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UNCERTAINTY PRINCIPLES FOR EVENTUALLY CONSTANT SIGN BANDLIMITED FUNCTIONS

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ABSTRACT. We study the uncertainty principles related to the generalized Logan problem in \mathbb{R}^d . Our main result provides the complete solution of the following problem: for a fixed $m \in \mathbb{Z}_+$, find

$$\sup\{|x|: (-1)^m f(x) > 0\} \cdot \sup\{|x|: x \in \text{supp } \widehat{f}\} \rightarrow \inf,$$

where the infimum is taken over all nontrivial positive definite bandlimited functions such that $\int_{\mathbb{R}^d} |x|^{2k} f(x) dx = 0$ for $k = 0, \dots, m-1$ if $m \geq 1$.

We also obtain the uncertainty principle for bandlimited functions related to the recent result by Bourgain, Clozel, and Kahane.

1. INTRODUCTION

1.1. Logan's problems. Logan stated and proved [31, 32] the following two extremal problems for real-valued positive definite bandlimited functions on \mathbb{R} . Since such functions are even, we state these problems for functions on $\mathbb{R}_+ = [0, \infty)$.

Problem A. Find the smallest $\lambda_0 > 0$ such that

$$f(x) \leq 0, \quad x > \lambda_0,$$

where f is a positive definite function of exponential type at most 1 satisfying

$$(1.1) \quad f(x) = \int_0^1 \cos xt d\mu(t), \quad d\mu \geq 0, \quad f(0) = 1.$$

Logan showed that admissible functions are integrable (even if the measure $d\mu$ is nonnegative in a neighborhood of the origin), $\lambda_0 = \pi$, and the unique extremizer is

$$f_0(x) = \frac{\cos^2(x/2)}{1 - x^2/\pi^2} = \frac{\pi}{2} \int_0^1 \sin \pi t \cos xt dt,$$

Note that f_0 satisfies $\int_{\mathbb{R}_+} f_0(x) dx = 0$.

Problem B. Find the smallest $\lambda_1 > 0$ such that

$$f(x) \geq 0, \quad x > \lambda_1,$$

where f is a positive definite integrable function satisfying (1.1) and having mean value zero.

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It turns out that admissible functions are integrable with respect to the weight x^2 , and $\lambda_1 = 3\pi$. Moreover, the unique extremizer is

$$f_1(x) = \frac{\cos^2(x/2)}{(1 - x^2/\pi^2)(1 - x^2/(3\pi)^2)} = \frac{3\pi}{4} \int_0^1 (\sin \pi t)^3 \cos xt \, dt,$$

This function satisfies $\int_{\mathbb{R}_+} x^2 f_1(x) \, dx = 0$.

We will study the multivariate generalization of Logan's problems for the Fourier transforms. In more detail, we consider the m -parameter problem, $m \in \mathbb{Z}_+ = \{0, 1, \dots\}$, so that, for $d = 1$, if $m = 0, 1$ we recover Problems A and B respectively.

Let $d \in \mathbb{N}$ and \mathbb{R}^d is d -dimensional Euclidean space with the scalar product $\langle x, y \rangle = x_1 y_1 + \dots + x_d y_d$ and the norm $|x| = \langle x, x \rangle^{1/2}$. Let $B_\tau^d = \{x \in \mathbb{R}^d : |x| \leq \tau\}$ be the ball of radius $\tau > 0$. Let $Q = \mathbb{R}^d$ or $Q = \mathbb{R}_+$. As usual, for a positive measure space $(Q, d\rho)$, let $L^p(Q, d\rho)$ denote the space of measurable functions such that $\|f\|_{p, d\rho} = (\int_Q |f(x)|^p d\rho(x))^{1/p} < \infty$, $L^\infty(Q)$ be the space of the essentially bounded measurable functions, and $C(Q)$ consists of continuous functions on Q . The Fourier transform of f is given by

$$\widehat{f}(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, y \rangle} \, dx, \quad y \in \mathbb{R}^d.$$

