} = B_{1,q}^\theta(\mathbb{R}^d)$$

(cf. [22, Corollary 11.7]). On the other hand, by (2.15), (2.18) and (2.16),

$$\begin{aligned} (L_1(\mathbb{R}^d), \text{Lip}_{1,q}^{(1,-b)}(\mathbb{R}^d))_{\theta,q} &= (L_1(\mathbb{R}^d), (L_1(\mathbb{R}^d), W_1^1(\mathbb{R}^d))_{(1,-b),q})_{\theta,q} \\ &= (L_1(\mathbb{R}^d), W_1^1(\mathbb{R}^d))_{\theta,q;\theta(-b+1/q)} = B_{1,q}^{\theta, \theta(-b+1/q)}(\mathbb{R}^d). \end{aligned}$$

Therefore, (5.29) results in

$$B_{1,q}^{\theta, \theta(-b+1/q)}(\mathbb{R}^d) \hookrightarrow B_{1,q}^{\theta}(\mathbb{R}^d),$$

which implies $-b + 1/q \geq 0$. By assumptions, this is not possible. \square

6. Characterization of Lipschitz spaces via wavelets

In order to describe our results, we briefly discuss wavelet bases. For full treatment, we refer the reader to [19], [48] and [59]. As usual, $C^u(\mathbb{R})$ with $u \in \mathbb{N}$ collects all (complex-valued) continuous functions on \mathbb{R} having continuous bounded derivatives up to order u . Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \quad u \in \mathbb{N}, \quad (6.1)$$

be real compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x) x^v dx = 0 \quad \text{for all } v \in \mathbb{N}_0 \text{ with } v < u.$$

Recall that ψ_F is called the *scaling function (father wavelet)* and ψ_M the *associated wavelet (mother wavelet)*. The extension of these wavelets from \mathbb{R} to \mathbb{R}^d , $d \geq 2$, is based on the usual tensor procedure. Let

$$G = (G_1, \dots, G_n) \in G^0 = \{F, M\}^d,$$

which means that G_r is either F or M . Let

$$G = (G_1, \dots, G_n) \in G^j = \{F, M\}^{d*}, \quad j \in \mathbb{N},$$

which means that G_r is either F or M where $*$ indicates that at least one of the components of G must be an M . Hence G^0 has 2^d elements, whereas G^j with $j \in \mathbb{N}$ has $2^d - 1$ elements. Let

$$\Psi_{G,m}^j(x) = 2^{jd/2} \prod_{r=1}^d \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j, \quad m \in \mathbb{Z}^d, \quad j \in \mathbb{N}_0.$$

We shall assume that ψ_F and ψ_M in (6.1) are normalized with respect to $L_2(\mathbb{R})$. Then the system

$$\Psi = \left\{ \Psi_{G,m}^j : j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^d \right\}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$ and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^d} \lambda_m^{j,G} 2^{-jd/2} \Psi_{G,m}^j$$

with

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jd/2} \int_{\mathbb{R}^d} f(x) \Psi_{G,m}^j(x) dx, \quad (6.2)$$

where $2^{-jd/2} \Psi_{G,m}^j$ are uniformly bounded functions (with respect to j and m).

Under certain conditions on the smoothness parameter u (see (6.1)), Besov spaces and Lebesgue and Sobolev spaces admit characterizations via wavelet decompositions. Next we introduce the related sequence spaces.

Let $\chi_{j,m}$ be the characteristic function of the dyadic cube $Q_{j,m} = 2^{-j}m + 2^{-j}(0,1)^d$ in \mathbb{R}^d with sides of length 2^{-j} parallel to the axes of coordinates and $2^{-j}m$ as the lower left corner. Let $-\infty < s, \xi < \infty, 1 < p < \infty$ and $0 < q \leq \infty$. The space $b_{p,q}^{s,\xi}$ is the collection of all sequences $\lambda = (\lambda_m^{j,G})$ with $j \in \mathbb{N}_0, G \in G^j$ and $m \in \mathbb{Z}^d$ such that

$$\|\lambda\|_{b_{p,q}^{s,\xi}} = \left(\sum_{j=0}^{\infty} 2^{j(s-d/p)q} (1+j)^{\xi q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^d} |\lambda_m^{j,G}|^p \right)^{q/p} \right)^{1/q} < \infty \quad (6.3)$$

with the usual modification if $q = \infty$. We write $f_{p,2}^s$ for the space of all sequences $\lambda = (\lambda_m^{j,G})$ with $j \in \mathbb{N}_0, G \in G^j$ and $m \in \mathbb{Z}^d$ such that

$$\|\lambda\|_{f_{p,2}^s} = \left\| \left(\sum_{j,G,m} 2^{js2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} < \infty. \quad (6.4)$$

