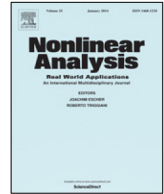




CENTRE DE RECERCA MATEMÀTICA

Title: *On the global dynamics and integrability of the Chemostat system*
Journal Information: *Nonlinear Analysis: Real World Applications*,
Author(s): Martínez Y.P., Valls C..
Volume, pages: 53 1, DOI:[10.1016/j.nonrwa.2019.103051]



On the global dynamics and integrability of the Chemostat system

Y. Paulina Martínez^{a,b,*}, Claudia Valls^c

^a Centre de Recerca Matemàtica, 08193 Bellaterra, Barcelona, Catalonia, Spain

^b Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile

^c Center for Mathematical Analysis, Geometry and Dynamical Systems, Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal



ARTICLE INFO

Article history:

Received 14 May 2019

Received in revised form 30 September 2019

Accepted 2 October 2019

Available online 23 October 2019

Keywords:

Chemostat systems

Phase portraits

Darboux polynomials

Puiseux series

Invariant algebraic curves

Liouvillian first integrals

ABSTRACT

We study a Chemostat system of the form

$$\dot{x} = -qx + \frac{\tilde{R}}{K+y}xy, \quad \dot{y} = (\tilde{c}-y)q - \frac{\tilde{R}}{\tilde{a}(K+y)}xy,$$

where $q > 0$, $\tilde{R} > 0$, $K > 0$, $\tilde{c} > 0$ and $\tilde{a} \neq 0$. This system appears in competition modelling in biology. We describe its global dynamics on the Poincaré disc and study its Liouvillian integrability. For the first topic we use the well-known Poincaré compactification theory and for the second one we make use of the Puiseux series to derive the structure of all the irreducible invariant algebraic curves.

© 2019 Elsevier Ltd. All rights reserved.

1. Introduction and statement of the main result

One of the more challenging aspects in mathematical biology is the competition modelling. Competition is essential in nature, but modelling it is really difficult in general since there are so many ways the populations can compete. However, one of the simplest ways for competition is when two or more populations are competing for the same resource (being a growth-limiting nutrient or common food supply). Chemostat (or also called continuous culture) is a type of competition which refers to a laboratory device, used in a culture environment (as a crop of bacteria) for growing microorganisms and has been considered in mathematical biology to study the modelling of competition in nature. This device is very important in ecological studies because the relevant experiments are possible and, although it can be very difficult, the mathematics are tractable. The name “chemostat” seems to have originated in the work of Novick and Szilard [1].

* Corresponding author at: Centre de Recerca Matemàtica, 08193 Bellaterra, Barcelona, Catalonia, Spain.

E-mail addresses: yohanna.martinez@uab.cat, ymartinez@ubiobio.cl (Y.P. Martínez), cvals@math.tecnico.ulisboa.pt (C. Valls).

Chemostat plays a really important role in theoretical ecology (see e.g. [2–9] and the references therein). It can also be used to study recombinant problems related to genetically altered microorganisms (see e.g. [10–12]), in analysis of antibiotic [13] and in problems of waste water treatment (see e.g. [14,15]). Recently Caraballo, Han and Kloeden (see [16,17]) worked with two standard assumptions for simplifying the chemostat models: that the availability of the nutrient and its supply rate are fixed, and the tendency of microorganisms to adhere to surfaces is not taken into account. Doing so, they analysed the existence and uniqueness of solutions as well as the existence of a random attractor associated to the random dynamical system generated by the solution.

Generally, the chemostat model can be described by the following equation

$$\dot{x} = -qx + \frac{\tilde{R}}{K+y}xy, \quad \dot{y} = (\tilde{c} - y)q - \frac{\tilde{R}}{\tilde{a}(K+y)}xy, \quad (1)$$

where x is the mass concentration of organisms at time t ; y is the substrate concentration at time t ; q is the dilution rate; \tilde{c} is the concentration of the feed substrate; \tilde{a} is the yield constant; K is the half-saturation constant (Michaelis–Menten constant); and \tilde{R} is the maximal consumption rate of the nutrient (also the maximal specific growth rate of microorganisms). We assume that q, \tilde{R}, K and \tilde{c} are positive and that $\tilde{a} \neq 0$ (mainly the cases in which the equations have biological meaning). See chapter 1.2 of [18] and the references therein as well as [19] for the derivation and analysis for the basic equations of growth.

In this work we deal with the quadratic rational differential systems in \mathbb{R}^2 defined in (1). Taking the notation $R = \tilde{R}/q > 0$ and re-parametrizing the time by $dt/d\tau = (K+y)/q$ we get the system

$$x' = -Kx + (R-1)xy, \quad y' = (\tilde{c} - y)(K+y) - \frac{R}{\tilde{a}}xy. \quad (2)$$

Doing now a change of variables $X = \alpha x$, $Y = \beta y$ and a reparametrization of the time $t = \gamma\tau$ with $\alpha = R/(\tilde{a}K)$, $\beta = -1/K$ and $\gamma = \beta$, system (2) can be written as

$$X' = X(1 + (R-1)Y), \quad Y' = \tilde{c}/K - (\tilde{c} - K)Y/K + XY - Y^2.$$

For simplicity we rename the old parameters $R, K, \tilde{c}, \tilde{a}$ as the new parameters a, c in the form

$$(R-1) \rightarrow a, \quad \tilde{c}/K \rightarrow c \quad \text{with } a > -1, c > 0, b \in \mathbb{R}.$$

Moreover, we rename (X, Y) again as (x, y) . Doing so, we obtain the following system which will be used along the paper.

$$x' = x(1 + ay) = P(x, y), \quad y' = -y^2 + xy + (1 - c)y + c = Q(x, y). \quad (3)$$

Instead of working with system (3) we will consider the more general system of the form

$$x' = x(1 + ay) = P(x, y), \quad y' = -y^2 + xy + by + c = Q(x, y), \quad (4)$$

in which $b \in \mathbb{R}$ and at the end we will consider the case $b = 1 - c$ obtaining all the results for the Chemostat system (3) as a particular case of this one.

The vector field associated to system (4), called \mathcal{X} , is

$$\mathcal{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$

The first objective in this paper is to describe the dynamics of the Chemostat system (4) in \mathbb{R}^2 adding the infinity, i.e. to describe its phase portraits on the Poincaré disc (see Section 2.1 for details). Roughly speaking the Poincaré disc \mathbb{D}^2 is the closed unit disc centred at the origin of \mathbb{R}^2 , where we identified \mathbb{R}^2 with its interior and the boundary \mathbb{S}^1 is established as the infinity of \mathbb{R}^2 . A polynomial differential system in \mathbb{R}^2 can be extended in a unique analytic sense to the boundary \mathbb{D}^2 , i.e. \mathbb{S}^1 as it was done by Poincaré in [20]. Then, we call the Poincaré compactification to this extension to the infinity of a polynomial differential system. For a definition of topological equivalence see again Section 2.1.

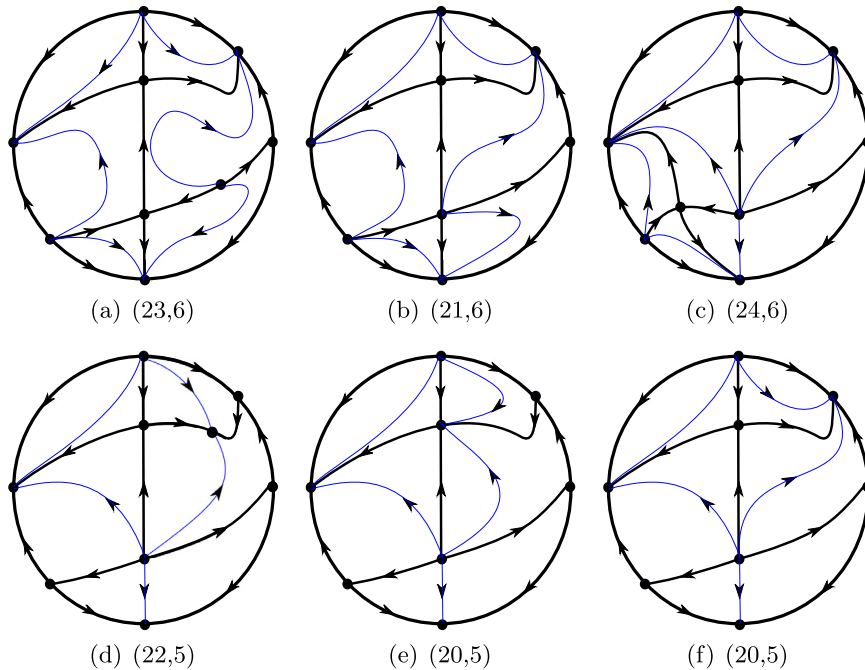


Fig. 1. Global phase portraits of system (4) on the Poincaré disc. (s, r) =(separatrices, canonical regions). $b^* = (-1 - 4a - 2a^2c + 4a^3c - a^4c^2)/(4a^2)$. (a): $a > 0$ and $b < (-1 + a^2c)/a$, (b): $a > 0$ and $b = (-1 + a^2c)/a$, (c): $a > 0$ and $b > (-1 + a^2c)/a$, (d): $a < 0$ and $b \neq (-1 + a^2c)/a$, (e): $a < 0$ and $b = (-1 + a^2c)/a$, (f): $a = 0$.

Theorem 1. The phase portraits on the Poincaré disc of system (4) are topologically equivalent to one of the phase portraits shown in Fig. 1.

Theorem 1 is proved in Section 3.

Note that we give the complete global dynamics in all the Poincaré ball, however for biological applications the dynamics to consider is which happens for $x > 0$ and $y > 0$ (the first quadrant of the Poincaré disc).

The second goal is the study of the integrability of system (4) (see Section 2.2 for details). We separate this study in two main results, the last one focused in the case $a = 0$.

Theorem 2. The following holds for system (4):

- (a) It has no polynomial first integrals.
- (b) It has no global analytic first integrals.
- (c) It has the following irreducible Darboux polynomials of degree 1:
 - (i) $f_{11} = a_{10}x$ with cofactor $k_{11} = 1 + ay$.
 - (ii) $f_{12} = a_{01}(c - \frac{1}{1+a}x + y)$ with cofactor $k_{12} = 1 - y$, for $b = 1 - c$.
- (d) If $a \notin \mathbb{Q}$, the unique invariant algebraic curves are f_{11} and f_{12} given in statement (c).
- (e) If $a = 1$ and $b \neq 1 - c$, the unique Darboux polynomials are $f_{11} = a_{10}x$ and $f_{21} = a_0(1 + 1/c xy)$.

Theorem 2 is proved in Section 4.

Theorem 3. The following hold for system (4) with $a = 0$:

- (a) The unique irreducible Darboux polynomial is x and:

Table 1
Darboux polynomials of system (4) with cofactor $k = M - 1 + y$ for $a = 0$.

Degree	Darboux polynomial	Cofactor	Parameters
2	$a_{11}((-6 + 5b - b^2) + (4 - 2b)x + (-3 + b)y - x^2 + xy)$	$2 - y$	$c = -2(b - 2)$
3	$-a_{21}((-60 + 47b - 12b^2 + b^3) + (36 - 21b + 3b^2)x - (-20 + 9b - b^2)y + (9 - 3b)x^2 - (8 - 2b)xy + x^2y - x^3)$	$3 - y$	$c = 9 - 3b$ and $b < 3$
4	$\frac{-(-7 + b)(-4 + b)}{b - 6} \frac{3}{2}x + \frac{-4(-4 + b)}{3(b - 6)(b - 5)} \frac{3}{2}xy + \frac{-7 + b}{b - 6} \frac{3}{2}y - \frac{2(b - 4)}{b - 6}x^2 + xy - \frac{4(b - 4)}{3(b - 6)(b - 5)}x^3 + \frac{x^2y}{b - 6} - \frac{1}{3(b - 6)(b - 5)}(x^4 - x^3y)$	$4 - y$	$c = -4(-4 + b)$

Table 2
Darboux polynomials of system (4) with cofactor $k = M + b + x - y$ for $a = 0$.

