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Remez-type inequalities for the hyperbolic cross polynomials

V. Temlyakov and S. Tikhonov *

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Abstract

In this paper we study the Remez-type inequalities for trigonometric polynomials with harmonics from hyperbolic crosses. The interrelation between the Remez and Nikolskii inequalities for individual functions and its applications are discussed.

1 Introduction

In many questions in analysis one deals with a problem of finding the best possible way to estimate the global norm $\|f\|_{X(\Omega)}$ in terms of local norms $\|f\|_{X(\Omega \setminus B)}$. In some cases, this problem can be reduced to the problem for certain approximation methods, in particular, polynomials. An important result in this topic is the Remez inequality.

For algebraic polynomials P_n , the Remez inequality establishes a sharp upper bound for $\|P_n\|_{L_\infty[-1,1]}$ if the measure of the subset of $[-1, 1]$, where the modulus of the polynomial is at most 1, is known [22]. A sharp multidimensional inequality for algebraic polynomials was obtained by Brudnyi and Ganzburg in [4].

In the case of trigonometric polynomials $T_n(x) = \sum_{|k| \leq n} c_k e^{ikx}$, $c_k \in \mathbb{C}$, the Remez inequality reads as follows: for any Lebesgue measurable set $B \subset \mathbb{T}$ we have

$$\|T_n\|_{L_\infty([0,2\pi])} \leq C(n, |B|) \|T_n\|_{L_\infty([0,2\pi] \setminus B)}. \quad (1.1)$$

*V. Temlyakov, University of South Carolina and Steklov Institute of Mathematics, S. Tikhonov, ICREA, Centre de Recerca Matemàtica, and UAB

In [8], (1.1) was proved with $C(n, |B|) = \exp(4n|B|)$ for $|B| < \pi/2$; the history of the question can be found in, e.g., [2, Ch. 5], [11, Sec. 3], and [16]. The constant can be sharpened as $C(n, |B|) = \exp(2n|B|)$, see [11, Th. 3.1].

In case when the measure $|B|$ is big, that is, when $\pi/2 < |B| < 2\pi$, one has

$$C(n, |B|) = \left(\frac{17}{2\pi - |B|} \right)^{2n},$$

see [11, 17] and references therein.

Asymptotics of the sharp constant in the Remez inequality was recently obtained in [21] and [12] for $|B| \rightarrow 0$ and $|B| \rightarrow 2\pi$, respectively.

Multidimensional variants of Remez' inequality for trigonometric polynomials

$$T_{\mathbf{n}}(x) = \sum_{|k_1| \leq n_1} \cdots \sum_{|k_d| \leq n_d} c_{\mathbf{k}} e^{i(\mathbf{k}, x)}, \quad c_{\mathbf{k}} \in \mathbb{C}, \quad x \in \mathbb{T}^d, \quad d \geq 1,$$

were obtained in [21]:

$$\|T_{\mathbf{n}}\|_{L_{\infty}(\mathbb{T}^d)} \leq \exp \left(2d \left(|B| \prod_{j=1}^d n_j \right)^{1/d} \right) \|T_{\mathbf{n}}\|_{L_{\infty}(\mathbb{T}^d \setminus B)}$$

for

$$|B| < \left(\frac{\pi}{2} \right)^d \frac{\left(\min_{1 \leq j \leq d} n_j \right)^d}{\prod_{j=1}^d n_j}.$$

This improves the previous results for the case of $n_1 = \cdots = n_d$ from [6] and [15].

It is worth mentioning that Remez inequalities for exponential polynomials

$$p(t) = \sum_{k=1}^n c_k e^{\lambda_k t}, \quad c_k, \lambda_k \in \mathbb{C},$$

are sometimes called the Turán inequality after Paul Turán [28] who studied related inequalities for algebraic complex-valued polynomials. In [17], Nazarov proved that for an interval $I \subset \mathbb{R}$ and a measurable set $E \subset I$ of positive Lebesgue measure one has

$$\sup_{t \in I} |p(t)| \leq e^{\mu(I) \max |\operatorname{Re} \lambda_k|} \left(\frac{A\mu(I)}{\mu(E)} \right)^{n-1} \sup_{t \in E} |p(t)|.$$

Here, $A > 0$ is an absolute constant, independent of n .

Many different applications of Remez type inequalities include extension theorems (see, e.g., [3, 30]) and polynomial inequalities (see, e.g., [11, 6, 15]). Moreover, Remez inequalities were used to obtain the uncertainty principle relations of the type

$$\|f\|_{L^2(\mathbb{R})}^2 \leq Ae^{A\mu(E)\mu(\Sigma)} \left(\int_{\mathbb{R} \setminus E} |f|^2 + \int_{\mathbb{R} \setminus \Sigma} |\widehat{f}|^2 \right)$$

for any function $f \in L^2(\mathbb{R})$ (see [17]) and Logvinenko–Sereda type theorems (see [14, 17]).

In [18], the authors used the Remez inequalities to derive sharp dimension-free estimates for the distribution of values of polynomials in convex subsets in \mathbb{R}^n , which allows to obtain interesting results about the distribution of zeroes of random analytic functions. This topic is closely related to the known Kannan-Lovász-Simonovits lemma. In addition, the Remez inequality turns out to be useful to deal with the Rademacher Fourier series

$$f(\theta) = \sum_{k \in \mathbb{Z}} \xi_k a_k e^{2\pi i k \theta},$$

where ξ_k are independent Rademacher random variables taking the values of ± 1 with probability $1/2$ and the coefficient sequence $\{a_k\} \in \ell^2$. In particular, in [19] the authors obtain L^p bounds for the logarithm of a Rademacher Fourier series.

Remez inequality is closely related to the so-called Bernstein type inequalities [10, 29], which have many applications in differential equations, potential theory, and dynamical systems, see [29].

The main goal of this paper is to prove the Remez-type inequalities for the hyperbolic cross trigonometric polynomials. We also establish connections between the Remez-type inequalities and the Nikol'skii-type inequalities in a general setting. We use the following definitions of these inequalities.