It turns out that $f_{p,2}^s$ can be identified with a complemented subspace of $L_p(\mathbb{R}^d; \ell_2^s(\ell_2))$, where

$$\|\lambda\|_{\ell_2^s(\ell_2)} = \left(\sum_{j=0}^{\infty} 2^{js2} \sum_{G \in G^j, m \in \mathbb{Z}^d} |\lambda_m^{j,G}|^2 \right)^{1/2} < \infty$$

(see (3.5)). Indeed, since

$$\|\lambda\|_{f_{p,2}^s} = \|(\lambda_m^{j,G} \chi_{j,m}(\cdot))\|_{L_p(\mathbb{R}^d; \ell_2^s(\ell_2))}, \quad (6.5)$$

it is plain to check that

$$Pf(x) = P((f_m^{j,G}))(x) = \left(\left(2^{jd} \int_{Q_{j,m}} f_m^{j,G}(y) dy \chi_{j,m}(x) \right)_{\substack{G \in G^j \\ m \in \mathbb{Z}^d}} \right)_{j \in \mathbb{N}_0}, \quad x \in \mathbb{R}^d,$$

defines a projection operator from $L_p(\mathbb{R}^d; \ell_2^s(\ell_2))$ onto a subspace of $L_p(\mathbb{R}^d; \ell_2^s(\ell_2))$, which is isometric to $f_{p,2}^s$.

We are now in a position to state the well-known characterizations of Lebesgue, Sobolev and Besov spaces via wavelets. Recall that $L_p(\mathbb{R}^d) = H_p^0(\mathbb{R}^d)$. For the proof and more general statements covering not only Besov and Triebel-Lizorkin spaces but also Besov spaces of generalized smoothness, the reader is referred to [59, Theorem 1.20] and [1].

Theorem 6.1.

(i) Let $1 < p < \infty$ and $-\infty < s < \infty$. Assume that (6.1) holds with $u > |s|$. Then $f \in H_p^s(\mathbb{R}^d)$ if and only if

$$f = \sum_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^d} \lambda_m^{j,G} 2^{-jd/2} \Psi_{G,m}^j, \quad (\lambda_m^{j,G}) \in f_{p,2}^s$$

(unconditional convergence being in $H_p^s(\mathbb{R}^d)$). This representation is unique, that is, the wavelet coefficients $(\lambda_m^{j,G})$ are given by (6.2), and the operator

$$I : f \mapsto (\lambda_m^{j,G})$$

defines an isomorphism from $H_p^s(\mathbb{R}^d)$ onto $f_{p,2}^s$.

(ii) Let $1 < p < \infty, 0 < q \leq \infty$, and $-\infty < s, \xi < \infty$. Assume that (6.1) holds with $u > |s|$. Then $f \in B_{p,q}^{s,\xi}(\mathbb{R}^d)$ if and only if

$$f = \sum_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^d} \lambda_m^{j,G} 2^{-jd/2} \Psi_{G,m}^j, \quad (\lambda_m^{j,G}) \in b_{p,q}^{s,\xi}$$

(unconditional convergence being in $\mathcal{S}'(\mathbb{R}^d)$). This representation is unique, that is, the wavelet coefficients $(\lambda_m^{j,G})$ are given by (6.2), and the operator

$$I : f \mapsto (\lambda_m^{j,G})$$

defines an isomorphism from $B_{p,q}^{s,\xi}(\mathbb{R}^d)$ onto $b_{p,q}^{s,\xi}$. If, in addition, $q < \infty$, then $\{\Psi_{G,m}^j\}$ is an unconditional basis in $B_{p,q}^{s,\xi}(\mathbb{R}^d)$.