Degree	Darboux polynomial	Cofactor	Parameters
2	$a_{01}(1 + y + 1/(1 + b)xy)$	$c + x - y$	$c = b + 1$
3	$a_{11}((3 + b) + x + (3 + b)/2 y + xy + x^2y)/(2(2 + b))$	$b + 2 + x - y$	$c = 2(2 + b)$
	$a_{11}(-7 + x - 7/2 y + xy - 1/16 x^2y)$	$-8 + x - y$	$b = -10$ and $c = -16$
	$a_{11}(4 + 2x + 2y + 2xy + x^2y)/2$	$1 + x - y$	$b = -1$ and $c = 2$
	$a_{11}(12 + 4x + 6y + 4xy + x^2y)/4$	$2 + x - y$	$b = 0$ and $c = 4$
	$a_{11}(176 + 48x + 88y + 48xy + 9x^2y)/48$	$8/3 + x - y$	$b = 2/3$ and $c = 16/3$
	$a_{11}(24 + 6x + 12y + 6xy + x^2y)/6$	$3 + x - y$	$b = 1$ and $c = 6$
4	$a_{11}(60 + 10x + 30y + 10xy + x^2y)/10$	$5 + x - y$	$b = 3$ and $c = 10$
	$a_{11}\left((b + 5) + 2x + \frac{b+5}{3}y + \frac{1}{b+4}x^2 + xy + \frac{1}{b+4}x^2y + \frac{1}{3(b+3)(b+4)}x^3y\right)$	$b + 3 + x - y$	$c = 3(b + 3)$
	$a_{11}(72 + 36x + 24y + 6x^2 + 18xy + 6x^2y + x^3y)/18$	$2 + x - y$	$b = -1$ and $c = 6$
	$a_{11}(7854 + 2772x + 2618y + 297x^2 + 1386xy + 297x^2y + 27x^3y)/1386$	$11/3 + x - y$	$b = 2/3$ and $c = 11$

(i) $f_2 = p_1(x)y + p_0(x)$ with cofactor $K = M - 1 - y$ where $M \in \mathbb{N}$, $M \geq 2$, whenever $c = 1 + b - M(2 + b) + M^2$ and where

$$p_1(x) = \frac{1}{c}((M - 1)p_0(x) - xp_0'(x)), \quad p_0(x) = 1 + \sum_{j=1}^{M-1} \frac{\prod_{k=1}^j (M - k)}{j! \prod_{k=1}^j (k + 2 + b - 2M)} x^j;$$

(ii) $f_3 = p_1(x)y + p_0(x)$ with cofactor $K = M + b + x - y$ where $M \in \mathbb{N}$, $M \geq 1$, whenever $c = M(b + M)$ and where

$$p_1(x) = \frac{1}{c}(((M + b) + x)p_0(x) - xp_0'(x)), \quad p_0(x) = 1 + \sum_{j=1}^{M-1} \frac{\prod_{k=1}^j (M - k)}{j! \prod_{k=1}^j (2M + b - k)} x^j;$$

(b) It admits a Liouvillian first integral if and only if either $c = 1 + b - M(2 + b) + M^2$ with $M \in \mathbb{N}$, $M \geq 2$, or $c = M(b + M)$, with $M \in \mathbb{N}$, $M \geq 1$.

Theorem 3 is proved in Section 5. Examples of the Darboux polynomials given in **Theorem 3** statement (a) for different values of M are given in **Tables 1** and **2**.

Now we consider the Chemostat system (3).

Theorem 4. *The following hold for system (3).*

- (a) *The phase portraits on the Poincaré disc are topologically equivalent to one of the phase portraits shown in Fig. 2. All orbits are heteroclinic. Except for $a < 0$, the ω -limit of all orbits is an infinite singular point and when $a < 0$ there is a finite ω -limit point. The α - and ω -limit sets of all orbits are drawn in Fig. 2.*
- (b) *It has neither polynomial first integrals nor analytic first integrals.*
- (c) *It has the irreducible Darboux polynomials of degree 1 f_{11} and f_{12} provided in statement (c) of Theorem 2.*

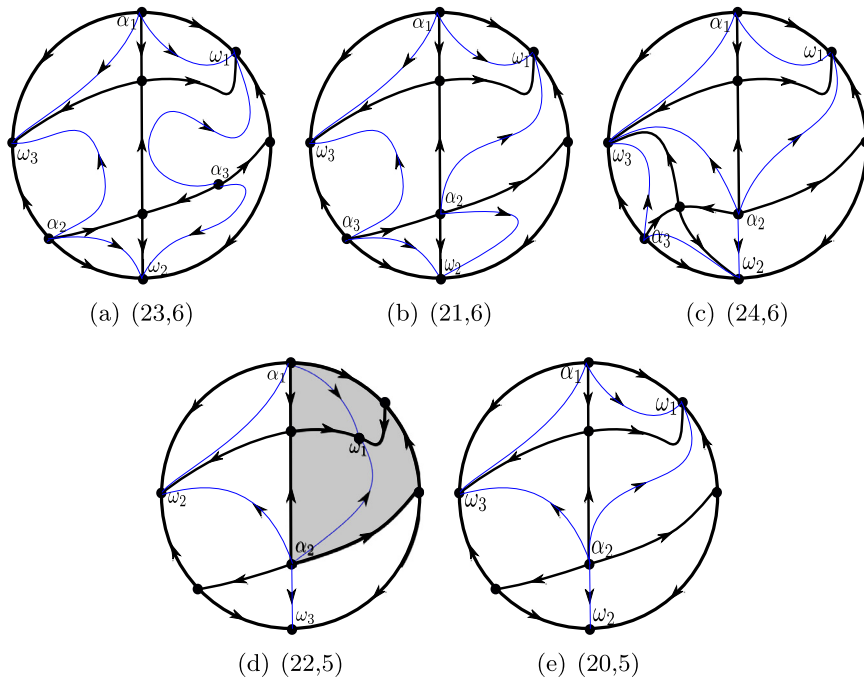


Fig. 2. Global phase portraits of system (3) on the Poincaré disc. (s, r) =(separatrices, canonical regions). α_i : α -limit points. ω_i : ω -limit points. $c^* = (1 + 2a)/a^2$. (a): $a > 0$ and $ac > 1$, (b): $a > 0$ and $ac = 1$, (c): $a > 0$ and $ac < 1$, (d): $a < 0$ (e): $a = 0$.

- (d) If $a \notin \mathbb{Q}$, the unique invariant algebraic curves are f_{11} and f_{12} given in statement (c).
 (e) If $a = 0$ it has an irreducible Darboux polynomial different from f_{11} and f_{12} if and only if $c = M$ with $M \in \mathbb{N}$, $M \geq 1$ and it is f_3/M where

$$f_3 = \sum_{j=0}^{M-1} \frac{M-j}{j!} x^j + y \sum_{j=0}^M \frac{1}{j!} x^j.$$

Moreover it admits a Liouvillian first integral if and only if $c = M$ with $M \in \mathbb{N}$, $M \geq 1$ and it can be taken as $e^x f_{12}/f_3$.

The proof of Theorem 4 is given in Section 6.

Since we have provided the α - and ω -limit sets of all the orbits, we can determine the initial and final evolution of the concentration of organisms and of the substrate concentration modelled by system (3). In particular the concentrations (x, y) in sector in grey in Fig. 2(d) will survive. Note that this region contains the first quadrant, which is the quadrant with biological meaning.

If the system is integrable, for example when $a = 0$ and $c = M$ with $M \in \mathbb{N}$, $M \geq 1$, then with the first integral we can describe completely the phase portraits of the system because the first integral allows to compute the explicit expression of the trajectories of the system. However the reverse is not true. Again, since from the first integral we know the α - and ω -limit sets of all the orbits, we can determine the biological meaning of the concentrations.

In Section 2 we provide the notations, basic definitions and results which will be useful in developing this article. We have also included an Appendix with the construction of the Puiseux series for system (4) that will be useful in the proofs of Theorems 2–4.

2. Preliminaries

In order to give a detailed proof of [Theorems 1–3](#) we give some definitions and results that will be useful in the next sections.

We consider a planar polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (5)$$

where the dot denotes the derivative with respect to the independent variable t , usually called *time*.

2.1. Phase portraits in the Poincaré disc

We describe for the polynomial differential system (5) in \mathbb{R}^2 (or equivalently its associated polynomial vector field $\mathcal{X} = (P, Q) = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$) the equations of the Poincaré compactification.

The degree \mathcal{X} is defined as $n = \max\{\deg(P), \deg(Q)\}$. In the disc \mathbb{D}^2 we consider the local charts (U_k, ϕ_k) and (V_k, ψ_k) for $k = 1, 2$ defined as follows

$$U_k = \{\mathbf{x} = (x, y) \in \mathbb{D}^2 : x_k > 0\}, \quad V_k = \{\mathbf{x} = (x, y) \in \mathbb{D}^2 : x_k < 0\},$$

the $\phi_k : U_k \rightarrow \mathbb{R}^3$ for $k = 1, 2$ are

$$\phi_1(\mathbf{x}) = \left(\frac{y}{x}, \frac{1}{x} \right) = (z_1, z_2), \quad \phi_2(\mathbf{x}) = \left(\frac{x}{y}, \frac{1}{y} \right) = (z_1, z_2),$$

and $\psi_k(\mathbf{x}) = -\phi_k(\mathbf{x})$.

The coordinates (z_1, z_2) have different meaning in each local chart, but the points of the infinity all have the coordinate $z_2 = 0$.

On the local chart U_1 of \mathbb{D}^2 the expression of the compactified analytical vector field $p(\mathcal{X})$ of the polynomial vector field \mathcal{X} of degree n is

$$z_2^n (-z_1 P + Q, -z_2 P), \quad (6)$$

where $P = P(1/z_2, z_2/z_1)$ and $Q = Q(1/z_2, z_2/z_1)$.

The expression of $p(\mathcal{X})$ in U_2 is

$$z_2^n (-z_1 Q + P, -z_2 Q), \quad (7)$$

where $P = P(z_1/z_2, 1/z_2)$ and $Q = Q(z_1/z_2, 1/z_2)$.

We call *infinite singular points* to the singular points of $p(\mathcal{X})$ which are on the boundary \mathbb{S}^1 of \mathbb{D}^2 , and we call *finite singular points* to the ones which are in the interior of \mathbb{D}^2 .

From (6) and (7) it follows that the infinity \mathbb{S}^1 of the Poincaré disc is invariant under the flow of the compactified vector field $p(\mathcal{X})$. Then for studying its infinite singular points we only need to study the singular points on the local chart U_1 and, in case that this be a singular point, the origin of the local chart U_2 .

The expression for $p(\mathcal{X})$ in the local chart V_k is the same as in U_k multiplied by $(-1)^{n-1}$. Therefore the infinite singular points appear on pairs diametrically opposite on \mathbb{S}^1 . For details of the Poincaré compactification see chapter 5 of [21].

We shall see how to characterize the phase portrait of a compactified vector field $p(\mathcal{X})$ in the Poincaré disc. For this it is necessary to introduce the following definition.

Definition 5. A *separatrix* of $p(\mathcal{X})$ being \mathcal{X} a polynomial vector field defined in \mathbb{R}^2 is an orbit which is either an equilibrium point, or a trajectory which lies in the boundary of a hyperbolic sector of a finite or an infinite equilibrium point, or any orbit contained at the infinity of the Poincaré disc, or a limit cycle.

Neumann [22] proved that the set formed by all separatrices of $p(\mathcal{X})$, denoted by $S(p(\mathcal{X}))$ is closed. We call canonical regions of \mathcal{X} or of $p(\mathcal{X})$ to the open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$. The union of $S(p(\mathcal{X}))$ plus one orbit chosen in each canonical region is called *separatrix configuration*. Two separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [23], Neumann [22] and Peixoto [24], who found it independently.

Theorem 6. *The phase portraits in the Poincaré disc \mathbb{D}^2 of two compactified polynomial vector fields $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent, if and only if, their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

2.2. Integrability

System (5) has a *first integral* on an open subset $\Omega \subset \mathbb{R}^2$, if there exists a continuous non-constant function $H(x, y) : \Omega \rightarrow \mathbb{R}$ constant on all solution curves $(x(t), y(t))$ of system (5) contained in Ω . The vector field \mathcal{X} associated to system (5) has a first integral H if

$$\mathcal{X}(H) = P \frac{\partial H}{\partial x} + Q \frac{\partial H}{\partial y} = 0. \quad (8)$$

The function H is said to be a *global analytic first integral* if $\Omega = \mathbb{R}^2$ and is said to be *local analytic first integral around a singularity* (x^*, y^*) if $\Omega \neq \mathbb{R}^2$ and it contains the singularity (x^*, y^*) .

The following result establishes conditions for the non existence of local analytic first integral, it was given by Poincaré in [25] and in [26] we can find a proof.

Theorem 7. *Assume that the eigenvalues $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ at some singular point of system (5) do not satisfy any resonant condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0 \text{ for } k_1, k_2 \in \mathbb{Z}_+ \text{ with } k_1 + k_2 > 0.$$

Then, system (5) has no local analytic first integral around the singular point.

A similar result for a zero eigenvalue can be found in [27], and is presented in the next theorem.

Theorem 8. *Assume that the eigenvalues λ_1 and λ_2 at some singular point (\bar{x}, \bar{y}) of system (5) satisfy that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then system (5) has no local analytic first integrals around the singular point (\bar{x}, \bar{y}) if it is isolated.*

2.3. Liouvillian integrability

A *Darboux polynomial* of system (5) is a polynomial f such that

$$\mathcal{X}(f) = P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = Kf, \quad (9)$$

where K is called the cofactor of f and it has degree at most $d - 1$, where d is the degree of system (5). The curve associated to $f = 0$ is called an *invariant algebraic curve*, and is invariant by the dynamics because a trajectory on it, always lies in the surface.