Definition 1.1. *We say that f satisfies the Remez-type inequality with parameters p, b, R (in other words, $RI(p, b, R)$ holds) if for any measurable $B \subset \Omega$ with measure $|B| \leq b$*

$$\|f\|_{L_p(\Omega)} \leq R \|f\|_{L_p(\Omega \setminus B)}. \quad (1.2)$$

Definition 1.2. For $p > q$ we say that f satisfies the Nikol'skii-type inequality with parameters p, q, C, m (in other words, $NI(p, q, C, m)$ holds) if

$$\|f\|_p \leq Cm^\beta \|f\|_q, \quad \|f\|_p := \|f\|_{L_p(\Omega)}, \quad \beta := 1/q - 1/p. \quad (1.3)$$

In Section 2 we establish that the Remez-type inequalities and the Nikol'skii-type inequalities are closely connected. A typical result, that shows that RI implies NI is Proposition 2.1, which gives for all $0 < q < p \leq \infty$

$$RI(\infty, b, R) \Rightarrow NI(p, q, R^{q\beta}, 1/b).$$

A typical result in the opposite direction, which shows that NI implies RI is Proposition 2.3 and Remark 2.6: we have for $0 < q < p \leq \infty$

$$NI(p, q, C, m) \Rightarrow RI(q, (C'm)^{-1}, 2^{\max(1, 1/q)}).$$

It is well known and easy to derive from the interpolation inequality

$$\|f\|_q \leq \|f\|_v^\theta \|f\|_p^{1-\theta}, \quad 0 < v < q < p \leq \infty, \quad \theta := (1/q - 1/p)(1/v - 1/p)^{-1}$$

that for $0 < v < q < p \leq \infty$

$$NI(p, q, C, m) \Rightarrow NI(q, v, C', m).$$

This indicates that the Nikol'skii-type inequalities "propagate" from bigger values of p, q to smaller values of q, v . In Section 2 we note that a similar effect holds for the Remez-type inequalities. For instance, we prove that (see Lemma 2.2) for $0 < q < p < \infty$

$$RI(p, b, R) \Rightarrow RI(q, b, R^{p/q}).$$

The main results of the paper are in Section 3, where we study the Remez-type inequalities for the hyperbolic cross trigonometric polynomials. The above discussion shows that the strongest RI are for $p = \infty$. In Section 3 we prove that (see Theorem 3.1) the $RI(\infty, b(N), R(N))$ holds for all polynomials from $\mathcal{T}(N)$ (see the definition in Section 3) for $b(N) \asymp (N(\log N)^{d-1})^{-1}$ and $R(N) \asymp (\log N)^{d-1}$. We also prove that the extra factor $R(N)$ cannot be substantially improved. Namely, Proposition 3.1 shows that even if we make a stronger assumption on $b(N) \asymp (N(\log N)^A)^{-1}$ with arbitrarily large fixed A , we still cannot replace $R(N) \asymp (\log N)^{d-1}$ by $R(N) \asymp (\log N)^{(d-1)(1-\delta)}$

with some $\delta > 0$. This indicates that the Remez-type inequalities for $p = \infty$ for the hyperbolic cross polynomials differ from their univariate counterparts. It is not surprising, because it is known (see [23]) that the same phenomenon holds for the Bernstein and Nikol'skii inequalities. In Section 3 we establish that contrary to the case $p = \infty$ in the case $p < \infty$ the RI has the form similar to the univariate case (see Theorem 3.2). In particular, this implies that Theorem 3.2 is sharp. The problem of sharpness of the RI in the case $p = \infty$ is open. For instance, we do not know what is the best rate of decay of $b(N)$, which guarantees the $RI(\infty, b(N), R(N))$ with the above $R(N) \asymp (\log N)^{d-1}$. Theorem 3.1 shows that it is sufficient to take $b(N) \asymp (N(\log N)^{d-1})^{-1}$. However, Theorem 3.4 shows that we can expect some improvements on the rate of $b(N)$. Here is other very interesting open problem.

Open problem. What is the best rate of $\{b(N)\}$ to guarantee that $RI(\infty, b(N), C(d))$ holds for $\mathcal{T}(N)$?

This problem might be related to the discretization problem discussed in Subsections 2.4 and 3.4.

As usual, $f \ll g$ for $f, g \geq 0$ means that $f \leq Cg$ with C independent of essential quantities, and $f \asymp g$ means that $f \ll g \ll f$.

2 Some general inequalities

In this section we show how the Remez-type inequality for an individual function f can be used to derive the Nikol'skii-type inequalities for f . Also, we show how the Nikol'skii-type inequalities imply the Remez-type inequalities. These results show that the Remez-type and the Nikol'skii-type inequalities are closely related. In addition, we show that the discretization inequality (see below for the definition) implies the Remez-type inequalities.

2.1 Remez-type inequalities

Suppose f is a continuous on a compact Ω function. Let μ be a normalized measure on Ω . Assume that the following Remez-type inequality holds: for any measurable $B \subset \Omega$ with measure $|B| \leq b$

$$\|f\|_{L^\infty(\Omega)} \leq R \|f\|_{L^\infty(\Omega \setminus B)}. \quad (2.1)$$

We now show that inequality (2.1) implies the Remez-type inequality for f in the $L_p(\Omega)$, $0 < p < \infty$.

Lemma 2.1. *We have for $0 < p < \infty$*

$$RI(\infty, b, R) \Rightarrow RI(p, b/2, 2^{1/p}R). \quad (2.2)$$

Proof. We prove inequality (1.2) for B , satisfying $|B| \leq b/2$. Take any set $B \subset \Omega$ with $|B| \leq b/2$ and estimate

$$\int_B |f|^p d\mu \leq |B| \|f\|_\infty^p, \quad \|f\|_\infty := \|f\|_{L_\infty(\Omega)}. \quad (2.3)$$

Define $B' \subset \Omega \setminus B$, $|B'| = b/2$, to be such that for all $x \in B'$ we have

$$|f(x)| \geq \sup_{u \in (\Omega \setminus B) \setminus B'} |f(u)|. \quad (2.4)$$

Denote $B'' := B \cup B'$. Then $|B''| \leq b$. By (2.1) and (2.4) for all $x \in B'$

$$|f(x)| \geq \sup_{u \in \Omega \setminus B''} |f(u)| \geq R^{-1} \|f\|_\infty.$$

Therefore,

$$\|f\|_\infty^p \leq \frac{2}{b} \int_{B'} (R|f|)^p d\mu. \quad (2.5)$$

Inequalities (2.3) and (2.5) imply

$$\begin{aligned} \int_\Omega |f|^p d\mu &= \int_B |f|^p d\mu + \int_{\Omega \setminus B} |f|^p d\mu \leq \int_{B'} (R|f|)^p d\mu + \int_{\Omega \setminus B} |f|^p d\mu \\ &\leq (R^p + 1) \int_{\Omega \setminus B} |f|^p d\mu. \end{aligned} \quad (2.6)$$

In other words

$$\|f\|_{L_p(\Omega)} \leq (R^p + 1)^{1/p} \|f\|_{L_p(\Omega \setminus B)} \leq 2^{1/p} R \|f\|_{L_p(\Omega \setminus B)} \quad (2.7)$$

for any B with $|B| \leq b/2$. \square

Remark 2.1. *Note that the reverse implication of relation (2.2) is not valid. More precisely, there are no positive constants C_1 and C_2 such that for $0 < p < \infty$*

$$RI(p, b, R) \Rightarrow RI(\infty, C_1 b, C_2 R).$$

This follows immediately from Proposition 3.1 and Theorem 3.2 below.