Remark 6.2. Note that (6.2) can be written as

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jd/2} \left(f, \Psi_{G,m}^j \right),$$

understood as the dual pairing. Then for spaces $A(\mathbb{R}^d)$ satisfying $\mathcal{S}(\mathbb{R}^d) \hookrightarrow A(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$, the expression (f, g) should be understood in the sense of the dual pairing in the context of $(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$. The well-definedness of such expressions has been considered in [58, Theorem 3.5] and, in a fairly general setting which also covers the above approach, in [39, Proposition 4.4].

The goal of this section is to provide the wavelet description of Lipschitz spaces. With this in mind, we define the sequence spaces $\text{lip}_{p,q}^{(\alpha,-b)}$ as follows.

Definition 6.3. Let $\alpha > 0, 1 < p < \infty, 0 < q \leq \infty$, and $b > 1/q$ ($b \geq 0$ if $q = \infty$). The space $\text{lip}_{p,q}^{(\alpha,-b)}$ is the collection of all $\lambda = (\lambda_m^{j,G})$ such that

$$\|\lambda\|_{\text{lip}_{p,q}^{(\alpha,-b)}} = \left(\sum_{k=0}^{\infty} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\alpha 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} < \infty \quad (6.6)$$

(with the usual modification if $q = \infty$).

It is not hard to show that $f_{p,2}^\alpha \hookrightarrow \text{lip}_{p,q}^{(\alpha,-b)} \hookrightarrow f_{p,2}^0$. In fact, our next result shows that $\text{lip}_{p,q}^{(\alpha,-b)}$ can be characterized in terms of the classical spaces $(f_{p,2}^0, f_{p,2}^\alpha)$ via limiting interpolation.

Lemma 6.4. Let $\alpha > 0, 1 < p < \infty, 0 < q \leq \infty$ and $b > 1/q$ ($b \geq 0$ if $q = \infty$). Then

$$\text{lip}_{p,q}^{(\alpha,-b)} = (f_{p,2}^0, f_{p,2}^\alpha)_{(1,-b),q}.$$

Proof. For $f = ((f_m^{j,G})_{G \in G^j, m \in \mathbb{Z}^d})_{j \in \mathbb{N}_0} \in L_p(\mathbb{R}^d; \ell_2(\ell_2))$, we can invoke Lemma 3.5 to get

$$\begin{aligned} & \|f\|_{(L_p(\mathbb{R}^d; \ell_2(\ell_2)), L_p(\mathbb{R}^d; \ell_2^\alpha(\ell_2)))_{(1,-b),q}} \\ & \asymp \left(\sum_{k=0}^{\infty} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\alpha 2} |f_m^{j,G}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}. \end{aligned}$$

Since $f_{p,2}^0$ and $f_{p,2}^\alpha$ are isometric to complemented subspaces of $L_p(\mathbb{R}^d; \ell_2(\ell_2))$ and $L_p(\mathbb{R}^d; \ell_2^\alpha(\ell_2))$, respectively, via (6.5), we can apply the theorem on interpolation of complemented subspaces [55, Theorem 1.17.1] to obtain

$$\begin{aligned} & \|\lambda\|_{(f_{p,2}^0, f_{p,2}^\alpha)_{(1,-b),q}} \\ & \asymp \left(\sum_{k=0}^{\infty} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\alpha 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} = \|\lambda\|_{\text{lip}_{p,q}^{(\alpha,-b)}}. \quad \square \end{aligned}$$

Now we are ready to establish the wavelet decomposition of Lipschitz spaces.

Theorem 6.5. *Let $\alpha > 0, 1 < p < \infty, 0 < q \leq \infty$ and $b > 1/q$. Assume that (6.1) holds with $u > \alpha$. Then $f \in \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ if and only if*

$$f = \sum_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^d} \lambda_m^{j,G} 2^{-jd/2} \Psi_{G,m}^j, \quad (\lambda_m^{j,G}) \in \text{lip}_{p,q}^{(\alpha,-b)} \quad (6.7)$$

(unconditional convergence being in $L_p(\mathbb{R}^d)$). This representation is unique, that is, the wavelet coefficients $(\lambda_m^{j,G})$ are given by (6.2), and the operator

$$I : f \mapsto (\lambda_m^{j,G})$$

defines an isomorphism from $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ onto $\text{lip}_{p,q}^{(\alpha,-b)}$. If, in addition, $q < \infty$, then $\{\Psi_{G,m}^j\}$ is an unconditional basis in $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$.