A function $F = \exp(g/f)$ is an *exponential factor* of system (5) if it satisfies

$$\mathcal{X}(F) = P \frac{\partial F}{\partial x} + Q \frac{\partial F}{\partial y} = LF, \quad (10)$$

where L is called the cofactor of F and it has degree $d - 1$. Furthermore f is constant or it is a Darboux polynomial, and $\mathcal{X}(g) = Kg + Lf$ being K the cofactor of f .

A function V is an *inverse integrating factor* of system (5) if it satisfies

$$\mathcal{X}(V) = P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} \right) V. \quad (11)$$

Note that if V is a polynomial, then it is a Darboux polynomial with cofactor the divergence of the system.

The following result shows how to find Darboux and Liouvillian first integrals (for a proof see [21]).

Theorem 9. Assume that a polynomial differential system \mathcal{X} of degree d defined in \mathbb{C}^2 admits p Darboux polynomials f_i with cofactors K_i , $i = 1, \dots, p$, and q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j , $j = 1, \dots, q$. Then, the following statements hold:

(a) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$$

if and only if the function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}, \quad (12)$$

is a first integral of \mathcal{X} . Such a function is called a Darboux function.

(b) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = \operatorname{div}(\mathcal{X})$$

if and only if the function of Darboux type (12) is an inverse integrating factor of \mathcal{X} . Here $\operatorname{div}(\mathcal{X})$ stands for the divergence of the system.

The following result proved in [28] will be useful to prove the results related with the Liouvillian first integrals. System (5) is said to be *Liouvillian integrable* if it has a Liouvillian first integral.

Theorem 10. A polynomial differential system of degree two defined in \mathbb{C}^2 has a Liouvillian first integral if and only if it has an integrating factor which is a Darboux function.

2.4. Irreducible invariant algebraic curves and puseux series

Given the differential system (5), we will use the Puiseux series in order to study its invariant Darboux polynomials (or its invariant algebraic curves). In this sense, we follow the method development in [29] (see also [30]) related with Puiseux series near infinite points, to analyse the maximum degree of all irreducible invariant algebraic curves and to obtain them. Although this method is not ours, we summarize it here for clearness.

A Puiseux series near $x = \infty$ is

$$y(x) = \sum_{k=0}^{+\infty} b_k x^{\frac{l_0}{n_0} - \frac{k}{n_0}}, \quad \text{where } l_0 \in \mathbb{Z}, n_0 \in \mathbb{N}. \quad (13)$$

The objective is to use Puiseux series that satisfies the following equation

$$P(x, y)y_x - Q(x, y) = 0, \quad (14)$$

where y is taken as dependent variable and x as independent, or the equation

$$P(x, y) - Q(x, y)x_y = 0, \quad (15)$$

where x is taken as dependent variable and y as independent.

It is possible construct the Puiseux series considering the Newton polygon related to Eqs. (14) and (15) (see details in section 3 of [30]). For this, we consider Eq. (14) (analogous analysis can be done for (15)) as the sum of monomials of the form

$$M[y(x), x] = cx^l y^{j_0} \left\{ \frac{dy}{dx} \right\}^{j_1}, \quad c \in \mathbb{C}, \quad j_1 = 0, 1. \quad (16)$$

Considering these monomials, we construct the Newton polygon on \mathbb{R}^2 with vertices given by the following map $q : M \mapsto \mathbb{R}^2$ that satisfies

$$cx^{q_1 \cdot q_2} \mapsto q = (q_1, q_2), \quad \frac{dy}{dx} \mapsto q = (-1, 1), \quad q(M_1 M_2) = q(M_1) + q(M_2), \quad (17)$$

where c is a constant and M_1, M_2 are different monomials. The set of all $p \in \mathbb{R}^2$ corresponding to the monomials is denoted by $S(E)$ and the convex hull of $S(E)$ is the Newton polygon of (14). To continue it is necessary introduce the following definition.

Definition 11. We say that an algebraic ordinary differential equation (14) has a dominant balance $E_0[y(x), x]$, where this expression denotes a polynomial in $x, y(x)$ and its derivatives, related to the point $x = \infty$ if the following conditions are satisfied:

- each differential monomial $M[y(x), x]$ appearing in $E_0[y(x), x]$ is also involved in the original equation (14);
- there exists a power function $y(x) = b_0 x^r$ with $b_0 \neq 0, r \in \mathbb{C}$ such that all the monomials $M[y(x), x]$ of $E_0[y(x), x]$ have the same exponent $s \in \mathbb{C}$ in the relation $M[b_0 x^r, x] = C_m x^s$;
- for all the monomials $L[y(x), x]$ of Eq. (14) that are not involved in $E_0[y(x), x]$ we obtain $L[b_0 x^r, x] = C_L x^{P_L}$, where $\operatorname{Re} P_L < \operatorname{Re} s$.

With the Newton polygon we can identify the balances $E_0[y(x), x]$ selecting all the different monomials that generate its vertices and edges, and obtain power solutions $y(x) = b_0 x^r$, where $r \in \mathbb{Q}$ of the equation $E_0[y(x), x] = 0$.

Since we are actually in \mathbb{R}^2 we consider the origin $(q_1, q_2) = (0, 0)$ and for an edge of the Newton polygon we denote ψ the angle between its external normal and the unit vector \vec{e}_{q_1} directed along the axis q_1 . On the other hand, for a vertex of the Newton polygon we consider the angle ψ between the vector $\delta(1, \operatorname{Re} r)$ and \vec{e}_{q_1} , where r is the exponent in $y = c x^r$, and $\delta = \pm 1$ is such that $\delta(1, \operatorname{Re} r)$ lies in the region bounded by the rays associated to the external normals of the edges attached to the vertex. The balance is related to the point $x = \infty$ if $0 \leq \psi \leq \pi$ and we can obtain $x \rightarrow \infty$ for the corresponding power asymptotics. Whenever the Newton polygon degenerates to an edge or a vertex it is necessary to consider both normals (for the edge) and both vectors $\pm(1, \operatorname{Re} r)$ (for the vertex). For more details of balances with power solutions related to $x = \infty$ see again [30].

To find the Puiseux series we have to calculate the Gâteaux derivative of the balance $E_0[y(x), x]$ at the solution $y(x) = b_0 x^r$:

$$\frac{\delta E_0}{\delta y}[b_0 x^r] = \lim_{t \rightarrow 0} \frac{E_0[b_0 x^r + t x^{r-j}, x] - E_0[b_0 x^r, x]}{t} = V(j) x^{\bar{r}} \quad (18)$$

The zeros of $V(j)$ are called *Fuchs indices* of the balance $E_0[y(x), x]$ and its power solution $b_0 x^r$. The Fuchs indices in \mathbb{Q}^+ take relevance on the construction of the Puiseux series. The coefficient l_0 of the

Puiseux series (13) is given by $l_0 = rn$, where $n = \text{lmc}(q_1, q_2, \dots, q_{m_0}, r_2)$ where the Fuchs indices $0 < j_1 < j_2 < \dots < j_{m_0} \in \mathbb{Q}^+$ are of the form $j_0 = p_m, q_m$ and $r = r_1/r_2$ (where $p_m, q_m \in \mathbb{N}$, $(p_m, q_m) = 1$, $1 \leq m \leq m_0$, $r_1 \in \mathbb{Z}$, and $r_2 \in \mathbb{N}$). Substituting series (13) into (5) one can find relations for the coefficients of the Puiseux series and found it explicitly.

The number and type of the Puiseux series satisfying (14) and (15) are very useful in order to establish a bound for the degree of irreducible invariant algebraic curves of system (5) and consequently to obtain them explicitly. In [31] it was established the relation between Puiseux series and an irreducible invariant algebraic curve. The results are the following:

Theorem 12. *Let $F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$, $F_y \not\equiv 0$ be an irreducible Darboux polynomial of the polynomial vector field \mathcal{X} and its related dynamical system (5). Then $F(x, y)$ takes the form*

$$F(x, y) = \left\{ \mu(x) \prod_{j=1}^N \{y - y_j(x)\} \right\}_+, \quad N \in \mathbb{N}, \quad (19)$$

where $\mu(x) \in \mathbb{C}[x]$ and $y_1(x), y_2(x), \dots, y_N(x)$ are pairwise distinct Puiseux series in a neighbourhood of the point $x = \infty$ that satisfy Eq. (14). The symbol $\{W(x, y)\}_+$ means that we take the polynomial part of the expression $W(x, y)$. Moreover, the degree of $F(x, y)$ with respect to y does not exceed the number of distinct Puiseux series of the form (13) satisfying Eq. (14) whenever the latter is finite.

Theorem 13. *Suppose that $y_1(x), y_2(x), \dots, y_N(x)$ are pairwise distinct Puiseux series in a neighbourhood of the point $x = \infty$ that satisfy Eq. (14). Let the polynomial $\mu(x) \in \mathbb{C}[x]$ be such that the following expression*

$$F(x, y) = \mu(x) \prod_{j=1}^N \{y - y_j(x)\} \quad (20)$$

is an irreducible in $\mathbb{C}[x, y]$ polynomial, i.e. the non-polynomials part in (20) vanished producing the polynomial $F(x, y)$, then $F(x, y)$ is an irreducible Darboux polynomial of the polynomial vector field \mathcal{X} and its related dynamical system (5).

3. Proof of Theorem 1

To prove Theorem 1 we study the phase portraits of (2) in the Poincaré disc. First we note that $\dot{x}|_{x=0} = 0$, so the y -axis is invariant, and $\dot{x}|_{y=0} = x$, i.e. for $x > 0$ the flow is increasing in the x direction and for $x < 0$ the flow is decreasing in this direction. On the other hand, $\dot{y}|_{y=0} = c > 0$ so in the x -axis the flow is increasing in the y -direction, thus the semidisc $y > 0$ do not have orbits leaving it.

We separate the study in two cases: $a = 0$ and $a \neq 0$.

3.1. Case $a \neq 0$

System (4) has three finite equilibria, namely $e_1 = (0, (b + \sqrt{b^4 + 4c})/2)$, $e_2 = (0, (b - \sqrt{b^4 + 4c})/2)$ and $e_3 = ((-1 - ab + a^2c)/a, -1/a)$. Note that e_1 and e_2 are always different since $c > 0$. On the other hand, if $b = (-1 + a^2c)/a$ and $a > 0$ it holds that $e_3 = e_2$; and if $b = (-1 + a^2c)/a$ and $a < 0$ then $e_3 = e_1$.

The linear part of system (4) is

$$\begin{pmatrix} ay + 1 & ax \\ y & b + x - 2y \end{pmatrix}. \quad (21)$$

Considering (21) we analyze the stability of the finite equilibria. The equilibrium $e_1 = (0, (b + \sqrt{b^4 + 4c})/2)$ has eigenvalues $\lambda_1 = -\sqrt{b^2 + 4c} < 0$ and $\lambda_2 = (a\sqrt{b^2 + 4c} + ab + 2)/2$. So it is a stable node if $-1 < a < 0$

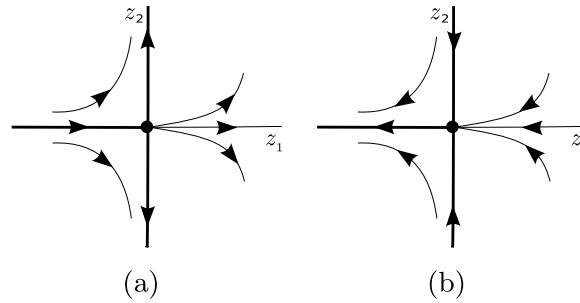


Fig. 3. Local phase portraits at the semi-hyperbolic singular point $(0, -1/a)$ of system (4) with $b = (-1 + a^2c)/a$. (a) when $a > 0$, and (b) when $a < 0$.

Table 3

Local phase portraits at the finite singular points in the order e_1 - e_2 - e_3 for $a \neq 0$. UN: Unstable node, SN: stable node, SF: Stable Focus, UF: Unstable Focus, S/N: saddle-node, S: saddle, S1: saddle with local stable manifold on x -axis, S2: saddle with local stable manifold on y -axis. $b^* = (-1 - 4a - 2a^2c + 4a^3c - a^4c^2)/(4a^2)$.

a	$b < b^*$	$b^* \leq b < (-1 + a^2c)/a$	$b = (-1 + a^2c)/a$	$b > (-1 + a^2c)/a$
$a > 0$	S1 - S2 - UF	S1 - S2 - UN	S1 - S/N	S1 - UN - S
$a < 0$	S1 - UN - SF	S1 - UN - SN	S/N - UN	SN - UN - S

and $b > (-1 + a^2c)/a$, and it is a saddle (with the local stable manifold located on the y -axis) if $-1 < a < 0$ and $b < (-1 + a^2c)/a$, or if $a > 0$. Due to the fact that the case $b = (-1 + a^2c)/a$ is special, we do it its study later.