Lemma 2.2. *We have for $0 < q < p < \infty$*

$$RI(p, b, R) \Rightarrow RI(q, b, R^{p/q}).$$

Proof. Let $p < \infty$ and let f satisfy the $RI(p, b, R)$: for any B , $|B| \leq b$ we have

$$\int_{\Omega} |f|^p \mu \leq R^p \int_{\Omega \setminus B} |f|^p d\mu. \quad (2.8)$$

Let B^* be a set of measure b such that for all $x \in B^*$ we have

$$|f(x)| \geq \sup_{u \in \Omega \setminus B^*} |f(u)| =: T.$$

Then, by (2.8) for $f \neq 0$ we have $T > 0$. It is clear that for any B , $|B| = b$ we have

$$\int_{\Omega \setminus B} |f|^q d\mu \geq \int_{\Omega \setminus B^*} |f|^q d\mu.$$

We estimate from below $\int_{\Omega \setminus B^*} |f|^q d\mu$. By (2.8) we get

$$R^p \int_{\Omega \setminus B^*} |f|^p d\mu \geq \int_{\Omega} |f|^p d\mu = \int_{\Omega \setminus B^*} |f|^p d\mu + \int_{B^*} |f|^p d\mu$$

and

$$(R^p - 1) \int_{\Omega \setminus B^*} |f|^p d\mu \geq \int_{B^*} |f|^p d\mu. \quad (2.9)$$

Using the inequalities $|f|/T \leq 1$ on $\Omega \setminus B^*$ and $|f|/T \geq 1$ on B^* we write

$$(R^p - 1) \int_{\Omega \setminus B^*} (|f|/T)^q d\mu \geq (R^p - 1) \int_{\Omega \setminus B^*} (|f|/T)^p d\mu$$

and continue by (2.9)

$$\geq \int_{B^*} (|f|/T)^p d\mu \geq \int_{B^*} (|f|/T)^q d\mu. \quad (2.10)$$

This implies

$$R^p \int_{\Omega \setminus B^*} |f|^q d\mu \geq \int_{\Omega} |f|^q d\mu,$$

which completes the proof. \square

Note that also as in Remark 2.1 the Remez inequality for $p < \infty$ keeps only "strong monotonicity" property with respect to parameters.

Remark 2.2. *There are no positive constants C_1 and C_2 such that for $0 < v < p < \infty$*

$$RI(v, b, R) \Rightarrow RI(p, C_1 b, C_2 R^{v/p}). \quad (2.11)$$

First we prove that the inequality

$$NI(p, q, C, m) \Rightarrow NI(q, v, C', m), \quad 0 < v < q < p \leq \infty,$$

mentioned in the introduction, is not invertible in the following sense.

Remark 2.3. *The implication*

$$NI(q, v, C', m) \Rightarrow NI(p, q, C, cm), \quad c > 0,$$

does not hold in general.

Note that the case $p = \infty$ follows easily from Theorems 3.7 and 3.8 below. In the case $p < \infty$ we will use the following sharp Nikol'skii inequalities for spherical harmonics recently obtained in [5]. Let \mathcal{H}_n^d be the space of all spherical harmonics of degree n on $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$, where $\|\cdot\|$ denotes the Euclidean norm of \mathbb{R}^d . In particular, it is proved in [5] that for $d \geq 3$

(i) if $1 \leq v \leq 2$ and $v < q \leq \frac{dv'}{d-2}$, then

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_v} \asymp n^{\frac{d-2}{2}(\frac{1}{v} - \frac{1}{q})}, \quad (2.12)$$

(ii) if $\frac{2d-2}{d-2} < q < p \leq \infty$, then

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_p}{\|Y_n\|_q} \asymp n^{(d-1)(\frac{1}{q} - \frac{1}{p})}. \quad (2.13)$$

Since $1 \leq v \leq 2$ always implies $\frac{2d-2}{d-2} < v'$, inequalities (2.12) and (2.13) give

$$NI(q, v, C_1(q, v), m) \not\Rightarrow NI(p, q, C_2(p, q), C_3 m)$$

for

$$\frac{2d-2}{d-2} < q < \frac{dv'}{d-2}$$

and

$$1 \leq v \leq 2 < q < p \leq \infty.$$

This completes the proof of Remark 2.8.

To prove Remark 2.2, we first note that Proposition 2.3 implies that

$$NI(q, v, C_1(q, v), m) \Rightarrow RI(v, \frac{C_1'(q, v)}{m}, 2^{\max(1, 1/v)}), \quad 0 < v < q \leq \infty.$$

On the other hand, Proposition 2.1 yields that

$$RI(p, \frac{C_1''(p, v, q)}{m}, C_2(p, v)) \Rightarrow NI(p, q, C_2(p, v), \frac{m}{C_1''(p, v, q)})$$

for $0 < q < p < \infty$. Combining these estimates with inequality (2.11) for $0 < v < p < \infty$, we finally get

$$NI(q, v, C_1(q, v), m) \Rightarrow NI(p, q, C_2(p, v), \frac{m}{C_1''(p, v, q)})$$

for $0 < v < q < p < \infty$. This contradicts Remark 2.8. Thus, the proof of Remark 2.2 is now complete.

2.2 Remez-type inequality implies Nikol'skii-type inequalities

First, we derive from (2.1) Nikol'skii-type inequalities for f . We begin with estimating $\|f\|_\infty$ in terms of $\|f\|_q$, $0 < q < \infty$. Let, as above, B^* be a set of measure b such that for all $x \in B^*$ we have

$$|f(x)| \geq \sup_{u \in \Omega \setminus B^*} |f(u)|.$$

Then by (2.1) we have for all $x \in B^*$

$$|f(x)| \geq R^{-1} \|f\|_\infty.$$

Therefore,

$$\|f\|_q^q \geq \int_{B^*} |f|^q d\mu \geq |B^*| (R^{-1} \|f\|_\infty)^q.$$

We obtain from here

$$\|f\|_\infty \leq Rb^{-1/q}\|f\|_q. \quad (2.14)$$

Let now $0 < q < p < \infty$. We have

$$\|f\|_p = \| |f|^{1-q/p} |f|^{q/p} \|_p \leq \|f\|_\infty^{1-q/p} \|f\|_q^{q/p}.$$

Using (2.14) we continue

$$\leq (Rb^{-1/q})^{1-q/p} \|f\|_q^{1-q/p} \|f\|_q^{q/p} = R^{q\beta} b^{-\beta} \|f\|_q, \quad \beta := 1/q - 1/p.$$

Thus we have proved the following statement.