Remark 6.6. Let $\beta > \alpha$. We make the following claim

$$\|f\|_{B_{p,q}^{\alpha,-b}(\mathbb{R}^d)} \asymp \left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\beta 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}. \quad (6.8)$$

We postpone the proof of this characterization until a little later, and meanwhile point out certain similarities between Besov, Lebesgue, Sobolev and Lipschitz spaces. Namely, it turns out that the family of norms given by

$$\left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\beta 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}, \quad \beta \geq \alpha,$$

(as usual, the sum should be replaced by the supremum if $q = \infty$) allows us to introduce in a unifying way the Besov spaces (see (6.8)), Lipschitz spaces (taking $\beta = \alpha$; see (6.6) and Theorem 6.5), Lebesgue spaces (taking $\beta = \alpha = 0, q = \infty$ and $b = 0$; see (6.4) and Theorem 6.1(i)) and Sobolev spaces (taking $\beta = \alpha, q = \infty$ and $b = 0$; see (6.4) and Theorem 6.1(i)).

Now let us show (6.8). We have

$$\left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\beta 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q}$$

$$\leq \left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left(\sum_{j=0}^k 2^{j\beta} \left\| \sum_{G \in G^j, m \in \mathbb{Z}^d} |\lambda_m^{j,G} \chi_{j,m}| \right\|_{L_p(\mathbb{R}^d)} \right)^q \right)^{1/q}. \quad (6.9)$$

We distinguish two possible cases. If $q \geq 1$ then we apply Hardy's inequality (noting that $\beta > \alpha$) to get

$$\begin{aligned} & \left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left(\sum_{j=0}^k 2^{j\beta} \left\| \sum_{G \in G^j, m \in \mathbb{Z}^d} |\lambda_m^{j,G} \chi_{j,m}| \right\|_{L_p(\mathbb{R}^d)} \right)^q \right)^{1/q} \\ & \lesssim \left(\sum_{k=0}^{\infty} 2^{k\alpha q} (1+k)^{-bq} \left\| \sum_{G \in G^j, m \in \mathbb{Z}^d} |\lambda_m^{j,G} \chi_{j,m}| \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} \asymp \|f\|_{B_{p,q}^{\alpha,-b}(\mathbb{R}^d)}. \end{aligned} \quad (6.10)$$

where the last step follows from (6.3) and Theorem 6.1(ii). On the other hand, if $q < 1$ then

$$\begin{aligned} & \left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left(\sum_{j=0}^k 2^{j\beta} \left\| \sum_{G \in G^j, m \in \mathbb{Z}^d} |\lambda_m^{j,G} \chi_{j,m}| \right\|_{L_p(\mathbb{R}^d)} \right)^q \right)^{1/q} \\ & \leq \left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \sum_{j=0}^k 2^{j\beta q} \left\| \sum_{G \in G^j, m \in \mathbb{Z}^d} |\lambda_m^{j,G} \chi_{j,m}| \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} \asymp \|f\|_{B_{p,q}^{\alpha,-b}(\mathbb{R}^d)} \end{aligned} \quad (6.11)$$

where we have also used that $\beta > \alpha$ in the last step.

Therefore, it follows from (6.9), (6.10) and (6.11) that

$$\left(\sum_{k=0}^{\infty} 2^{k(\alpha-\beta)q} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{Z}^d} 2^{j\beta 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} \lesssim \|f\|_{B_{p,q}^{\alpha,-b}(\mathbb{R}^d)}.$$

On the other hand, the converse estimate can be easily shown by taking into account that the supports of the characteristic functions $\chi_{j,m}$ with j fixed are pairwise disjoint. Hence we arrive at the desired claim (6.8).

Proof of Theorem 6.5. According to Theorem 6.1, $L_p(\mathbb{R}^d)$ and $H_p^\alpha(\mathbb{R}^d)$ are isomorphic to $f_{p,2}^0$ and $f_{p,2}^\alpha$ via $I : f \mapsto (\lambda_m^{j,G})$. Hence, applying (2.13) and Lemma 6.4, we derive that

$$I : \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) = (L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d))_{(1,-b),q} \rightarrow (f_{p,2}^0, f_{p,2}^\alpha)_{(1,-b),q} = \text{lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$$

is also an isomorphism.