The equilibrium e_2 has the eigenvalues $\lambda_1 = 1 + a(b - \sqrt{b^2 + 4c})/2$ and $\lambda_2 = \sqrt{b^2 + 4c} > 0$. So e_2 is an unstable node if $-1 < a \leq 0$, or if $a > 0$ and $b > (-1 + a^2c)/a$, and a saddle (with the local unstable manifold located on the y -axis) if $b < (-1 + a^2c)/a$ and $a > 0$.

Finally, the equilibrium e_3 has eigenvalues

$$\frac{1 + a^2c \pm \sqrt{1 + a(4 + 4ab + ac(2 + a(-4 + ac)))}}{2a}.$$

So for $a > 0$ it is an unstable node if $(-1 - 4a - 2a^2c + 4a^3c - a^4c^2)/(4a^2) \leq b < (-1 + a^2c)/a$, a stable node if $b > (-1 + a^2c)/a$, and an unstable focus (counterclockwise) if $b < (-1 - 4a - 2a^2c + 4a^3c - a^4c^2)/(4a^2)$. On the other hand, for $a < 0$ the singular point e_3 is a stable node if $(-1 - 4a - 2a^2c + 4a^3c - a^4c^2)/(4a^2) \leq b < (-1 + a^2c)/a$, a saddle if $b > (-1 + a^2c)/a$, and a stable focus (counterclockwise) if $b < (-1 - 4a - 2a^2c + 4a^3c - a^4c^2)/(4a^2)$.

Now, we complete the study of the finite equilibria taking $b = (-1 + a^2c)/a$. If $a > 0$ it holds that $e_1 = (0, ac)$ and $e_2 = (0, -1/a)$. Recall that under this condition we have only two finite equilibria. Thus, it follows from (21) that e_1 is a saddle (with eigenvalues $1 + a^2c$ and $-(1/a) - ac$), and e_2 is semi-hyperbolic (the non zero eigenvalue is $1/a + ac > 0$). In order to obtain the local phase portrait of e_2 we apply Theorem 2.19 in [21]. Doing so, we get that it is a saddle-node, and since $a > 0$, it has one attracting sector for $x > 0$, and two hyperbolic sectors for $x < 0$. This local phase portrait is shown in Fig. 3(a). On the other hand, if $a < 0$ we have that $e_2 = (0, ac)$ is an unstable node (as before its eigenvalues are $1 + a^2c$ and $-(1/a) - ac$), and $e_1 = (0, -1/a)$ is semi-hyperbolic (the non zero eigenvalue is $1/a + ac < 0$). Using again Theorem 2.19 in [21] we get that e_1 is a saddle-node, and since the flow on the x -axis is decreasing, and the flow on the y -axis is attracting we get that the local phase portrait is as shown in Fig. 3(b).

We summarize the local phase portraits of the finite equilibria for $a \neq 0$ in Table 3.

Next, we study the infinite equilibria. Considering the Poincaré compactification, the associated system on the local chart U_1 is

$$z'_1 = z_1 - (1 + a)z_1^2 + (-1 + b)z_1z_2 + cz_2^2, \quad z'_2 = -z_2(az_1 + z_2). \quad (22)$$

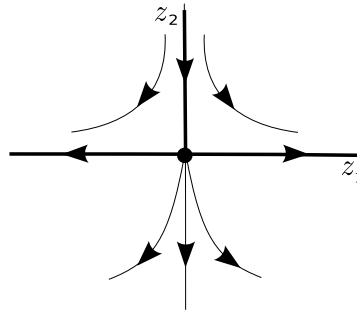


Fig. 4. Local phase portrait at the semi-hyperbolic singular point $(0,0)$ of system (22).

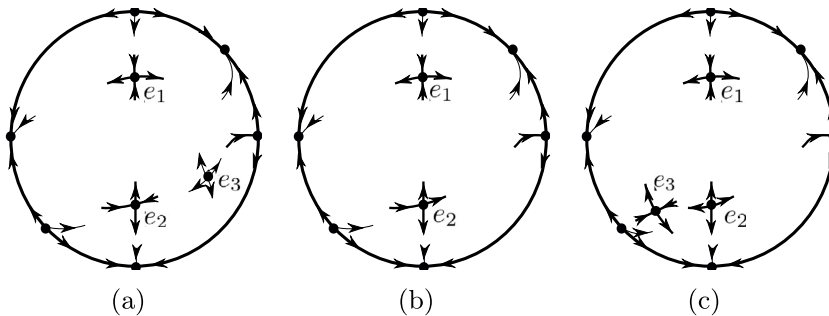


Fig. 5. Local phase portraits at the equilibrium of system (4) on the Poincaré disc for $a > 0$. (a): $b < (-1 + a^2c)/a$ (if $b < b^*$ then e_3 is a focus), (c): $b = (-1 + a^2c)/a$, (d): $b > (-1 + a^2c)/a$.

At infinity, for $z_2 = 0$, system (22) has two equilibria, the origin and $(1/(1+a), 0)$. The linear part at $z_2 = 0$ is

$$\begin{pmatrix} 1 - 2(a+1)z_1 & (b-1)z_1 \\ 0 & -az_1 \end{pmatrix}.$$

The origin is a semi-hyperbolic equilibrium, and $(1/(1+a), 0)$ is a stable node for $a > 0$ and a saddle for $a < 0$. For the origin of U_1 , applying Theorem 2.19 of [21] we get that it is a saddle-node, and since the flow is decreasing in the z_1 -axis, its local phase portrait is as shown in Fig. 4.

In U_2 the associated system is

$$z_1' = z_1(1 + a - z_1 + z_2 - bz_2 - cz_2^2), \quad z_2' = -z_2(-1 + z_1 + bz_2 + cz_2^2) \quad (23)$$

The origin of U_2 is an unstable node (the associated eigenvalues are 1 and $1+a$). In the local charts V_1 and V_2 we have the same phase portraits than in U_1 and U_2 but with the opposite stability due to the fact that the maximum degree of system (4) is two.

From the previous analysis, we have the local phase portraits in the Poincaré disc, as shown in Figs. 5 and 6.

To complete the global phase portraits we recall that the y -axis is invariant under the flow (so the orbits cannot cross it), and that $\dot{y}|_{y=0} = c > 0$. These informations will be useful in order to establish the connections of the separatrices on the Poincaré disc.

First, we focus on the region $x < 0$. Note that the finite equilibria e_1 , when it is a saddle ($a > 0$, or $a < 0$ and $b < (-1 + a^2c)/a$), or a saddle-node ($a < 0$ and $b = (-1 + a^2c)/a$), we can see from Figs. 5 and 6 that the unstable separatrix on $x < 0$ (when there are no other finite equilibria on the second quadrant) must connect with the attracting sector generated by the saddle-node of the origin of U_1 located at the end of the negative x -axis (note that it cannot connect with e_2 because the flow on the x -axis in the y -direction

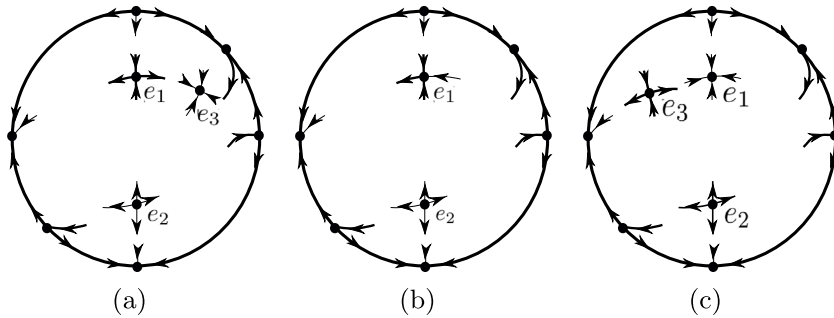


Fig. 6. Local phase portraits at the equilibrium of system (4) on the Poincaré disc for $a < 0$. (a): $b < (-1 + a^2c)/a$ (if $b < b^*$ then e_3 is a focus), (c): $b = (-1 + a^2c)/a$, (d): $b > (-1 + a^2c)/a$.

is increasing). For $a > 0$ and $b \leq (-1 + a^2c)/a$ it holds that e_2 has a stable separatrix on this region, so it must connect with the other infinite equilibrium which is a repeller. For $a < 0$ the finite equilibrium e_2 is an unstable node, and the infinite equilibrium on $U_2 \cap V_2$ is a saddle. Then its stable separatrix (which is not at infinity) must connect with e_2 .

Finally when $b > (-1 + a^2c)/c$ the finite equilibrium e_3 is also in $x < 0$. If additionally $a > 0$ the equilibrium e_3 is in the third quadrant, then as before, the unstable separatrix of e_1 must connect with the stable node at the end of the negative x -axis, because these orbits cannot cross the x -axis. In the third quadrant the saddle e_3 must connect one stable separatrix with e_2 which is an unstable node, and since there are no more finite equilibria in this region, the other separatrices of e_3 must connect with the infinite equilibria.

On the other hand, if $a < 0$, e_3 is a saddle located in the second quadrant, so one unstable separatrix of e_3 must connect with the stable node e_1 , and one stable separatrix of e_3 must connect with the unstable node e_2 . The remaining separatrices must connect with the infinite equilibria on the second quadrant. Finally the separatrix of the infinite equilibria on $U_2 \cap V_2$ must connect with the finite unstable node e_2 .

Now we consider the region $x > 0$. We can do a similar analysis. First note that for $a > 0$ and in the first quadrant, the unstable separatrix of e_1 must connect with the infinite stable node in $U_1 \cap V_1$, because there are no more finite equilibria and the orbits cannot cross the x -axis. In a similar way for $a < 0$ and in the fourth quadrant, the stable separatrix of the saddle-node located on $U_1 \cap V_2$ must connect with the unstable node e_2 .

Now we consider the case in which either e_3 does not exist or is in the region $x > 0$ studied above. For $b = (-1 + a^2c)/a$ there are no more finite equilibria than e_1 and e_2 . Thus when $a > 0$ the stable separatrix at the infinite saddle located at the positive end of the x -axis must connect with the repelling sector of e_2 on $x > 0$. This completes the analysis of the global phase portrait for $a > 0$ and $b = (-1 + a^2c)/a$ and it is shown in Fig. 1(b). Under the hypothesis $b = (-1 + a^2c)/a$ but when $a < 0$ the unstable separatrix of the saddle e_1 must connect with the infinite saddle in $U_1 \cap V_1$. This completes the analysis of the global dynamics under these conditions and its phase portrait is topologically equivalent to the one shown in Fig. 1(e). Analogously for $b > (-1 + a^2c)/a$, since there are no finite equilibria for $x > 0$ it holds that the remaining separatrix at infinity must connect with the node e_2 for $a > 0$, and with the node e_1 for $a < 0$. The global dynamics are topologically equivalent to the phase portraits shown in Figs. 1(c) and 1(d).

Now we consider the cases in which e_3 is in the region $x > 0$. If $b < b^*$ the equilibrium e_3 is a focus. For $a > 0$ is unstable and it is in the fourth quadrant. Then the stable separatrix of e_2 must connect with this unstable focus, and the same α -limit has the stable separatrix of the infinite equilibria given by the origin of U_1 . Then the global phase portrait of system (4) for $a > 0$ and $b < b^*$ is as shown in Fig. 1(a). Analogously, for $a < 0$ and $b < b^*$ we have a stable focus on the first quadrant. Then the unstable separatrix of e_1 and the unstable separatrix of the infinite equilibrium at $U_1 \cap V_1$ go to e_3 . So, the global phase portrait is as in

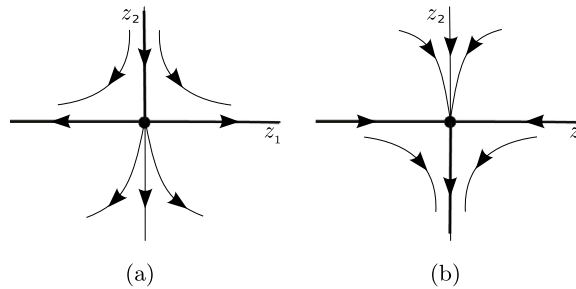


Fig. 7. Local phase portrait at the semi-hyperbolic singular points of system (25). (a) for $(0, 0)$, and (b) for $(1, 0)$.

Fig. 1(a) for $a > 0$ and Fig. 1 (d) for $a < 0$. Finally, if $b^* \leq b(-1 + a^2c)/c$ the finite equilibrium e_3 is a node. For $a > 0$ is a stable node located in the fourth quadrant, and the separatrix of e_2 on this quadrant and the separatrix of the infinite saddle must connect with e_3 . Analogously for $a < 0$ (reversing orientation) the equilibrium e_3 is a stable node in the first quadrant and it is the ω -limit of the unstable separatrices of e_1 and the infinite saddle on $U_1 \cap U_2$. The global phase portrait is topologically equivalent to one of the those shown in Fig. 1(a) and 1(d), for $a > 0$ and $a < 0$, respectively, with a focus on e_3 .