Proposition 2.1. *Remez-type inequality (2.1) implies Nikol'skii-type inequality*

$$\|f\|_p \leq R^{q\beta} b^{-\beta} \|f\|_q, \quad \beta := 1/q - 1/p \quad (2.15)$$

for all $0 < q < p \leq \infty$. In other words,

$$RI(\infty, b, R) \Rightarrow NI(p, q, R^{q\beta}, 1/b).$$

Second, we consider the case of $RI(p, b, R)$ with $p < \infty$.

Proposition 2.2. *For $0 < q < p < \infty$ we have*

$$RI(p, b, R) \Rightarrow NI(p, q, R, 1/b).$$

Proof. We use the same notations as in the above proof of Lemma 2.2. First, we bound from above the thresholding parameter T . We have

$$\|f\|_q^q \geq \int_{B^*} |f|^q d\mu \geq T^q b,$$

which implies

$$T \leq \|f\|_q b^{-1/q}. \quad (2.16)$$

Second, we estimate

$$\int_{\Omega \setminus B^*} (|f|/T)^q d\mu \geq \int_{\Omega \setminus B^*} (|f|/T)^p d\mu \geq T^{-p} R^{-p} \|f\|_p^p. \quad (2.17)$$

Relations (2.17) and (2.16) imply

$$\|f\|_p^p \leq R^p T^{p-q} \|f\|_q^q \leq R^p b^{-(p-q)/q} \|f\|_q^q \quad (2.18)$$

and

$$\|f\|_p \leq Rb^{-\beta} \|f\|_q.$$

□

Note that the statements of Propositions 2.1 and 2.2 are sharp in the following sense.

Remark 2.4. For $0 < q < p \leq \infty$ the implication

$$NI(p, q, R, 1/b) \Rightarrow RI(p, C_1 b, C_2 R(q, \beta)), \quad C_1, C_2 > 0,$$

does not hold in general.

In particular, this follows from Proposition 3.1 and Theorem 3.7 taking $p = \infty$ and $0 < q \leq 1$.

Remark 2.5. In light of Lemmas 2.1 and 2.2, one can ask if for $0 < q < p \leq \infty$ the following implication

$$RI(q, b, R) \Rightarrow NI(p, q, C_1 R(p, q), C_2/b), \quad C_1, C_2 > 0,$$

holds, which is stronger than the one stated in Propositions 2.1 and 2.2. Again, Proposition 3.1 and Theorem 3.7 with $p = \infty$ and $0 < q \leq 1$ show that this is not the case.

2.3 Nikol'skii inequality implies Remez inequality

We prove here the following statement.

Proposition 2.3. Suppose that a function f satisfies the Nikol'skii inequality

$$\|f\|_p \leq C(p, q) m^\beta \|f\|_q, \quad 1 \leq q < p \leq \infty, \quad \beta := 1/q - 1/p. \quad (2.19)$$

Then there exists a constant $C'(p, q)$ such that for any set $B \in \Omega$, $|B| \leq (C'(p, q)m)^{-1}$ we have

$$\|f\|_{L_q(\Omega)} \leq 2\|f\|_{L_q(\Omega \setminus B)}. \quad (2.20)$$

Proof. Denote $B^c := \Omega \setminus B$ and χ_A the characteristic function of a set A . Then

$$\|f\|_q \leq \|f\chi_{B^c}\|_q + \|f\chi_B\|_q. \quad (2.21)$$

Further, by Hölder inequality with parameter p/q and our assumption (2.19) we obtain

$$\|f\chi_B\|_q \leq \|f\|_p |B|^\beta \leq C(p, q) m^\beta |B|^\beta \|f\|_q. \quad (2.22)$$

Making measure $|B|$ small enough to satisfy $C(p, q) m^\beta |B|^\beta \leq 1/2$ we derive from (2.22) and (2.21) the required inequality. \square

Remark 2.6. *Proposition 2.3 holds for all $0 < q < p \leq \infty$ with 2 replaced by $2^{1/q}$ in (2.20) in case $q < 1$.*

Remark 2.7. *Remark 2.5 shows that the reverse statement to Proposition 2.3 does not hold in general.*

Propositions 2.2 and 2.3 yield the following result.

Remark 2.8. *Let W_m , $m \in \mathbb{N}$ or $m > 0$, be a collection of subclasses of $L_r(\Omega)$, $A < r < B$. The following two conditions are equivalent:*

(i) *for any $A < q < p < B$ we have*

$$\sup_{f \in W_m} \frac{\|f\|_p}{\|f\|_q} \asymp \lambda(m)^{\frac{1}{q} - \frac{1}{p}};$$

(ii) *for any $A < r < B$ we have*

$$\sup_{f \in W_m} \sup \left\{ |B| : \|f\|_{L_r(\Omega)} \leq R \|f\|_{L_r(\Omega \setminus B)} \right\} \asymp \frac{1}{\lambda(m)}.$$

In many cases the Nikol'skii type inequalities are known. Proposition 2.3 and Remark 2.6 allow us to derive the Remez type inequalities from these known results. We illustrate this on some examples.

Example 2.1. (i). *Taking into account the results from [20], for each trigonometric polynomial*

$$T(x) = \sum_{k \in \text{supp } \hat{T}} c_k \exp(ikx), \quad \text{supp } \hat{T} = \{k \in \mathbb{Z}^d : \hat{T}(k) = c_k \neq 0\}$$

we have

$$\|T\|_{L_p(\mathbb{T}^d)} \leq 2^{\max(1, 1/p)} \|T\|_{L_p(\mathbb{T}^d \setminus B)},$$

where

$$|B| \leq \frac{1}{C} \begin{cases} 1/N(\text{supp}(\hat{T})), & 0 < p < 2; \\ 1/N(p_0 \text{Conv}(\text{supp}(\hat{T}))), & 2 \leq p < \infty, \end{cases}$$

$N(X)$ is the number of lattice points in $X \subset \mathbb{R}^d$, p_0 is the smallest integer not less than $p/2$, and $\text{Conv}(\text{supp}(\hat{T}))$ denotes the convex hull of $\text{supp}(\hat{T})$.

(ii). For each trigonometric polynomial

$$T_n(x) = \sum_{k=1}^n c_k \exp(in_k x), \quad n_k \in \mathbb{Z},$$

we have

$$\|T_n\|_{L_p(\mathbb{T})} \leq 2^{\max(1, 1/p)} \|T_n\|_{L_p(\mathbb{T} \setminus B)},$$

where

$$|B| \leq \frac{1}{C} \begin{cases} 1/n, & 0 < p \leq 2; \\ 1/n^{p/2}, & 2 < p < \infty. \end{cases}$$

This follows from results of Belinskii [1].