A function f defined on \mathbb{R}^d is positive-definite if for each N

$$\sum_{i,j=1}^N c_i \overline{c_j} f(x_i - x_j) \geq 0, \quad \forall c_1, \dots, c_N \in \mathbb{C}, \quad \forall x_1, \dots, x_N \in \mathbb{R}^d.$$

Recall that for a continuous function f , by Bochner's theorem, f is positive definite if and only if

$$(1.2) \quad f(x) = \int_{\mathbb{R}^d} e^{i\langle x, y \rangle} d\mu(y),$$

where μ is a finite positive Borel measure (see, e.g., [15, 9.2.8]). In particular, if $f \in L^1(\mathbb{R}^d)$, then $d\mu(x) = (2\pi)^{-d/2} \widehat{f}(x) \, dx$ and $\widehat{f} \geq 0$.

In this paper we deal with continuous even functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$, which are constant sign outside of a ball B_λ^d . Denote by $\lambda(f)$ the smallest radius of a ball such that f is non-positive outside of this ball, that is,

$$\lambda(f) = \sup\{|x| : f(x) > 0\}.$$

Thus, functions with $\lambda(-f) < \infty$ are eventually nonnegative.

A function f is bandlimited if the distributional Fourier transform \widehat{f} (or the measure μ in (1.2)) has a compact support. Let

$$\tau(f) = \sup\{|x| : x \in \text{supp } \widehat{f}\}.$$

By the Paley–Wiener–Schwarz theorem, bandlimited functions f are restrictions of complex-valued entire functions of spherical exponential type $\tau(f)$ to \mathbb{R}^d (see, e.g., [34]).

As in the original Logan's problems we are interested in the smallest value of $\lambda(\pm f)$ for continuous positive definite functions f with finite type $\tau(f)$. We also assume that the following orthogonality condition holds:

$$\int_{\mathbb{R}^d} |x|^{2k} f(x) \, dx = 0, \quad k = 0, \dots, k_m, \quad f \in L^1(\mathbb{R}^d, |x|^{2k_m} \, dx),$$

for some integer k_m , cf. the condition in Problem B. This condition is equivalent to

$$\Delta^k \widehat{f}(0) = 0, \quad k = 0, \dots, k_m,$$

where Δ is the Laplace operator, $\Delta^0 = \text{Id}$.

One of the main goals of this paper is to solve the following

Problem C. For $d \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, find

$$\inf \lambda((-1)^m f) \tau(f),$$

where the infimum is taken over all nontrivial continuous positive definite bandlimited functions on \mathbb{R}^d such that additionally if $m \geq 1$, $f \in L^1(\mathbb{R}^d, |x|^{2m-2} dx)$ and $\Delta^k \hat{f}(0) = 0$, $k = 0, \dots, m-1$.

It is worth mentioning that admissible functions in problem C as well as the expression $\lambda(\pm f) \tau(f)$ are invariant with respect to the dilation $f_a(x) = f(ax)$, $a > 0$, since $\lambda(\pm f_a) = a^{-1} \lambda(\pm f)$ and $\tau(f_a) = a \tau(f)$. Note that in Problems A and B we have $\tau(f) = 1$.

Problem C has various applications, in particular, to investigate Odlyzko's question on zeros of the Dedekind zeta function (see [32] and [8, Sec. 4]). For $m = 0$ it plays an important role in several extremal problems in approximation theory (see, e.g., [7, 20]).

To formulate our main result, for $\alpha \geq -1/2$ we introduce the even entire function of exponential type 2

$$(1.3) \quad f_{\alpha,m}(t) = \frac{j_\alpha^2(t)}{(1 - t^2/q_{\alpha,1}^2) \cdots (1 - t^2/q_{\alpha,m+1}^2)}, \quad t \in \mathbb{R}_+,$$

where $j_\alpha(t) = \Gamma(\alpha+1)(2/t)^\alpha J_\alpha(t)$ is the normalized Bessel function and $q_{\alpha,1} < q_{\alpha,2} < \cdots$ are positive zeros of J_α .