3.2. Case $a = 0$

System (4) for $a = 0$ takes the form

$$x' = x = P(x, y), \quad y' = -y^2 + xy + by + c = Q(x, y). \quad (24)$$

It has two finite equilibria, namely $e_1 = (0, (b + \sqrt{b^4 + 4c})/2)$ and $e_2 = (0, (b - \sqrt{b^4 + 4c})/2)$. Considering the linear part given in (21), we obtain that the eigenvalues of e_1 are 1 and $-\sqrt{b^2 + 4c}$. Thus it is a saddle with stable separatrix on the y -axis. For e_2 we have that is an unstable node with eigenvalues 1 and $\sqrt{b^2 + 4c}$,

For the study of the infinite equilibria we consider the Poincaré compactification. The associated system on the local chart U_1 is

$$z_1' = z_1 - z_1^2 + (-1 + b)z_1z_2 + cz_2^2, \quad z_2' = -z_2^2. \quad (25)$$

As in the case $a \neq 0$, in the infinity, for $z_2 = 0$ system (25) has two equilibria, the origin and $(1, 0)$. The linear part at $z_2 = 0$ is

$$\begin{pmatrix} 1 - 2z_1 & (b - 1)z_1 \\ 0 & 0 \end{pmatrix}.$$

The origin and $(1, 0)$ are semi-hyperbolic. Note that the non zero eigenvalue is $1 > 0$ for the origin, and $-1 < 0$ for $(1, 0)$. Applying Theorem 2.19 of [21], we get that they are both saddle-nodes that decrease in the z_1 -axis, but $(0, 0)$ is unstable on the z_2 -axis while $(1, 0)$ is stable in the z_2 -axis (see Fig. 7 for their local phase portrait).

The origin of U_2 is an unstable node, whose associated eigenvalues are both equal to 1. From the previous analysis, we have the local phase portrait on the Poincaré disc, as shown in Fig. 8.

To complete the global phase portrait we note that the y -axis is invariant under the flow, and that the flow is increasing in the y -direction on the x -axis. Then, due to the fact that there are no more finite equilibria we have that the unstable separatrices of e_1 must connect with the stable nodes at infinity in each region $x > 0$ and $x < 0$. On the other hand the stable separatrices of the infinite saddles on V_2 must connect with the unstable node e_2 . These informations force that the global phase portrait is as shown in Fig. 1(f).

The proof of Theorem 1 is completed.

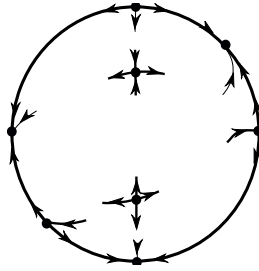


Fig. 8. Local phase portrait of system (24).

4. Proof of Theorem 2

We separate the proof in different subsections.

4.1. Proof of statements (a) and (b) of Theorem 2

First we prove statement (a). Let $g(x, y)$ be a polynomial first integral of degree $m \in \mathbb{N}$ of system (2). We write $g = \sum_{i=1}^m g_i(x, y)$, where g_i is a homogeneous polynomial of degree i with $g_m \neq 0$.

The differential equation of degree $m+1$ of $\mathcal{X}(g) = 0$ is

$$y \left(ax \frac{\partial g_m}{\partial x} + (-y+x) \frac{\partial g_m}{\partial y} \right) = 0, \quad (26)$$

whose solution is

$$g_m(x, y) = C_m \left(x^{1/a} \frac{-x+y+ay}{1+a} \right),$$

for some arbitrary function C_m . Since g_m is a homogeneous polynomial of degree m , we have

$$g_m(x, y) = c_m x^{l/a} (-x+y+ay)^l,$$

where $l \in \mathbb{N}$, $c_m \neq 0$, $l/a + l = m$, or equivalently, $l = am/(1+a)$.

Concerning the terms of degree m we have the partial differential equation

$$y \left(ax \frac{\partial g_{m-1}}{\partial x} + (-y+x) \frac{\partial g_{m-1}}{\partial y} \right) + x \frac{\partial g_m}{\partial x} + by \frac{\partial g_m}{\partial y} = 0.$$

Solving it, we get

$$\begin{aligned} g_{m-1} &= -\frac{a+1}{a} c_m l x^{l/a} (ay-x+y)^{l-1} \left(a(b-1) + (1+a) {}_2F_1 \left(1, \frac{1}{1+a}; 1 + \frac{1}{1+a}; \frac{-x}{-x+y+ay} \right) \right) \\ &+ C_{m-1} \left(\frac{x^{1/a}(ay-x+y)}{a+1} \right), \end{aligned}$$

where C_{m-1} is an arbitrary function, and

$${}_2F_1 \left(1, \frac{1}{1+a}; 1 + \frac{1}{1+a}; \frac{-x}{-x+y+ay} \right) = \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{R}} \left(\frac{-x}{-x+ay+y} \right)^n,$$

is called the Hypergeometric function. Since $a > -1$ we have $1/(1+a) > 0$, so the Hypergeometric function ${}_2F_1$ is not a polynomial. Hence, we must necessarily have

$$(1+a)c_m l = 0,$$

but since $a > -1$, this implies that $g_m = 0$ or $g_m = k$. Therefore, there does not exist a polynomial first integral g . This proves statement (a) of [Theorem 2](#).

Now, we focus on statement (b) of [Theorem 2](#). In this sense, if $a = 0$, the singular point e_2 is always an unstable node, so equation of [Theorem 7](#) cannot be satisfied. This implies that system (4) for $a = 0$ has no local first integral around e_2 and consequently it has no global analytic first integrals.

Assume now $a \neq 0$ and consider the singular points e_2 and e_3 of system (4). The equilibrium point e_3 can satisfy the equation of [Theorem 7](#) only if $b > (-1 + a^2c)/a$ (because e_3 is a saddle under this hypothesis), whereas the equilibrium point e_2 can satisfy the equation of [Theorem 7](#) only if $a > 0$ and $b < (-1 + a^2c)/a$ (in which case it is a saddle). It is clear that the singular points e_2 and e_3 cannot have a local first integral simultaneously. Thus, we can conclude that there does not exist a global first integral when either $b > (-1 + a^2c)/a$, or $a > 0$ and $b < (-1 + a^2c)/a$. Note that for $b = (-1 + a^2c)/a$ with $a \neq 0$ and $a \in (-1, 0)$ with $b < (-1 + a^2c)/a$ the system does not have a finite saddle. Furthermore, for the case $b = (-1 + a^2c)/a$ with $a \neq 0$ it holds that for $a > 0$ the finite equilibrium e_1 (for $a < 0$ the finite equilibrium e_2) is semi-hyperbolic and isolate, and so it follows from [Theorem 8](#) that it cannot have a local analytic first integral. Finally for $a \in (-1, 0)$ with $b < (-1 + a^2c)/a$ the finite equilibrium e_2 is an unstable node. Then the equation on [Theorem 7](#) cannot be satisfied, implying that there does not exist a local first integral around this point. Since in any case we have at least one singular point for which there does not exist a local analytic first integral, it is clear that there does not exist a global analytic first integral for system (4). This proves statement (b) of [Theorem 2](#).

4.2. Proof of statements (c), (d) and (e) of [Theorem 2](#)

The proof of statement (c) follows in a direct way.

To prove statements (d) and (e) we will use the form of the Puiseux series obtained in the [Appendix](#) and [Theorem 13](#) (see Eq. (20)).

Take $a \notin \mathbb{Q}$. Using the [Appendix](#) and [Theorem 12](#) we get that $F(x, y)$ can be written as the following two forms

$$F(x, y) = \mu(x) \left(y + \frac{c}{x} + \frac{c(b+1)}{x^2} + \dots \right)^{j_1} \left(y - \frac{x}{1+a} - (b-1) + \dots \right)^{j_2}, \quad (27)$$

where $j_i = 0$ or 1 , for $i = 1, 2$, and

$$F(x, y) = \nu(y) (x - (1+a)y + (1+a)(b-1) + \dots)^{j_1}, \quad (28)$$

where $j_1 = 0$ or 1 .

With these informations we can conclude that in this case a Darboux polynomial has at most degree two in y and degree 1 in x . Taking $F(x, y) = \alpha_1 y^2 + \alpha_2 xy + \alpha_3 y + \alpha_4 x + \alpha_5$ with $\alpha_i \in \mathbb{C}$ and imposing that it satisfies relation (9), we can prove readily statement (d) in [Theorem 2](#).

Now consider $a = 1$ and $b \neq 1 - c$. Again it follows from the [Appendix](#) and [Theorem 12](#) that

$$F(x, y) = \mu(x) \left(y + \frac{c}{x} + \frac{c(b+1)}{x^2} + \dots \right)^{j_1}, \quad (29)$$

where $j_1 = 0$ or 1 .

On the other hand, from the Puiseux series studied with $x(y)$ (see the [Appendix](#)) we found one Puiseux series completely determined and three Puiseux series with an indeterminate coefficient. Thus this Puiseux series does not provide good information. From (29) we have that a Darboux polynomial has at most degree one in y . We can write it as $F(x, y) = P_0(x) + yP_1(x)$. Then, substituting this expression on $P(x, y)F_x + Q(x, y)F_y = kF$, with cofactor $k = k_0 + k_1x + k_2y$ and comparing coefficients we get

$$y^2(P_1'(x)x - P_1(x)) = y^2(k_2P_1(x)) \text{ that is } P_1(x) = k_0x^{k_2+1}.$$

Now we consider the coefficient associated to y

$$P'_0(x)x + P'_1(x)x + (b+x)P_1(x) = k_2P_0(x)$$

that is

$$P_0(x) = x^{k_2}(c_1 + (-1 - b + k_0 - k_2)x^1 + (k_1 - 1)/2x^2).$$

From the relations with the Puiseux series in (29) we have $k_1 = 1$ and $b = -1$. Then

$$P_0(x) = x^{k_2}(c_1 + k_0x - k_2y).$$

Since $P_0(x) + yP_1(x) = F(x, y) = x^{k_2+1}y + c_1x^{k_2-2} + x^{k_2+1}(k_0 - k_2)$ and we know that $F(x, y) = x^{k_2+1}(y - c/x) = x^{k_2+1}y - cx^{k_2}$, we get $k_2 = k_0 = 0$ and $c_1 = -c$. Then, among f_{11} , there exists a Darboux polynomial with cofactor x and it is

$$f_{21} = a_0 \left(1 + \frac{1}{c}xy \right).$$

This proves statement (e) of Theorem 2 and consequently concludes the proof of Theorem 2.

5. Proof of Theorem 3

As in the previous section we still use the Appendix and Theorem 13 to construct all the irreducible invariant algebraic curves $F(x, y)$.

We will separate the proof in two subsections.

5.1. Proof of statement (a) of Theorem 3

We recall that since $c > 0$ and $k \leq M - 1$, we cannot have $k + 2 + b - 2M = 0$ for some b and so the polynomial $p_0(x)$ in statement (a.i) is well-defined. Analogously, again since $c > 0$ and $k \leq M - 1$, we cannot have $b - k + 2M = 0$ for some b and so the polynomial $g_0(x)$ in statement (a.ii) is well-defined.

It follows from (47), (48), (53) and (55) that we have the relations

$$F(x, y) = \mu(x) \left(y + \frac{c}{x} + \frac{c(b+1)}{x^2} + \dots \right)^{j_1} \left(y - x - \frac{c+b-1}{x} + \frac{(b-3)(c+b-1)}{x^2} + \dots \right)^{j_2} \quad (30)$$

where $j_i = 0$ or 1 for $i = 1, 2$, and also

$$F(x, y) = \nu(y) \left(x - y - (b-1) - \frac{1-b-c}{y} + \frac{2(1-b-c)}{y^2} + \dots \right)^{j_1} \prod_{i=1}^{N-j_1} \left\{ x - k_0^i \left(1 + \frac{1}{y} + \frac{1+b+k_0^i}{2y^2} + \dots \right) \right\} \quad (31)$$

where $j_1 = 0$ or 1 . We thus conclude that a Darboux polynomial for $a = 0$ has at most degree 2 on y . Hence, F has the form

$$F(x, y) = P_0(x) + P_1(x)y + P_2(x)y^2,$$

for some $P_i \in \mathbb{C}[x]$, for $i = 0, 1, 2$.

Now we impose that F is a Darboux polynomial, i.e. that satisfies

$$P(x, y)F_x + Q(x, y)F_y = (k_0 + k_1x + k_2y)F(x, y), \quad k_i \in \mathbb{C}, \quad i = 0, 1, 2. \quad (32)$$

Computing the coefficient of y^3 in (32) we get

$$2P_2 - k_2P_2 = P_2(2 - k_2) = 0, \quad (33)$$

and so either $P_2(x) = 0$ or $k_2 = 2$ (and $P_2 \neq 0$). We consider both cases separately.