(iii). Sharp Nikol'skii inequalities for spherical harmonics given by inequalities (2.12) and (2.13) imply that for any $Y_n \in \mathcal{H}_n^d$, $d \geq 3$, we have that $Y_n \in RI(p, 1/b, 2^{\max(1, 1/p)})$ with

$$|B| \leq \frac{1}{C} \begin{cases} 1/n^{\frac{d-2}{2}}, & 0 < p < 2; \\ 1/n^{d-1}, & \frac{2d-2}{d-2} < p < \infty. \end{cases}$$

(iv). Let $\Lambda_n = \{\lambda_0 < \lambda_1 < \dots < \lambda_n\}$ be a set of real numbers. Let us denote by $E(\Lambda_n)$ the collection of all linear combination of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over \mathbb{R} . Then the sharp Nikol'skii inequality from [9] imply that

$$\|P\|_{L_p([a, b])} \leq 2^{\max(1, 1/p)} \|P\|_{L_p([a, b] \setminus B)}, \quad P \in E(\Lambda_n), \quad 0 < p < \infty,$$

where

$$|B| \leq \frac{1}{C \left(n^2 + \sum_{j=1}^n |\lambda_j| \right)}.$$

(v). For functions f such that $\text{supp } \hat{f}$ is compact, using [20], we have that

$$\|f\|_{L_p(\mathbb{R}^d)} \leq 2^{\max(1, 1/p)} \|f\|_{L_p(\mathbb{R}^d \setminus B)},$$

where

$$|B| \leq \frac{1}{C} \begin{cases} 1/\mu(\text{supp}(\hat{f})), & 0 < p < 2; \\ 1/p_0^n \mu(\text{Conv}(\text{supp}(\hat{T}))), & 2 \leq p < \infty, \end{cases}$$

$\mu(X)$ is the Lebesgue measure of X and p_0 is the smallest integer not less than $p/2$.

Note that Remark 2.8 shows that if we have sharp Nikol'skii inequalities, which is the case in (ii), (iii), and (iv), then the corresponding Remez inequalities obtained above are also sharp.

2.4 Discretization inequality implies Remez inequality

We prove the following theorem here.

Theorem 2.1. *Let f be a continuous periodic function on \mathbb{T}^d . Assume that there exists a set $X_m = \{\mathbf{x}^j\}_{j=1}^m \subset \mathbb{T}^d$ such that for all functions $f_{\mathbf{y}}(\mathbf{x}) := f(\mathbf{x} - \mathbf{y})$ we have the discretization inequality*

$$\|f_{\mathbf{y}}\|_{\infty} \leq D \max_{1 \leq j \leq m} |f_{\mathbf{y}}(\mathbf{x}^j)|. \quad (2.23)$$

Then for any B with $|B| < 1/m$ we have (2.1) with $R = D$.

Proof. Consider the function

$$g(\mathbf{y}) := \sum_{j=1}^m \chi_B(\mathbf{x}^j - \mathbf{y}).$$

At each point \mathbf{y} either $g(\mathbf{y}) = 0$ or $g(\mathbf{y}) \geq 1$. We prove that for B , $|B| < 1/m$ there is a point \mathbf{y}^* such that $g(\mathbf{y}^*) = 0$. We prove this by contradiction. If such point \mathbf{y}^* does not exist then $g(\mathbf{y}) \geq 1$ for all $\mathbf{y} \in \mathbb{T}^d$ and

$$\int_{\mathbb{T}^d} g d\mu \geq 1.$$

On the other hand

$$\int_{\mathbb{T}^d} g d\mu = m|B| < 1.$$

The obtained contradiction proves the existence of \mathbf{y}^* such that $g(\mathbf{y}^*) = 0$. This implies in turn that for all j we have $\chi_B(\mathbf{x}^j - \mathbf{y}^*) = 0$ or, in other words, $\mathbf{x}^j - \mathbf{y}^* \in B^c := \mathbb{T}^d \setminus B$. Next, by (2.23)

$$\|f\|_{\infty} = \|f_{\mathbf{y}^*}\|_{\infty} \leq D \max_{1 \leq j \leq m} |f_{\mathbf{y}^*}(\mathbf{x}^j)| = D \max_{1 \leq j \leq m} |f(\mathbf{x}^j - \mathbf{y}^*)| \leq D \sup_{\mathbf{x} \in B^c} |f(\mathbf{x})|.$$

This completes the proof. \square

3 Hyperbolic cross polynomials

3.1 Remez inequality

Let

$$\Gamma(N) := \{\mathbf{k} = (k_1, \dots, k_d) : \prod_{j=1}^d \bar{k}_j \leq N, \quad \bar{k}_j := \max(1, |k_j|)\}$$

be the hyperbolic cross and

$$\mathcal{T}(N) := \{f(\mathbf{x}), \mathbf{x} = (x_1, \dots, x_d) : f(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma(N)} c_{\mathbf{k}} e^{i(\mathbf{k}, \mathbf{x})}\}.$$

Theorem 3.1. *There exist two positive constants $C_1(d)$ and $C_2(d)$ such that for any set $B \subset \mathbb{T}^d$ of normalized measure $|B| \leq (C_2(d)N(\log N)^{d-1})^{-1}$ and for any $f \in \mathcal{T}(N)$ we have*

$$\|f\|_{\infty} \leq C_1(d)(\log N)^{d-1} \sup_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})|. \quad (3.1)$$

Proof. Denote by \mathcal{V}_N the de la Vallée Poussin kernel for the hyperbolic cross $\Gamma(N)$: $\hat{\mathcal{V}}_N(\mathbf{k}) = 1$ for $\mathbf{k} \in \Gamma(N)$ and $\hat{\mathcal{V}}_N(\mathbf{k}) = 0$ for $\mathbf{k} \notin \Gamma(2^d N)$. It is known (see, for instance, [23], Chapter 1) that there exists a kernel \mathcal{V}_N with the following properties:

$$\|\mathcal{V}_N\|_1 \leq C'(d)(\log N)^{d-1}, \quad \|\mathcal{V}_N\|_{\infty} \leq C''(d)N(\log N)^{d-1}. \quad (3.2)$$

Then for any $f \in \mathcal{T}(N)$ we have $f = f * \mathcal{V}_N$, where $*$ means convolution.