The uniqueness and the unconditional convergence in $L_p(\mathbb{R}^d)$ of the representation (6.7) are immediate consequences of the embeddings

$$\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \quad \text{and} \quad \text{lip}_{p,q}^{(\alpha,-b)} \hookrightarrow f_{p,2}^0$$

together with the corresponding assertion for $L_p(\mathbb{R}^d)$ spaces given in Theorem 6.1. The fact that $\{\Psi_{G,m}^j\}$ is an unconditional basis in $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$, $q < \infty$, follows easily from (6.6). \square

Remark 6.7. Our method also works with periodic wavelets. Before we state the periodic analogue of Theorem 6.5, let us fix some notation. Let $L \in \mathbb{N}$ be fixed such that

$$\Psi_{G,m}^{j,L}(x) = 2^{(j+L)d/2} \prod_{r=1}^d \psi_{G_r}(2^{j+L}x_r - m_r)$$

satisfies

$$\text{supp } \Psi_{G,m}^{j,L} \subset \{x \in \mathbb{R}^d : |x| < 1/2\}$$

and let $\{\Psi_{G,m}^{j,L,\text{per}}\}$ be the orthonormal system in $L_2(\mathbb{T}^d)$ obtained from $\{\Psi_{G,m}^{j,L}\}$ via standard periodization arguments. See [59, Section 1.3.2] for full details. The periodic counterparts of the sequence spaces $f_{p,2}^\alpha$ (see (6.4)) and $\text{lip}_{p,q}^{(\alpha,-b)}$ (see (6.6)) are introduced as follows. Let

$$\mathbb{P}_j^d = \{m \in \mathbb{Z}^d : 0 \leq m_r < 2^{j+L}\}, \quad j \in \mathbb{N}_0.$$

Let $\alpha > 0, 1 < p < \infty, 0 < q \leq \infty$, and $b > 1/q$ ($b \geq 0$ if $q = \infty$). The space $f_{p,2}^{\alpha,\text{per}}$ is formed by all $\lambda = (\lambda_m^{j,G})$ such that

$$\|\lambda\|_{f_{p,2}^{\alpha,\text{per}}} = \left\| \left(\sum_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{P}_j^d} 2^{j\alpha 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{T}^d)} < \infty,$$

where $\chi_{j,m}$ is the characteristic function of a cube with the left corner $2^{-j-L}m$ and of side-length 2^{-j-L} . The space $\text{lip}_{p,q}^{(\alpha,-b),\text{per}}$ is the collection of all λ such that

$$\|\lambda\|_{\text{lip}_{p,q}^{(\alpha,-b),\text{per}}} = \left(\sum_{k=0}^{\infty} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k \sum_{G \in G^j, m \in \mathbb{P}_j^d} 2^{j\alpha 2} |\lambda_m^{j,G} \chi_{j,m}(\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{T}^d)}^q \right)^{1/q} < \infty$$

(with the usual modification if $q = \infty$).

The periodic counterpart of Theorem 6.5 reads as follows.

Theorem 6.8. *Let $\alpha > 0, 1 < p < \infty, 0 < q \leq \infty$ and $b > 1/q$. Assume that (6.1) holds with $u > \alpha$. Then $f \in \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{T}^d)$ if and only if*

$$f = \sum_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{P}_j^d} \mu_m^{j,G} 2^{-(j+L)d/2} \Psi_{G,m}^{j,L,\text{per}}, \quad (\mu_m^{j,G}) \in \text{lip}_{p,q}^{(\alpha,-b),\text{per}}$$

(unconditional convergence being in $L_p(\mathbb{T}^d)$). This representation is unique, that is, the wavelet coefficients $(\mu_m^{j,G})$ are given by

$$\mu_m^{j,G} = 2^{(j+L)d/2} \int_{\mathbb{T}^d} f(x) \Psi_{G,m}^{j,L,\text{per}}(x) \, dx,$$

and the operator

$$I : f \mapsto (\mu_m^{j,G})$$

defines an isomorphism from $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{T}^d)$ onto $\text{lip}_{p,q}^{(\alpha,-b),\text{per}}$. If, in addition, $q < \infty$, then $\{\Psi_{G,m}^{j,L,\text{per}}\}$ is an unconditional basis in $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{T}^d)$.