Case 1: $P_2(x) = 0$. In this case $F(x, y) = P_0(x) + P_1(x)y$. Computing the coefficient of y^2 in (32) we get

$$-P_1(x)(1 + k_2) = 0.$$

So, either $P_1 = 0$ or $k_2 = -1$. We again consider both subcases separately.

Subcase 1.1: $P_1(x) = 0$. In this case, computing the coefficients of y and the independent term in (32) we obtain

$$-k_2P_0(x) = 0 \text{ and } -(k_0 + k_1x)P_0(x) + xP_0'(x) = 0. \quad (34)$$

Since F is a Darboux polynomial it cannot be constant and so $P_0 \neq 0$. It then follows from the first relation in (34) that $k_2 = 0$. Therefore, the second relation in (34) is a linear differential equation for P_0 . Solving it we get that $P_0(x) = c_0 e^{k_1 x} x^{k_0}$ for some constant $c_0 \in \mathbb{C}$. Since $P_0 \neq 0$ we can take $c_0 = 1$. Since P_0 must be a polynomial we get $k_1 = 0$ and then $F = x^{k_0}$, but since F must be irreducible we conclude that $k_0 = 1$ and so $F = x$.

Subcase 1.2: $k_2 = -1$ and $P_1(x) \neq 0$. In this case we have the expressions

$$(P_0(x) + bP_1(x) - k_0P_1(x) + xP_1(x) - k_1xP_1(x) + xP_1'(x))y = 0 \quad (35)$$

and

$$-(k_0 + k_1x)P_0(x) + cP_1(x) + xP_0'(x) = 0. \quad (36)$$

From (36) we get

$$P_1(x) = \frac{1}{c} ((k_0 + k_1x)P_0(x) - xP_0'(x)). \quad (37)$$

Substituting P_1 of (37) into (35) we get

$$\begin{aligned} P_0(x)(c + bk_0 - k_0^2 + k_0x + k_1x + bk_1x - 2k_0k_1x + k_1x^2 - k_1^2x^2) \\ + P_0'(x)(-x - bx + 2k_0x - x^2 + 2k_1x^2) = -x^2P_0''(x). \end{aligned} \quad (38)$$

Assume that

$$P_0(x) = x^{M-1} + B_{M-2}x^{M-2} + B_{M-3}x^{M-3} + \cdots + B_2x^2 + B_1x + B_0. \quad (39)$$

Note that $B_0 \neq 0$ otherwise P_0 would be reducible (divisible by x) and from (37) we would have that P_1 would also be divisible by x and then F would be reducible, which is not possible. So, $B_0 \neq 0$.

Note that the coefficient of x^{M+1} in (39) yields $k_1(1 - k_1) = 0$. So, either $k_1 = 0$ or $k_1 = 1$. We consider both subcases separately.

Subcase 1.2.1: $k_1 = 0$. Computing the coefficient of x^M in (39) yields $k_0 - M + 1 = 0$. So, either $k_0 = M - 1$. Computing the constant term in (39) we get $c - 1 - b + M(2 + b) - M^2 = 0$ and so $c = 1 + b - M(2 + b) + M^2$. Computing the coefficients of x^i for $i = 1, 2, 3, \dots$ in (39) we get the relations

$$B_j = \frac{B_0 \prod_{k=1}^j (M - k)}{j! \prod_{k=1}^j (k + 2 + b - 2M)}, \quad j = 1, \dots, M - 1,$$

for $k + 2 + b - 2M \neq 0$. Note that since $c > 0$ and $k \leq M - 1$ we cannot have $k + 2 + b - 2M = 0$ and so the relation above is always well defined and we obtain the Darboux polynomials in statement (i). Note that if

$M = 1$ then $1 \leq k < 0$ which is not possible. So, $M \geq 2$. Furthermore, since p_0 and p_1 are multiplied by B_0 , we can set it equal to one.

Subcase 1.2.2: $k_1 = 1$. Computing the coefficient of x^M in (39) yields $k_0 = M + b$ and the constant term in (39) yields $c = M(b + M)$. Computing the coefficients of x^i for $i = 1, 2, 3, \dots$ in (39) we get the relations

$$B_j = \frac{B_0 \prod_{k=1}^j (M - k)}{j! \prod_{k=1}^j (2M + b - k)}.$$

Note that since $c > 0$ and $k \leq M - 1$ we cannot have $2M + b - k = 0$ and so the relation above is always well defined and we obtain the Darboux polynomials in statement (ii). Furthermore, since p_0 and p_1 are multiplied by B_0 , we can set it equal to one.

Case 2: $P_2(x) \neq 0$ and $k_2 = 2$. The Darboux polynomial has degree two in y . Hence it follows from (30) that $j_1 = j_2 = 2$ and so again it follows from (30) that if we write $\mu(x) = \sum_{j=0}^N b_j x^j$ for some $b_j \in \mathbb{C}$ (we can assume that $b_N = 1$) then the Darboux polynomial must take the form (see again (30))

$$F(x, y) = y^2 x^M - y x^{M+1} + y^2 A_{M-1}(x) + y B_M(x) + C_M(x) \quad (40)$$

where $A_{M-1}(x)$ means powers in x at most $M - 1$, $B_M(x)$ and $C_M(x)$ mean powers of x at most M .

Now we write the Darboux polynomial F as a polynomial in the variable x as

$$F(x, y) = \sum_{i=1}^M q_i(y) x^i, \quad q_i \in \mathbb{C}[x]. \quad (41)$$

If we consider the biggest power of x in Eq. (32) we get that it is x^{M+1} and the coefficient must satisfy $q'_M(y)y = k_1 q_M(y)$. Solving this linear differential equation we get

$$q_M(y) = \alpha y^{k_1}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Without loss of generality we can consider $\alpha = 1$. Since the degree of F in y is two, we get that $k_1 \in \{0, 1, 2\}$. Note that this implies that in the representation of $F(x, y)$ in (31) we have $\nu(y) = y^{k_1}$ with $k_1 \in \{0, 1, 2\}$. We consider the three Subcases separately.

Subcase 2.1: $k_1 = 0$. In this case it follows from (31) that the degree in y is at most $j_1 \leq 1$ and so this case is not possible.

Subcase 2.2: $k_1 = 1$. In this case, it follows from (31) that $j_1 = 1$ (for F with degree 2 on y), then $F(x, y)$ can be written as

$$y^2 x^{N-1} - y x^N + y^2 D_{N-2}(x) + y E_{N-1}(x) + F_N(x) \quad (42)$$

where $D_{N-2}(x)$ means powers in x at most $N - 2$, $E_{N-1}(x)$ means powers in x at most $N - 1$ and $F_N(x)$ mean powers of x at most N . Comparing (40) and (42) we reach to a contradiction, so this case is also not possible.

Subcase 2.3: $k_1 = 2$. It follows from (31) that if $j_1 = 1$ then the degree of y is three which is not possible and so $j_1 = 0$. Then again it follows from (31) that $F(x, y)$ can be written as

$$y^2 x^N - y x^{N+1} + y^2 D_{N-1}(x) + y E_N(x) + F_N(x) \quad (43)$$

(we recall that F must be a polynomial) where $D_{N-1}(x)$ means powers in x at most $N - 1$ and $E_N(x)$ and $F_N(x)$ mean powers of x at most N . Comparing (40) and (43) we reach to a contradiction, so this case is also not possible. This concludes the proof of statement (a) [Theorem 3](#).

5.2. Proof statement (b) of Theorem 3

First note that e^x is always an exponential factor of system (4) with $a = 0$ and has cofactor x .

Assume first that $c = 1 + b - M(2 + b) + M^2$ for some $M \in \mathbb{N}$. In view of Theorem 3(a)(i) there exist two distinct irreducible Darboux polynomials x and f_2 . The cofactors take the form $K_1 = 1$ and $K_2 = M - 1 - y$ accordingly. The divergence of system (4) is $1 + b + x - 2y$. Clearly

$$V = x^{3+b-2M} f_2^2 e^x$$

is an inverse integrating function and by Theorem 10 system (4) admits a Liouvillian first integral.

Assume now that $c = M(b + M)$ for some $M \in \mathbb{N}$. In view of Theorem 3(a)(ii) there exist two distinct irreducible Darboux polynomial x and f_3 . The cofactors take the form $K_1 = 1$ and $K_2 = M + b + x - y$ accordingly. The divergence of system (4) is $1 + b + x - 2y$. Clearly

$$V = x^{1-b-2M} f_3^2 e^{-x}$$

is an inverse integrating function and by Theorem 10 system (4) admits a Liouvillian first integral.

Now we shall show that in any other case systems (4) do not admit a Liouvillian first integral. Let now $c \neq 1 + b - M(2 + b) + M^2$ and $c \neq M(b + M)$ for any $M \in \mathbb{N}$. In this case the system can have an exponential factor of the form $E = \exp(g/x^n)$ with $g \in \mathbb{C}[x, y]$, $n > 0$ and g coprime with x . We will show that this is not possible. Assume that this is the case. Then, g satisfies

$$x \frac{\partial g}{\partial x} + (c + by + xy - y^2) \frac{\partial g}{\partial y} - ng = Lx^n,$$

where $L = \beta_0 + \beta_1 x + \beta_2 y$ is the cofactor of E being $\beta_i \in \mathbb{C}$. Evaluating the above equation on $x = 0$ and setting $\bar{g} = g|_{x=0}$ we get that $\bar{g} \neq 0$ (otherwise g would be coprime with x) and it satisfies

$$(c + by - y^2) \frac{d\bar{g}}{dy} = n\bar{g}.$$

Solving this linear differential equation we get

$$\bar{g} = \kappa \left(\frac{-b - 2y + \sqrt{b^2 + 4c}}{-b - 2y - \sqrt{b^2 + 4c}} \right)^{n/\sqrt{b^2 + 4c}}, \quad \kappa \in \mathbb{C} \setminus \{0\}.$$

Taking into account that \bar{g} must be a polynomial and that $\kappa n \neq 0$ we reach to a contradiction. Hence, such an exponential factors cannot exist. Therefore, the unique exponential factors are of the form $E = \exp(g)$ with $g \in \mathbb{C}[x, y]$. In this case g satisfies

$$x \frac{\partial g}{\partial x} + (c + by + xy - y^2) \frac{\partial g}{\partial y} = L,$$

where again $L = \beta_0 + \beta_1 x + \beta_2 y$ is the cofactor of E being $\beta_i \in \mathbb{C}$. Take $h = g - \beta_1 x$ and set $F = \exp(h) = \exp(g - \beta_1 x) = E \exp(-\beta_1 x)$. We will show that $h = \kappa$ with $\kappa \in \mathbb{C}$ and so the unique exponential factors are of the form $E = \exp(\kappa - \beta_1 x)$. Note that h satisfies

$$x \frac{\partial g}{\partial x} + (c + by + xy - y^2) \frac{\partial g}{\partial y} = \beta_0 + \beta_2 y. \quad (44)$$

Evaluating (44) on $x = 0$ and $y = \frac{-b \pm \sqrt{b^2 + 4c}}{2}$ we get

$$\beta_0 + \beta_2 \frac{-b + \sqrt{b^2 + 4c}}{2} = 0 \quad \text{and} \quad \beta_0 + \beta_2 \frac{-b - \sqrt{b^2 + 4c}}{2} = 0.$$

Table 4

Local phase portraits at the finite singular points in the order e_1 - e_2 - e_3 for $a \neq 0$ of system (3). UN: Unstable node, SN: stable node, S/N: saddle-node, S: saddle, S1: saddle with local stable manifold on x -axis, S2: saddle with local stable manifold on y -axis.

a	$ac > 1$	$ac = 1$	$ac < 1$
$a > 0$	$S1 - S2 - UN$	$S1 - S/N$	$S1 - UN - S$
$a < 0$	$S1 - UN - SN$	$S/N - UN$	$SN - UN - S$

Taking into account that $c = 0$ we conclude that $\beta_2 = \beta_4 = 0$ but then in view of (44) we get that h is either constant or a polynomial first integral which in view of statement (a) of Theorem 2 this is not possible. Hence $h = \kappa$, $\kappa \in \mathbb{C}$ and so the unique exponential factors are in fact $E = \exp(\kappa + \beta_1 x)$.

Now we will show that in this case there are no Liouvillian first integrals. In view of Theorems 9 and 10 if there exists a Liouvillian first integral then it must be of the form

$$G = x^{\lambda_1} E^{\lambda_2}$$

and its cofactor must be of the form $1 + b + x - 2y$. However the cofactor of G is equal to $\lambda_1 + \lambda_2 \beta_1 x$ which is never equal to $1 + b + x - 2y$. So in this case there are no Liouvillian first integrals. This concludes the proof of statement (b) of Theorem 3 and so the proof of Theorem 3.