Theorem 1.1. For $d \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, we have

$$\inf \lambda((-1)^m f) \tau(f) = 2q_{d/2-1,m+1},$$

where the infimum is taken over all admissible functions in Problem C. The function $f_{d/2-1,m}(|x|)$ is the unique extremizer up to a positive constant. Moreover, this function satisfies $\Delta^m \hat{f}(0) = 0$.

We note that the same statement is valid not only for positive definite functions but also for even functions with nonnegative Fourier transforms in a neighborhood of the origin. The positive definiteness of $f_{d/2-1,m}$ for $m = 0, 1$ was established by Yudin [39, 41]. In the case $m = 0, 1$ Theorem 1.1 was proved in [20]. We prove Theorem 1.1 by solving a more general problem for the Dunkl transform \mathcal{F}_k (see Section 6). In its turn, the corresponding problem for the Dunkl transform can be reduced to the one-dimensional problem for the Hankel transform \mathcal{H}_α , $\alpha \geq -1/2$, in $(\mathbb{R}_+, \lambda^{2\alpha+1} d\lambda)$.

The key step in the proof of Theorem 1.1 is to show the positive definiteness of $f_{d/2-1,m}$ for $m \geq 2$. Note that since the normalized Bessel function $j_{d/2-1}(|x|)$ is positive definite it is enough to verify that $g_{d/2-1,m}(|x|)$ is positive definite, where

$$(1.4) \quad g_{\alpha,m}(t) = \frac{j_\alpha(t)}{(1 - t^2/q_{\alpha,1}^2) \cdots (1 - t^2/q_{\alpha,m+1}^2)}.$$

This remarkable fact has been recently established by Cohn and de Courcy-Ireland [12, Proposition 3.1]. The method of the proof is based on the Mehler–Heine formula on interrelation between the Bessel functions and Gegenbauer polynomials as well as the important result from the paper [10] stating that the polynomial

$$\frac{P_n^{(\alpha,\alpha)}(z)}{(z - r_{1,n}) \cdots (z - r_{k,n})}$$

is a linear combination of $P_0^{(\alpha,\alpha)}(z), \dots, P_{n-k}^{(\alpha,\alpha)}(z)$ with nonnegative coefficients for each $k \leq n$, where $r_{1,n} > r_{2,n} > \dots > r_{n,n}$ are zeros of the Jacobi polynomial $P_n^{(\alpha,\alpha)}(z)$ (in the case $k = 1, 2$, this was proved in [19]). Cohn and de Courcy-Ireland used the function $f_{d/2-1,m}$ to obtain lower bounds for energy in the Gaussian core model (see [12, Sect. 6]).

To solve Logan's problem for the Hankel transform \mathcal{H}_α , one should show that $g_{\alpha,m}$ is positive definite with respect to \mathcal{H}_α for any $\alpha \geq -1/2$ and $m \geq 0$. For $\alpha = -1/2$, $m = 0, 1$, we arrive at the cosine Fourier transform considered by Logan. We will give two proofs of positive definiteness of the function $g_{\alpha,m}$. The first one is the direct proof using the Sturm theorem on number on zeros of linear combinations of eigenfunctions (see Section 7). In particular, following this approach, one can obtain the monotonicity of the Hankel transform of the function $g_{\alpha,m}$ on $[0, 1]$. The second proof extends the one by Cohn and de Courcy-Ireland for the case of any α (not necessarily half-integer) and is given in Section 8.

Remark 1.1. Note that the functions $g_{d/2-1+\theta,m}(|x|)$ and $f_{d/2-1+\theta,m}(|x|)$ are positive definite on \mathbb{R}^d for any $\theta \geq 0$ and $m \in \mathbb{Z}_+$. This follows from (2.16) below and the fact that for any $\alpha \geq -1/2$ and $m \in \mathbb{Z}_+$, $g_{\alpha,m}$ and $f_{\alpha,m}$ are positive definite with respect to Hankel transform. This result answers the question by M. Buhmann and is related to the theory of radial basis functions (see, e.g., [9]).