Let B be a set of small measure. We have for $f \in \mathcal{T}(N)$

$$\begin{aligned} \|f\|_{\infty} &= \|(2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{u}) \mathcal{V}_N(\mathbf{x} - \mathbf{u}) d\mathbf{u}\|_{\infty} \\ &= \|(2\pi)^{-d} \int_{\mathbb{T}^d \setminus B} f(\mathbf{u}) \mathcal{V}_N(\mathbf{x} - \mathbf{u}) d\mathbf{u} + (2\pi)^{-d} \int_B f(\mathbf{u}) \mathcal{V}_N(\mathbf{x} - \mathbf{u}) d\mathbf{u}\|_{\infty} \\ &\leq \max_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})| \|\mathcal{V}_N\|_1 + |B| \|f\|_{\infty} \|\mathcal{V}_N\|_{\infty} \\ &\leq C'(d)(\log N)^{d-1} \max_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})| + C''(d)N(\log N)^{d-1} |B| \|f\|_{\infty}. \end{aligned}$$

If $C''(d)N(\log N)^{d-1}|B| \leq 1/2$ then

$$\|f\|_{\infty} \leq 2C'(d)(\log N)^{d-1} \max_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})|.$$

This completes the proof with $C_1(d) := 2C'(d)$ and $C_2(d) := 2C''(d)$. \square

Theorem 3.1 cannot be improved in a certain sense. The following statement holds for $d \geq 2$.

Proposition 3.1. *The following statement is false: There exist $\delta > 0$, A , c , and C such that for any $f \in \mathcal{T}(N)$ and any set $B \subset \mathbb{T}^d$ of measure $|B| \leq (cN(\log N)^A)^{-1}$ the Remez-type inequality holds*

$$\|f\|_\infty \leq C(\log N)^{(d-1)(1-\delta)} \sup_{\mathbf{u} \in \mathbb{T}^d \setminus B} |f(\mathbf{u})|. \quad (3.3)$$

Proof. We use Proposition 2.1 with $p = \infty$. Our assumption (3.3) gives (2.1) with $b = (cN(\log N)^A)^{-1}$ and $R = C(\log N)^{(d-1)(1-\delta)}$. Therefore, by Proposition 2.1 with $p = \infty$ (see also (2.14)) we get for all $f \in \mathcal{T}(N)$

$$\|f\|_\infty \leq Rb^{-1/q} \|f\|_q, \quad 1 \leq q < \infty. \quad (3.4)$$

It is known (see [23] and Theorem 3.5 below) that it should be

$$Rb^{-1/q} \geq C(d, q)N^{1/q}(\log N)^{(d-1)(1-1/q)}. \quad (3.5)$$

Substituting our b and R expressed in terms of N and choosing large enough q and N , we obtain a contradiction in (3.5). \square

By (2.7) Theorem 3.1 implies the following Remez-type inequality for all $0 < p < \infty$.

Corollary 3.1. *There exist two positive constants $C_1(d, p)$ and $C_2(d)$ (this constant is from Theorem 3.1) such that for any set $B \subset \mathbb{T}^d$ of normalized measure $|B| \leq (2C_2(d)N(\log N)^{d-1})^{-1}$ and for any $f \in \mathcal{T}(N)$ we have*

$$\|f\|_p \leq C_1(d, p)(\log N)^{d-1} \|f\|_{L_p(\mathbb{T}^d \setminus B)}. \quad (3.6)$$

Proposition 2.3, Remark 2.6, and the Nikol'skii inequalities in Theorem 3.8, allow us to improve the above Corollary 3.1.

Theorem 3.2. *For $0 < q < \infty$ there exist two positive constants $C_1(d, q)$ and $C_2(d, q)$ such that for any set $B \subset \mathbb{T}^d$ of normalized measure $|B| \leq (C_2(d, q)N)^{-1}$ and for any $f \in \mathcal{T}(N)$ we have*

$$\|f\|_q \leq C_1(d, q) \|f\|_{L_q(\mathbb{T}^d \setminus B)}. \quad (3.7)$$

Theorems 3.2 and Theorem 3.8 imply the following combination of Nikol'skii-type and Remez-type inequalities.

Theorem 3.3. *For $0 < q \leq p < \infty$ there exist two positive constants $C_1 = C_1(d, p, q)$ and $C_2 = C_2(d, p, q)$ such that for any set $B \subset \mathbb{T}^d$ of normalized measure $|B| \leq (C_2N)^{-1}$ and for any $f \in \mathcal{T}(N)$ we have*

$$\|f\|_p \leq C_1 N^\beta \|f\|_{L_q(\mathbb{T}^d \setminus B)}, \quad \beta := 1/q - 1/p. \quad (3.8)$$

3.2 Improved Remez inequality in case $d = 2$

Proposition 3.1 shows that we cannot substantially improve on the additional factor $(\log N)^{d-1}$ in (3.1) of Theorem 3.1. In this subsection we will improve the bound b on the measure of a set B . Our technique is based on the Riesz products. It works in the case $d = 2$. We introduce some notations. Let $\mathbf{s} = (s_1, \dots, s_d)$ be a vector with nonnegative integer coordinates ($\mathbf{s} \in \mathbb{Z}_+^d$) and

$$\rho(\mathbf{s}) := \{\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_+^d : [2^{s_j-1}] \leq |k_j| < 2^{s_j}, \quad j = 1, \dots, d\}$$

where $[a]$ denotes the integer part of a number a . Denote for a natural number n

$$Q_n := \cup_{\|\mathbf{s}\|_1 \leq n} \rho(\mathbf{s}); \quad \Delta Q_n := Q_n \setminus Q_{n-1} = \cup_{\|\mathbf{s}\|_1 = n} \rho(\mathbf{s})$$

with $\|\mathbf{s}\|_1 = s_1 + \dots + s_d$ for $\mathbf{s} \in \mathbb{Z}_+^d$. We call a set ΔQ_n *hyperbolic layer*. For a set $L \subset \mathbb{Z}^d$ denote

$$\mathcal{T}(L) := \{f \in L_1 : \hat{f}(\mathbf{k}) = 0, \mathbf{k} \in \mathbb{Z}^d \setminus L\}.$$

For any two integers $a \geq 1$ and $0 \leq b < a$, we shall denote by $AP(a, b)$ the arithmetical progression of the form $al + b$, $l = 0, 1, \dots$. Set

$$H_n(a, b) := \{\mathbf{s} = (s_1, s_2) : \mathbf{s} \in \mathbb{Z}_+^2, \quad \|\mathbf{s}\|_1 = n, \quad s_1, s_2 \geq a, \quad s_1 \in AP(a, b)\}.$$

Define

$$\rho'(\mathbf{s}) := \{\mathbf{m} = (m_1, m_2) : [2^{s_i-2}] \leq |m_i| < 2^{s_i}, i = 1, 2\}.$$