6. Proof of Theorem 4

The proof of statement (a) follows from Theorem 1 taking $b = 1 - c$. We summarize here the analysis under these conditions. The finite equilibria of system (3) are $e_1 = (0, 1)$, $e_2 = (0, -c)$ and $e_3 = ((1+a)(-1+ac)/a, -1/a)$. Taking into account that $c > 0$, we have that e_1 and e_2 are always different. Moreover, $e_3 = e_2$ if $ac = 1$. The linear part of system (3) is

$$\begin{pmatrix} ay + 1 & ax \\ y & 1 - c + x - 2y \end{pmatrix}.$$

The local phase portrait at these equilibria is as follows. The equilibrium $e_1 = (0, 1)$ has eigenvalues $\lambda_1 = 1 + a > 0$ and $\lambda_2 = -1 - c < 0$, so it is a saddle (with the local stable manifold located on the y -axis). The equilibrium e_2 has the eigenvalues $\lambda_1 = 1 - ac$ and $\lambda_2 = 1 + c > 0$. So it is an unstable node if $ac < 1$, a saddle (with the local unstable manifold located on the y -axis) if $ac > 1$ and semi-hyperbolic if $ac = 1$. In this last case, in order to obtain the local phase portrait of e_2 , we apply Theorem 2.19 in [21]. Doing so, we get that it is a saddle-node, and since $a > 0$, it has one attracting sector for $x > 0$, and a two hyperbolic sectors for $x < 0$. This local phase portrait is shown in Fig. 3. Finally, the equilibrium e_3 has eigenvalues $(a+1)/a$ and $ac - 1$. So it is an stable node if $a < 0$, an unstable node if $a > 0$ and $ac > 1$, and a saddle if $a > 0$ and $ac < 1$. The local phase portraits at the finite equilibria of system (3) are summarized in Table 4. Note that the bifurcation value is $c = 1/a$ for $a > 0$ and that for $a = 0$ there only exist two finite equilibria: a saddle at e_1 and an unstable node at e_2 .

The dynamics at infinity was completely studied in Section 3. The local phase portraits at each equilibrium at the Poincaré disc are shown in Fig. 9. Each local phase portrait shown in Fig. 9 coincides with one of the local phase portraits given for system (4) in Fig. 5. Then we can establish the connection of separatrices analogously as we did in Section 3. It follows that the global phase portraits at the Poincaré disc of system (3) are topologically equivalent to one of the ones shown in Fig. 2. Statement (a) of Theorem 4 is proved.

The proof of statements (b), (c) and (d) follows directly from Theorem 2 taking $b = 1 - c$.

Regarding the proof of statement (e) we note that setting $b = 1 - c$ in Theorem 3(a)(i), we get that in order to have an irreducible Darboux polynomial different from x when $a = 0$ we must have $M = 2$ or

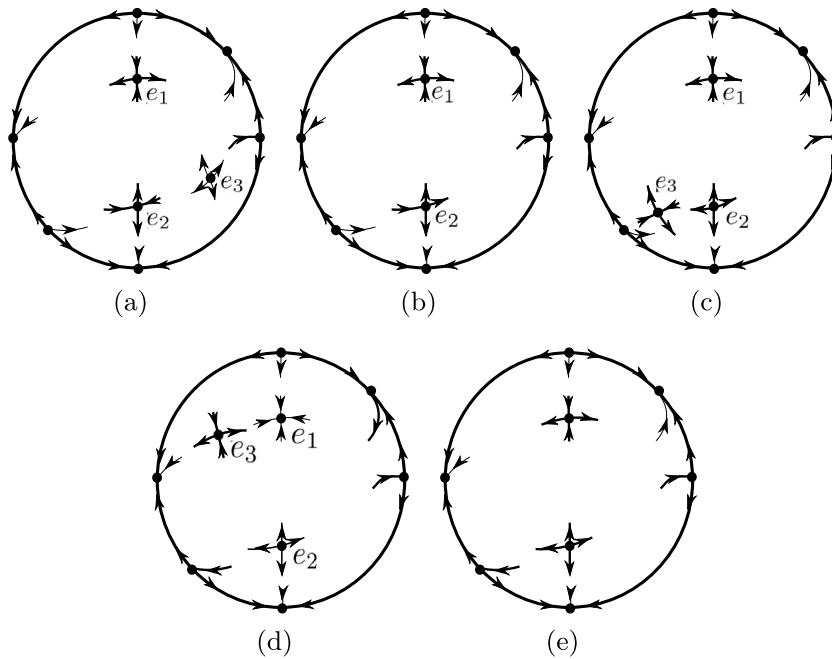


Fig. 9. Local phase portraits at the equilibrium of system (4) on the Poincaré disc. (a): $a > 0$ and $ac > 1$, (b): $a > 0$ and $ac = 1$, (c): $a > 0$ and $ac < 1$, (d): $a < 0$, (e): $a = 0$.

$b = M$, that is $c = 1 - M$ for any $M \in \mathbb{N}$ with $M \geq 1$. Note that if $M \geq 1$ then $c \leq 0$ which is not possible. So, $M = 2$, but in this case we get that the irreducible Darboux polynomial is

$$p_0 + p_1 y = 1 - \frac{1}{c}x + \frac{y}{c}(p_0 - xp'_0) = \frac{1}{c} \left(c - x + y \left(1 - \frac{x}{c} + \frac{x}{c} \right) \right) = \frac{1}{c}(c - x + y)$$

which is f_{12} with $a = 0$ and $a_{01} = 1/c$.

On the other hand, setting $b = 1 - c$ in Theorem 3(a)(ii) we get that $c = M$ with $M \in \mathbb{N}$, $M \geq 1$. Now, the irreducible Darboux polynomial is $p_0 + p_1 y$. Note that

$$p_0 = 1 + \sum_{j=1}^{M-1} \frac{M-j}{Mj!} x^j = \sum_{j=0}^{M-1} \frac{M-j}{Mj!} x^j$$

and

$$\begin{aligned} p_1 &= \frac{1}{M}((1+x)p_0 - xp'_0) = \frac{1}{M} \left(\sum_{j=0}^{M-1} \frac{M-j}{Mj!} (1-j)x^j + \sum_{j=0}^{M-1} \frac{M-j}{Mj!} x^{j+1} \right) \\ &= \frac{1}{M} \left(1+x + \sum_{j=2}^{M-1} \left(\frac{M-j}{Mj!} (1-j) + \frac{M-j+1}{M(j-1)!} \right) x^j + \frac{x^M}{M!} \right) \\ &= \frac{1}{M} \left(1+x + \sum_{j=2}^{M-1} \frac{x^j}{j!} + \frac{x^M}{M!} \right) = \frac{1}{M} \sum_{j=0}^M \frac{x^j}{j!}. \end{aligned}$$

Hence

$$p_0 + p_1 y = \frac{1}{M} \left(\sum_{j=0}^{M-1} \frac{M-j}{j!} x^j + y \sum_{j=0}^M \frac{x^j}{j!} \right) = \frac{f_3}{M}.$$

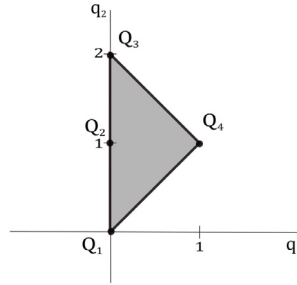


Fig. 10. Newton polygon related to Eq. (45).

Finally, note that it follows from Theorem 3(b) with $c = 1 - b$ that the Chemostat system (3) has a Liouvillian first integral if and only if $c = M$ with $M \in \mathbb{N}$, $M \geq 1$. It is straightforward to see that one Liouvillian first integral can be taken as $e^x f_{12}/f_3$. This completes the proof of Theorem 4.

Acknowledgements

The authors wish to thank the reviewers for their comments and suggestions that helped to improve the paper.

The first author is supported by CONICYT PCHA/POSTDOCTORADO BECAS CHILE/2018 - 74190062. The second author is supported by FCT/Portugal through UID/MAT/04459/2013.

Appendix. Puiseux series

In this appendix we follow the method described in Section 2.4 to construct the Puiseux series related to system (4). Through the development of this section we show that the construction of irreducible invariant algebraic curves of system (4) is not easy and requires a careful analysis, even more the construction must be separated in different branches according to the parameters. However, we will detail the form of the Puiseux series related to Eqs. (14) and (15) in any case.

We consider Eq. (14) that takes the form

$$(x - axy)y_x - (c + by + xy - y^2) = 0. \quad (45)$$

There exist three dominant balances producing Puiseux series near to $x = \infty$. The Newton polygon is presented in Fig. 10.

The balances giving power asymptotic at $x \rightarrow \infty$ and their power solutions take the form

$$\begin{aligned} (Q_1, Q_4) : \quad & -c - xy = 0, & y(x) &= \frac{-c}{x}, \\ (Q_3, Q_4) : \quad & axyy_x + y^2 - xy = 0, & y(x) &= \frac{x}{1+a}, \quad a \neq 1, \\ Q_3 : \quad & axyy_x + y = 0, & y(x) &= x^{-1/a}. \end{aligned} \quad (46)$$

I. In the first case there does not exist a Fuchs index and then, the corresponding Puiseux series associated to (Q_1, Q_4) is

$$(I) : y_1(x) = -\frac{c}{x} + \frac{c(b+1)}{x^2} - \frac{c(2+b(3+b)+(-1+a)c)}{x^3} + \dots \quad (47)$$

II. In the second balance associated to the edge (Q_3, Q_4) the Fuchs index depends on a , this is $j = (a+1)/a$. We denote by α_{2j} , β_{2j} and γ_{3j} real coefficients depending on a, b and c . We need to consider different cases:

- For $a \notin \mathbb{Q}$ or $-1 < a \leq 0$ the Fuchs index do not change n_0 , and the Puiseux series is

$$(II.1) : y_2(x) = \frac{x}{1+a} + (b-1) + \frac{(1+a)(-1+b+c)}{(-1+a)} \frac{1}{x} + \frac{(1+a)^2(3+a(-1+b)-b)(-1+b+c)}{(-1+a)(-1+2a)} \frac{1}{x^2} + \dots \quad (48)$$

- Note that for $a = 1/N$ with $N \in \mathbb{N}$ we do not have Fuchs index, but in this case the Puiseux series acquires the form

$$(II.2) : y_2^1(x) = \frac{x}{1+a} + (b-1) + \alpha_{22} \left(\frac{1}{x^{1/a}} + \frac{\beta_{22}(b)}{x^{(1/a)+1}} - \frac{\gamma_{22}(b)}{x^{(1/a)+2}} + \dots \right), \quad (49)$$

under the condition $c = 1 - b$.

- If $a \in \mathbb{N} \setminus \{1\}$, then $j = (a+1)/a$ and $n_0 = a$. In this case we have the Puiseux series

$$(II.3) : y_2^a(x) = \frac{x}{a+1} + (b-1) + \frac{\alpha_{23}}{x^{1/a}} + \frac{\beta_{23}(b,c)}{x} + \frac{\alpha_{23}\beta_{23}}{x^{(1/a)+1}} + \frac{\gamma_{23}(b,c)}{x^2} + \dots \quad (50)$$

III. For the balance associated to the vertex point Q_3 the Fuchs index is $j = 0$. We denote by α_{3j} , β_{3j} and γ_{3j} real coefficients depending on a, b and c . The Puiseux series depends on a and takes the following forms:

- If $a \notin \mathbb{Q}$ then this Puiseux series does not exist.
- If $a = N_1/N_2$, with $N_1, N_2 \in \mathbb{N}$, then

$$(III.1) : y_3^+(x) = -\frac{c}{x} + \frac{c(b+1)}{x^2} + \dots,$$

i.e. it is the same as $y_1(x)$.

- If $a = -1/N$ with $N \in \mathbb{N}$, the Puiseux series takes the form

$$(III.2) : y_3^-(x) = \alpha_{32}x^{-1/a} + \frac{x}{1+a} + b + \frac{1}{a} + \alpha_{32} \left(\frac{\beta_{32}(b,c)}{x^{-(1/a)-1}} + \frac{\gamma_{32}(b,c,k_0)}{x^{-1/a}} + \dots \right).$$

- If $a = -N_1/N_2$ with $N_1, N_2 \in \mathbb{N}$, $N_1 \neq 1$.

$$(III.3) : y_3^{-,2}(x) = \alpha_{33}x^{-1/a} + \frac{x}{1+a} + b + a + \alpha_{33} \left(\frac{\beta_{33}(b,c)}{x^{-a-1}} + \frac{\gamma_{33}(b,c,k_0)}{x^{-a}} + \dots \right).$$

Now we put the attention in Eq. (15), i.e.

$$(x - axy) - (c + by + xy - y^2)x_y = 0. \quad (51)$$

In this case the Newton polygon is as in three previous cases but translated, as we present in Fig. 11

The balances related to the point $x = \infty$ and their solutions take the form

$$\begin{aligned} (P_1, P_4) : & \quad -cx_y - xyx_y = 0, & x(y) = k \text{ and } x(y) = -\frac{c}{y}, \\ (P_3, P_4) : & \quad axy + y^2x_y - xyx_y = 0, & x(y) = (1+a)y, \\ P_3 : & \quad axy + y^2x_y = 0, & x(y) = ys^{-a}. \end{aligned} \quad (52)$$

i. The balance $x(y) = k$ generated by the edge (P_1, P_4) has associated the Gâteaux derivative $V(j) = j(k-y)y^{-j}$ (see (18)) and the Fuchs index is $j = 0$. The associated Puiseux series for $a \in \mathbb{N}$ is

$$(i) : x_1^{\mathbb{N}}(y) = \hat{\alpha}_1 \left(\frac{1}{y^a} + \frac{1+ab}{x^{a+1}} + \frac{\hat{\beta}}{x^{a+2}} + \dots \right), \quad (53)$$

where $\hat{\alpha}_1$ and $\hat{\beta}_1$ are real constants. If $a \notin \mathbb{N}$ the Puiseux series vanishes.