1.2. Uncertainty principle relations. Recently, Bourgain, Clozel, and Kahane [8] have studied the following uncertainty principle problem: find

$$A_d^+ = \frac{1}{2\pi} \inf \lambda(-f)\lambda(\widehat{f}),$$

where infimum is taken over all even real-valued (nontrivial) functions f such that $f, \widehat{f} \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $f(0) \leq 0, \widehat{f}(0) \leq 0$. They established

$$(1.5) \quad \frac{d}{2\pi e} < A_d^+ < \frac{d+2}{2\pi}, \quad d \in \mathbb{N}.$$

For further results, see [13, 18]. Cohn and Gonçalves [13] proved that

$$A_{12}^+ = 2.$$

Moreover, the authors considered the following problem: $A_d^- = (2\pi)^{-1} \inf \lambda(-f)\lambda(\widehat{f})$ for $f(0) \geq 0, \widehat{f}(0) \leq 0$ and found

$$A_1^- = 1, \quad A_8^- = 2, \quad A_{24}^- = 4.$$

This question is closely related to the linear programming bound for the sphere packing problem, which has been recently solved in dimensions 8 and 24 [11, 38].

In [18, Theorem 1.4], it was shown that an extremizer in the problem A_d^\pm exists and it is a radial function such that $(2\pi)^{d/2} \widehat{f}(2\pi x) = \pm f(x)$ and $f(0) = 0$. In particular, this implies that the support of \widehat{f} is not compact.

We study problems similar to that of finding A_d^\pm for bandlimited functions and obtain the following uncertainty principle.

Theorem 1.2. *Let $d \in \mathbb{N}$, $m, s \in \mathbb{Z}_+$. We have*

$$\inf \lambda((-1)^m f) \tau(f) = 2q_{d/2+s,m+1},$$

where the infimum is taken over all nontrivial even continuous bandlimited functions $f \in L^1(\mathbb{R}^d, |x|^{2m} dx)$ such that

$$\begin{cases} \Delta^k \widehat{f}(0) = 0, & k = 0, \dots, m-1, \\ \Delta^l f(0) = 0, & l = 0, \dots, s-1, \end{cases}$$

(for $m = 0$ or $s = 0$ the corresponding conditions are not assumed) and

$$\Delta^m \widehat{f}(0) \geq 0, \quad \Delta^s f(0) \leq 0.$$

Each extremizer $f(x)$ has the form $r(x)f_{d/2+s,m}(|x|)$, where

$$(1.6) \quad r(x) = \sum_{j=0}^{s+1} |x|^{2s+2-2j} h_{2j}(x) \geq 0, \quad |x| \geq q_{d/2+s,m+1},$$

and $h_{2j}(x)$ are even harmonic polynomials of order at most $2j$ such that $h_0 > 0$, $h_{2j}(0) = 0$, $j = 1, \dots, s+1$. Moreover, $\Delta^m \widehat{f}(0) = \Delta^s f(0) = 0$.

Remark 1.2. (1) We also obtain the following result (see Theorem 6.1 (iii)):

$$(1.7) \quad \inf \lambda((-1)^m f) \tau(f) = 2q_{d/2+s-1,m+1},$$

where the infimum is taken over all nontrivial even continuous bandlimited functions $f \in L^1(\mathbb{R}^d, |x|^{2m+2s} dx)$ such that

$$(1.8) \quad \Delta^k \widehat{f}(0) = 0, \quad k = s, \dots, m+s-1, \quad \Delta^{m+s} \widehat{f}(0) \geq 0.$$

The function $f_{d/2+s-1,m+1}(|x|)$ is the unique (up to a positive constant) extremizer. Moreover, this function satisfies $\Delta^{m+s} \widehat{f}(0) = 0$.

(2) For $s = 0$ all admissible functions in problem C satisfy condition (1.8). Moreover, the positive definite function $f_{d/2-1,m+1}(|x|)$ is the unique extremizer in both problems C and (1.7).

(3) If the polynomial $r(x)$ given by (1.6) is nonnegative on \mathbb{R}^d , then it is an even homogeneous polynomial of order $2s+2$.