Let us define the polynomials $\mathcal{A}_{\mathbf{s}}(\mathbf{x})$ for $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}_0^d$

$$\mathcal{A}_{\mathbf{s}}(\mathbf{x}) := \prod_{j=1}^d \mathcal{A}_{s_j}(x_j),$$

with $\mathcal{A}_{s_j}(x_j)$ defined as follows:

$$\mathcal{A}_0(x) := 1, \quad \mathcal{A}_1(x) := \mathcal{V}_1(x) - 1, \quad \mathcal{A}_s(x) := \mathcal{V}_{2^{s-1}}(x) - \mathcal{V}_{2^{s-2}}(x), \quad s \geq 2,$$

where \mathcal{V}_m are the de la Vallée Poussin kernels. Then for $d = 2$

$$\mathcal{A}_{\mathbf{s}} \in \mathcal{T}(\rho'(\mathbf{s})).$$

For a subspace Y in $L_2(\mathbb{T}^d)$ we denote by Y^\perp its orthogonal complement. We need the following lemma on the Riesz product, which is Lemma 2.1 from [25].

Lemma 3.1. *Take any trigonometric polynomials $t_{\mathbf{s}} \in \mathcal{T}(\rho'(\mathbf{s}))$ and form the function*

$$\Phi(\mathbf{x}) := \prod_{\mathbf{s} \in H_n(a,b)} (1 + t_{\mathbf{s}}).$$

Then for any $a \geq 6$ and any $0 \leq b < a$ this function admits the representation

$$\Phi(\mathbf{x}) = 1 + \sum_{\mathbf{s} \in H_n(a,b)} t_{\mathbf{s}}(\mathbf{x}) + g(\mathbf{x})$$

with $g \in \mathcal{T}(Q_{n+a-6})^\perp$.

We remind that we restrict ourselves to $d = 2$. Denote

$$t_{\mathbf{s}} := \mathcal{A}_{\mathbf{s}}/M, \quad M := \max_{\mathbf{s} \in H_n(a,b)} \|\mathcal{A}_{\mathbf{s}}\|_\infty \asymp 2^n.$$

Consider the Riesz product

$$\Phi := \prod_{\mathbf{s} \in H_n(a,b)} \left(1 + \frac{it_{\mathbf{s}}}{\sqrt{N}} \right), \quad N := |H_n(a,b)|.$$

Then it is easy to derive from the inequality $\left| 1 + \frac{it_{\mathbf{s}}}{\sqrt{N}} \right| \leq \left(1 + \frac{1}{N} \right)^{1/2}$ that (see Remark 2.1 from [26])

$$|\Phi| \leq C.$$

Moreover, by Lemma 3.1 we have

$$\Phi = 1 + \frac{i}{\sqrt{N}} \sum_{\mathbf{s} \in H_n(a,b)} t_{\mathbf{s}} + w, \quad w \in \mathcal{T}(Q_{n+a-6})^\perp.$$

Thus,

$$\left\| \sum_{\mathbf{s} \in H_n(a,b)} t_{\mathbf{s}} + N^{1/2} \text{Im}(w) \right\|_\infty \leq CN^{1/2}. \quad (3.9)$$

We now bound $\|w\|_1$. We introduce some more notations. Denote

$$H_n^k := \{(\mathbf{s}^1, \dots, \mathbf{s}^k) : \mathbf{s}^j \in H_n(a,b), j = 1, \dots, k, \text{ are distinct}\}$$

$$h_n := AP(a,b) \cap [a, n-a].$$

We have

$$\begin{aligned}
w &= \sum_{k=2}^N \left(\frac{i}{N^{1/2}M} \right)^k \sum_{(\mathbf{s}^1, \dots, \mathbf{s}^k) \in H_n^k} \prod_{j=1}^k \mathcal{A}_{\mathbf{s}^j} \\
&= \sum_{k=2}^N \left(\frac{i}{N^{1/2}M} \right)^k \sum_{s_1^1 \in h_n} \sum_{s_1^2 \in h_n: s_1^2 < s_1^1} \cdots \sum_{s_1^k \in h_n: s_1^k < s_1^{k-1}} \prod_{j=1}^k \mathcal{A}_{\mathbf{s}^j}.
\end{aligned}$$

Therefore,

$$\|w\|_1 \leq \sum_{k=2}^N \left(\frac{1}{N^{1/2}M} \right)^k \sum_{s_1^1 \in h_n} \sum_{s_1^2 \in h_n: s_1^2 < s_1^1} \cdots \sum_{s_1^k \in h_n: s_1^k < s_1^{k-1}} \left\| \prod_{j=1}^k \mathcal{A}_{\mathbf{s}^j} \right\|_1. \quad (3.10)$$

Next,

$$\begin{aligned}
\left\| \prod_{j=1}^k \mathcal{A}_{\mathbf{s}^j} \right\|_1 &\leq \|\mathcal{A}_{s_1^1}(x_1)\|_1 \prod_{j=2}^k \|\mathcal{A}_{s_1^j}(x_1)\|_\infty \|\mathcal{A}_{s_2^2}(x_2)\|_1 \prod_{j=1}^{k-1} \|\mathcal{A}_{s_2^j}(x_2)\|_\infty \\
&\leq C 2^{s_1^2 + \dots + s_1^k + n - s_1^1 + \dots + n - s_1^{k-1}}.
\end{aligned} \quad (3.11)$$

Inequalities (3.10) and (3.11) imply

$$\|w\|_1 \leq C \sum_{k=2}^N \left(\frac{1}{N^{1/2}M} \right)^k N 2^{n(k-1)} \ll 2^{-n}.$$

Therefore,

$$\|MN^{1/2}\text{Im}(w)\|_1 \ll N^{1/2}. \quad (3.12)$$

Bounds (3.12) and (3.9) with $a = 6$ imply that there exist a function $t \in \mathcal{T}(Q_n)^\perp$ such that

$$\left\| \sum_{\mathbf{s} \in H_n(a,b)} \mathcal{A}_{\mathbf{s}} - t \right\|_1 \ll n, \quad (3.13)$$

and

$$\left\| \sum_{\mathbf{s} \in H_n(a,b)} \mathcal{A}_{\mathbf{s}} - t \right\|_\infty \ll n^{1/2} 2^n. \quad (3.14)$$

Consider

$$\Delta \mathcal{V}_n := \sum_{\mathbf{s}: n \leq \|\mathbf{s}\|_1 \leq n+2} \mathcal{A}_{\mathbf{s}}.$$

Note that for any $f \in \mathcal{T}(\Delta Q_n)$ we have $f * \Delta \mathcal{V}_n = f$. The above inequalities (3.13) and (3.14) imply the following assertion.