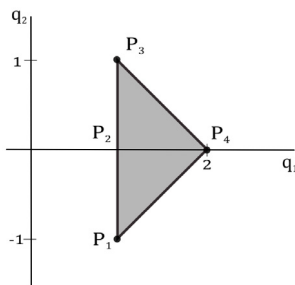


Fig. 11. Newton polygon related to Eq. (51).

ii. The Gâteaux derivative of the second balances produced by the same edge (P_1, P_4) , i.e. $x = -c/y$, is $V(j) = cy^{-2-j}$. So this balance does not have Fuchs index. The Puiseux series in this case only exists for $a = 1$, is uniquely determined, and it is

$$(ii) : x_2^1 = -\frac{c}{y} - \frac{c(b+1)}{y^2} - \dots, \quad a = 1 \quad (54)$$

iii. The Fuchs index for the $x = (1+a)y$ (associated to the edge (P_3, P_4)) is $j = (a+1)/a$. We denote by $\hat{\alpha}_{3j}$, $\hat{\beta}_{3j}$ and $\hat{\gamma}_{3j}$ real coefficients depending on a, b and c . We separate the study of the Puiseux series in three cases

- If $a \notin \mathbb{Q}$, or $-1 < a \leq 0$, then the Puiseux series is

$$(iii.1) : x_3^1 = (1+a)y - (1+a)(b-1) + \frac{\hat{\alpha}_{31}(a, b, c)}{y} + \frac{\hat{\beta}_{31}(a, b, c)}{y^2} + \frac{\hat{\gamma}_{31}(a, b, c)}{y^3} + \dots \quad (55)$$

- If $a = 1/N$ with $N \in \mathbb{N}$ the Puiseux series is

$$(iii.2) : x_3^2 = (1+a)y - (1+a)(b-1) + \frac{\hat{\alpha}_{32}(a, b, c)}{y} + \frac{\hat{\beta}_{32}(a, b, c)}{y^2} + \dots + \mu \left(\frac{\hat{\gamma}_{32}(a, b, c)}{y^N} + \dots \right). \quad (56)$$

This series exists if $b = 1 - c$ and then $\hat{\alpha}_{32} = \hat{\alpha}_{32} = \dots = 0$, or if c satisfies a relation in the form $c = \frac{A_0 + A_1b + A_2b^2 + A_3b^3}{B_0 + B_1b}$, where A_i, B_j are real constants for $i = 0, 1, 2$ and $j = 0, 1$.

- If $a = N$ with $N \in \mathbb{N}$ the Puiseux is given by

$$(iii.3) : x_3^3 = (1+a)y + (1+a)(b-1) + \frac{\hat{\alpha}_{33}}{y^{1/a}} + \frac{\hat{\beta}_{33}(b+c-1)}{y} + \frac{ab - (2a+1)}{y^{(a+1)/a}} + \frac{\hat{\gamma}_{33}}{y^{(a+2)/a}} + \dots \quad (57)$$

- If $a = N_1/N_2$ with $N_1, N_2 \in \mathbb{N}$, then the Fuchs index is $j = (N_1 + N_2)/N_1$, and the Puiseux series has the form

$$(iii.4) : x_3^3 = (1+a)y + \frac{\hat{\alpha}_{33}}{y^{1/a}} + \frac{\hat{\beta}_{33}(b+c-1)}{y} + \frac{ab - (2a+1)}{y^{(N_1+1)/a}} + \frac{\hat{\gamma}_{33}}{y^{(N_1+2)/a}} + \dots \quad (58)$$

iv. Finally, for the balance associated to the vertex point P_3 the Fuchs index is $j = 0$. The Puiseux series depends on a . Note that the series exists only if $a \in \mathbb{Q}$. We denote by $\hat{\alpha}_{4j}$ and $\hat{\beta}_{4j}$ real coefficients depending on a, b and c . We need to consider separately the following cases.

- If $a = -1/N$ with $N \in \mathbb{N}$ the Puiseux series starts with a positive exponent and takes the form

$$(iv.1) : x_3^1 = \hat{\alpha}_{41} \left(y^{-a} - \frac{\hat{\alpha}_{41}}{y^{1+2a}} - \frac{-ab-1}{y^{1+a}} + \dots + \frac{\hat{\beta}_{41}}{y^{1-a}} + \dots \right). \quad (59)$$

- If $a = -N_1/N_2$ then the Puiseux series acquires the similar form

$$(iv.2) : x_3^2 = \hat{\alpha}_{42} \left(y^{-a} - \frac{\hat{\alpha}_{42}}{y^{1+2a/N_1}} - \frac{-ab-1}{y^{1+a/N_1}} + \cdots + \frac{\hat{\beta}_{42}}{y^{1-a/N_1}} + \cdots \right). \quad (60)$$

- If $a = 1/N$ with $N \in \mathbb{N}$, the Puiseux series starts with a negative exponent again and takes the form

$$(iv.3) : x_3^3 = \hat{\alpha}_{43} \left(y^{-a} + \frac{1+ab}{y^{a+1}} + \frac{\hat{\beta}_{43}}{y^{1+2a}} + \cdots \right). \quad (61)$$

- If $a = N_1/N_2$ with $N_1, N_2 \in \mathbb{N}$, the Puiseux series starts with a negative exponent again and takes the form

$$(iv.4) : x_3^4 = \hat{\alpha}_{44} \left(y^{-a} + \frac{1+ab}{y^{1+aN_1}} + \frac{\hat{\beta}_{44}}{y^{1+2aN_1}} + \cdots \right). \quad (62)$$

- If $a = N$ with $N \in \mathbb{N}$ the Puiseux series starts with a negative exponent and takes the form

$$(iv.5) : x_3^5 = \hat{\alpha}_{45} \left(\frac{1}{y^a} + \frac{1+ab}{y^{a+1}} + \frac{\hat{\beta}_{45}}{y^{a+2}} + \cdots \right). \quad (63)$$

The construction of irreducible invariant algebraic curves from the Puiseux series is not easy, even more for $a \in \mathbb{Q} \setminus \{1\}$ because the function (20) can have infinite factors. Despite this, knowing the Puiseux series enables us to establish some properties of irreducible invariant algebraic curves that will be used in the paper.

References

- [1] A. Novick, L. Szilard, Experiments with the chemostat on spontaneous mutations of bacteria, *Proc. Natl. Acad. Sci. USA* 36 (12) (1950) 708–719.
- [2] H.R. Bungay, in: W.W. Umbreit, D. Perlman, F.M. Richards (Eds.), *Microbial Interactions in Continuous Culture*, in: *Advances in Applied Microbiology*, vol. 10, Academic Press, 1968, pp. 269–290.
- [3] A. Cunningham, R.M. Nisbet, Transients and oscillations in continuous culture, in: *Mathematics in Microbiology*, Academic Press, London, 1983, pp. 77–103.
- [4] A.G. Fredrickson, G. Stephanopoulos, Microbial competition, *Science* 213 (4511) (1981) 972–979.
- [5] H. Jannasch, R. Mateles, Experimental bacterial ecology studied in continuous culture, *Adv. Microb. Physiol.* 11 (1974) 165–212.
- [6] I. Karafyllis, Z.-P. Jiang, Reduced order dead-beat observers for the chemostat, *Nonlinear Anal. RWA* 14 (1) (2013) 340–351.
- [7] P.A. Taylor, P.J.L. Williams, Theoretical studies on the coexistence of competing species under continuous-flow conditions, *Can. J. Microbiol.* 21 (1) (1975) 90–98, PMID: 1116041.
- [8] H. Veldkamp, Ecological studies with the chemostat, in: M. Alexander (Ed.), *Advances in Microbial Ecology*, Springer US, Boston, MA, 1977, pp. 59–94.
- [9] P. Waltman, Competition models in population biology, in: *CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 45, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1983, p. v+77.
- [10] R. Freter, Mechanisms that control the microflora in the large intestine, in: D. Hentges (Ed.), *Human Intestinal Microflora in Health and Disease*, Academic Press, 1983, pp. 33–54, (Chapter 2).
- [11] R. Freter, H. Brickner, S. Temme, An understanding of colonization resistance of the mammalian large intestine requires mathematical analysis, *Microecol. Therapy* 16 (1986) 147–155.
- [12] G. Robledo, F. Grogard, J.-L. Gouzé, Global stability for a model of competition in the chemostat with microbial inputs, *Nonlinear Anal. RWA* 13 (2) (2012) 582–598.
- [13] M. Lin, H.-F. Huo, Y.-N. Li, A competitive model in a chemostat with nutrient recycling and antibiotic treatment, *Nonlinear Anal. RWA* 13 (6) (2012) 2540–2555.
- [14] G. D'ans, P.V. Kokotović, D. Gottlieb, A nonlinear regulator problem for a model of biological waste treatment, *IEEE Trans. Automat. Control* AC-16 (1971) 341–347.
- [15] J.W.M. La Rivière, Microbial ecology of liquid waste treatment, in: M. Alexander (Ed.), *Advances in Microbial Ecology*, Springer US, Boston, MA, 1977, pp. 215–259.
- [16] T. Caraballo, X. Han, P. Kloeden, Chemostats with time-dependent inputs and wall growth, *Appl. Math. Inf. Sci.* 9 (5) (2015) 2283–2296.

- [17] T. Caraballo, X. Han, P.E. Kloeden, Chemostats with random inputs and wall growth, *Math. Methods Appl. Sci.* 38 (16) (2015) 3538–3550.
- [18] H.L. Smith, P. Waltman, Dynamics of microbial competition, in: *The Theory of the Chemostat*, in: *Cambridge Studies in Mathematical Biology*, vol. 13, Cambridge University Press, Cambridge, 1995, p. xvi+313.
- [19] P. Waltman, S.P. Hubbell, S.B. Hsu, Theoretical and experimental investigations of microbial competition in continuous culture, in: *Modeling and Differential Equations in Biology (Conf., Southern Illinois Univ., Carbondale, Ill., 1978)*, in: *Lecture Notes in Pure and Appl. Math.*, vol. 58, Dekker, New York, 1980, pp. 107–152.
- [20] H. Poincaré, Sur les courbes définies par une équation différentielle, *Oeuvres Complètes*. 1 (1928) Theorem XVII.
- [21] F. Dumortier, J. Llibre, J.C. Artés, Qualitative Theory of Planar Differential Systems, in: *Universitext*, Springer-Verlag, Berlin, 2006, p. xvi+298.
- [22] D.A. Neumann, Classification of continuous flows on 2-manifolds, *Proc. Amer. Math. Soc.* 48 (1975) 73–81.
- [23] L. Markus, Global structure of ordinary differential equations in the plane, *Trans. Amer. Math. Soc.* 76 (1954) 127–148.
- [24] M.M. Peixoto, On the classification of flows on 2-manifolds, in: *Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, Academic Press, New York, 1973, pp. 389–419.
- [25] H. Poincaré, Sur les courbes tracées sur les surfaces algébriques, *Ann. Sci. École Norm. Sup.* (3) 27 (1910) 55–108.
- [26] S.D. Furta, On non-integrability of general systems of differential equations, *Z. Angew. Math. Phys.* 47 (1) (1996) 112–131.
- [27] W. Li, J. Llibre, X. Zhang, Local first integrals of differential systems and diffeomorphisms, *Z. Angew. Math. Phys.* 54 (2) (2003) 235–255.
- [28] M.F. Singer, Liouvillian first integrals of differential equations, *Trans. Amer. Math. Soc.* 333 (2) (1992) 673–688.
- [29] M.V. Demina, Invariant algebraic curves for Liénard dynamical systems revisited, *Appl. Math. Lett.* 84 (2018) 42–48.
- [30] M.V. Demina, N.A. Kudryashov, Meromorphic solutions in the FitzHugh-Nagumo model, *Appl. Math. Lett.* 82 (2018) 18–23.
- [31] M.V. Demina, Novel algebraic aspects of Liouvillian integrability for two-dimensional polynomial dynamical systems, *Phys. Lett. A* 382 (20) (2018) 1353–1360.