The proof of Theorem 6.8 follows the same arguments used to show Theorem 6.5 except for some minor modifications. Specifically, under assumptions of Theorem 6.8, we have that I is an isomorphism from $L_p(\mathbb{T}^d)$ onto $f_{p,2}^{0,\text{per}}$ and from $H_p^\alpha(\mathbb{T}^d)$ onto $f_{p,2}^{\alpha,\text{per}}$ (see [59, Theorem 1.37]). On the other hand, the periodic analogue of Lemma 6.4, that is,

$$\text{lip}_{p,q}^{(\alpha,-b),\text{per}} = (f_{p,2}^{0,\text{per}}, f_{p,2}^{\alpha,\text{per}})_{(1,-b),q},$$

also holds true. This is a consequence of the facts that $f_{p,2}^{0,\text{per}}$ and $f_{p,2}^{\alpha,\text{per}}$ are complemented subspaces of $L_p(\mathbb{T}^d; \ell_2(\mathbb{N}_0; \mathbb{R}^{2^{(j+L)d}}))$ and $L_p(\mathbb{T}^d; \ell_2^\alpha(\mathbb{N}_0; \mathbb{R}^{2^{(j+L)d}}))$, respectively, together with the generalization of the periodic counterpart of Lemma 3.5 to a family of Banach spaces $(A_j)_{j \in \mathbb{N}_0}$, i.e.,

$$\|(f_j)\|_{(L_p(\mathbb{T}^d; \ell_r(A_j)), L_p(\mathbb{T}^d; \ell_r^\alpha(A_j)))_{(1,-b),q}} \asymp \left(\sum_{k=0}^{\infty} (1+k)^{-bq} \left\| \left(\sum_{j=0}^k 2^{j\alpha r} \|f_j(\cdot)\|_{A_j}^r \right)^{1/r} \right\|_{L_p(\mathbb{T}^d)}^q \right)^{1/q}.$$

Remark 6.9. The method of proof of Theorem 6.5 can also be carried out to obtain the characterization of Lipschitz spaces in terms of Haar wavelet bases. We shall not record here the construction of Haar wavelet bases and we refer the interested reader to [59, Section 2.5.1] and [60, Section 2.3] for further explanations and related literature. Let $\{H_{G,m}^j\}$ be an orthonormal Haar wavelet basis in $L_2(\mathbb{R}^d)$. The characterization for Lipschitz spaces in terms of $\{H_{G,m}^j\}$ reads as follows.

Theorem 6.10. *Let $1 < p < \infty$, $0 < \alpha < \min\{1/p, 1/2\}$, $0 < q \leq \infty$ and $b > 1/q$. Then $f \in \text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ if and only if*

$$f = \sum_{j \in \mathbb{N}_0, G \in G^j, m \in \mathbb{Z}^d} \lambda_m^{j,G} 2^{-jd/2} H_{G,m}^j, \quad (\lambda_m^{j,G}) \in \text{lip}_{p,q}^{(\alpha,-b)}$$

(unconditional convergence being in $L_p(\mathbb{R}^d)$). This representation is unique, that is, the wavelet coefficients $(\lambda_m^{j,G})$ are given by

$$\lambda_m^{j,G} = 2^{jd/2} \int_{\mathbb{R}^d} f(x) H_{G,m}^j(x) dx,$$

and the operator

$$I : f \mapsto (\lambda_m^{j,G})$$

defines an isomorphism from $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ onto $\text{lip}_{p,q}^{(\alpha,-b)}$. If, in addition, $q < \infty$, then $\{H_{G,m}^j\}$ is an unconditional basis in $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$.

The proof of this theorem follows line by line the arguments of the proof of Theorem 6.5 and is safely left to the reader. However, note that unlike Theorem 6.5, we obtain unconditional Haar wavelet bases in $\text{Lip}_{p,q}^{(\alpha,-b)}(\mathbb{R}^d)$ under the additional assumption $\alpha < \min\{1/p, 1/2\}$. This restriction comes from the fact that if $\alpha > 0$ and $1 < p < \infty$ then $\{H_{G,m}^j\}$ is an unconditional basis in $H_p^\alpha(\mathbb{R}^d)$ if, and only if, $\alpha < \min\{1/p, 1/2\}$. See [60, Corollary 2.23] and [53, 54].

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