Lemma 3.2. *There exists $T \in \mathcal{T}(Q_n)^\perp$ such that*

$$\|\Delta\mathcal{V}_n - T\|_1 \ll n, \quad \|\Delta\mathcal{V}_n - T\|_\infty \ll n^{1/2}2^n.$$

In the same way as Theorem 3.1 was derived from inequalities (3.2) the following theorem can be derived from Lemma 3.2.

Theorem 3.4. *Let $d = 2$. There exist two positive constants C_1 and C_2 such that for any set $B \subset \mathbb{T}^2$ of normalized measure $|B| \leq (C_2 2^n n^{1/2})^{-1}$ and for any $f \in \mathcal{T}(\Delta Q_n)$ we have*

$$\|f\|_\infty \leq C_1 n \sup_{\mathbf{u} \in \mathbb{T}^2 \setminus B} |f(\mathbf{u})|. \quad (3.15)$$

3.3 The Nikol'skii inequalities

The following two theorems are from [23], Ch.1, Section 2.

Theorem 3.5. *Suppose that $1 \leq q < \infty$. Then*

$$\sup_{f \in \mathcal{T}(N)} \|f\|_\infty / \|f\|_q \asymp N^{1/q} (\log N)^{(d-1)(1-1/q)}.$$

Theorem 3.6. *Suppose that $1 \leq q \leq p < \infty$. Then*

$$\sup_{f \in \mathcal{T}(N)} \|t\|_p / \|f\|_q \asymp N^{1/q-1/p}.$$

In this subsection we extend the above two theorems to the range of parameters $0 < q < p \leq \infty$. We begin with the case $p = \infty$.

Theorem 3.7. *Suppose that $0 < q < \infty$. Then*

$$\sup_{f \in \mathcal{T}(N)} \|f\|_\infty / \|f\|_q \asymp N^{1/q} (\log N)^{(d-1)(1-1/q)_+}.$$

Proof. We prove the upper bound in the case $0 < q < 1$. The corresponding lower bounds in this case follow from the univariate case. We derive the required inequality from Theorem 3.5 with $q = 1$. Let $f \in \mathcal{T}(N)$. Then

$$\|f\|_1 = \| |f|^{1-q} |f|^q \|_1 \leq \| |f|^{1-q} \|_\infty \| |f|^q \|_1 = \|f\|_\infty^{1-q} \|f\|_q^q.$$

Applying Theorem 3.5 with $q = 1$ we continue

$$\leq C(d)N \|f\|_1 \|f\|_\infty^{-q} \|f\|_q^q.$$

This implies

$$\|f\|_\infty^q \leq C(d)N\|f\|_q^q \quad \text{and} \quad \|f\|_\infty \leq (C(d)N)^{1/q}\|f\|_q. \quad (3.16)$$

which completes the proof. \square

Theorem 3.8. *Suppose that $0 < q < p < \infty$. Then*

$$\sup_{f \in \mathcal{T}(N)} \|t\|_p / \|f\|_q \asymp N^{1/q-1/p}.$$

Proof. We prove the upper bound in the case $0 < q < 1$. We have

$$\|f\|_p = \| |f|^{1-q/p} |f|^{q/p} \|_p \leq \|f\|_\infty^{1-q/p} \|f\|_q^{q/p}.$$

Using Theorem 3.7 we continue

$$\leq (C(d)N)^{(1-q/p)/q} \|f\|_q^{1-q/p} \|f\|_q^{q/p} = (C(d)N)^\beta \|f\|_q, \quad \beta := 1/q - 1/p.$$

The sharpness of Nikolskii's inequality, i.e., the part " \gg ", follows from the one-dimensional Jackson kernel example:

$$T(x) = \left(\frac{\sin \frac{nt}{2}}{n \sin \frac{t}{2}} \right)^{2r}, \quad r \in \mathbb{N}.$$

See [27, §4.9] for $1 \leq p \leq \infty$; in the case of $0 < p < 1$ it is enough to take r large enough ($r > \frac{1}{2p}$). \square

3.4 Discretization

An operator T_n with the following properties was constructed in [24]. The operator T_n has the form

$$T_n(f) = \sum_{j=1}^m f(\mathbf{x}^j) \psi_j(\mathbf{x}), \quad m \leq c(d)2^n n^{d-1}, \quad \psi_j \in \mathcal{T}(Q_{n+d})$$

and

$$T_n(f) = f, \quad f \in \mathcal{T}(Q_n), \quad (3.17)$$

$$\|T_n\|_{L_\infty \rightarrow L_\infty} \asymp n^{d-1}. \quad (3.18)$$

Properties (3.17) and (3.18) imply that all $f \in \mathcal{T}(Q_n)$ satisfy the discretization inequality (see [13] and [7], subsection 2.5)

$$\|f\|_\infty \leq C(d)n^{d-1} \max_{1 \leq j \leq m} |f(\mathbf{x}^j)|. \quad (3.19)$$

Note that Theorem 2.1 and the discretization inequality (3.19) give other proof of Theorem 3.1. Theorem 2.1 and Proposition 3.1 imply the following assertion.

Proposition 3.2. *The following statement is false: There exist $\delta > 0$, A , c , and C such that there exists a set $X_m = \{\mathbf{x}^j\}_{j=1}^m$ with $m \leq c2^n n^A$, which provides the discretization inequality for $\mathcal{T}(Q_n)$:*

$$\|f\|_\infty \leq Cn^{(d-1)(1-\delta)} \max_{1 \leq j \leq m} |f(\mathbf{x}^j)|, \quad f \in \mathcal{T}(Q_n).$$

Thus, an extra factor n^{d-1} in the discretization inequality for $\mathcal{T}(Q_n)$ cannot be substantially improved, if we limit ourselves to the number of $m \ll 2^n n^A$ points. It is proved in [13] (see [7], subsection 2.5, for a discussion) that in the case $d = 2$ in order to drop the extra factor n in (3.19) we need to use at least $2^{n(1+c_0)}$, $c_0 > 0$, points. It is clear that the necessary condition for the discretization inequality (3.19) to hold with some extra factor is $m \geq |Q_n| = \dim \mathcal{T}(Q_n) \asymp 2^n n^{d-1}$. Therefore, the way from discretization inequality to the Remez inequality, provided by Theorem 2.1, cannot give a better bound than $b(Q_n) \asymp (2^n n^{d-1})^{-1}$. However, the direct proof of the Remez inequality in Theorem 3.4 gives for $d = 2$ a better bound $b(\Delta Q_n) \asymp (2^n n^{1/2})^{-1}$.

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