Remark 1.3. Let us compare problems A_d^\pm and Theorem 1.2 with $m = s = 0$. From the observations above we note that $A_d^\pm = (2\pi)^{-1} \inf \lambda(f) \lambda(\pm \widehat{f})$ with $f(0) = \widehat{f}(0) = 0$. For bandlimited f , we have $\lambda(\pm \widehat{f}) \leq \tau(f)$ and therefore, $A_d^\pm \leq (2\pi)^{-1} \inf \lambda(f) \tau(f)$. In particular, we get $A_d^\pm \leq \pi^{-1} q_{d/2,1}$ for any $d \in \mathbb{N}$. If $d = 1$ we arrive at the sharp bound $A_d^\pm \leq 1$. If $d \rightarrow \infty$, we derive

$$A_d^\pm \leq \frac{d}{2\pi} (1 + o(1)).$$

The latter corresponds to (1.5) but it is less interesting since $q_{\alpha,1} = \alpha + c\alpha^{1/3} + O(\alpha^{-1/3})$, where $c = 1.855 \dots$ [6, Sec. 7.9].

Remark 1.4. It is also worth mentioning the related results in metric geometry. Let $L \subset \mathbb{R}^d$ be a lattice of rank d , $\lambda_1(L)$ be the first successive minimum of L , $\mu(L)$ be the covering radius of L , and L^* be a dual lattice. One of the important problems is to find the infimum of $\mu(L) \lambda_1(L^*)$. There exists a self-dual lattice L_d such that [4]

$$\frac{d}{2\pi e} (1 + o(1)) \leq \mu(L_d) \lambda_1(L_d^*) \quad \text{as } d \rightarrow \infty.$$

Yudin showed [40] that $\mu(L) \lambda_1(L^*) \leq (2\pi)^{-1} \lambda(f) \tau(f)$ for any admissible function in Problem C with $m = 0$. This and Theorem 1.1 imply

$$\frac{d}{2\pi e} (1 + o(1)) \leq \mu(L) \lambda_1(L^*) \leq \frac{d}{2\pi} (1 + o(1)),$$

cf. (1.5) (see also [4]).

1.3. Structure of the paper. Section 2 contains some auxiliary results on the Hankel transform \mathcal{H}_α as well as the Gauss- and Radau-type quadrature formulas with zeros of Bessel functions as nodes.

In Section 3, we give the solution of the generalized Logan problem for Hankel transform (see Theorem 3.1). Section 4 provides the uncertainty principle relations for bandlimited functions in $(\mathbb{R}_+, t^{2\alpha+1} dt)$ (see Theorem 4.1).

In Section 5, we study the problem of finding the smallest interval containing at least n zeros of functions represented by $f(\lambda) = \int_0^1 j_\alpha(\lambda t) d\sigma(t)$ with a nonnegative bounded Stieltjes measure $d\sigma$. We will see that extremizers in this problem and Problem C are closely related (Remark 5.1).

In Section 6, we solve the multidimensional Logan problem for the Dunkl transform (see Theorem 6.1) reducing this problem to the corresponding problems for the Hankel transforms (Theorems 3.1 and 4.1). Theorems 1.1 and 1.2 dealing with for the Fourier transform are partial cases of Theorem 6.1.

In Section 7, we prove that the normalized Bessel functions form the Chebyshev system. Section 8 contains the proof of positive definiteness of the function $g_{\alpha,m}$ based on the Mehler–Heine formula for Jacobi polynomials.

2. NOTATION AND AUXILIARY RESULTS

Useful facts on harmonic analysis involving Hankel transform \mathcal{H}_α in $(\mathbb{R}_+, t^{2\alpha+1} dt)$, $\alpha \geq -1/2$, can be founded in [6, 22, 30]. For the reader's convenience we recall some of them.

Let

$$(2.1) \quad B_\alpha = \frac{1}{t^{2\alpha+1}} \left(\frac{d}{dt} t^{2\alpha+1} \frac{d}{dt} \right) = \frac{d^2}{dt^2} + \frac{2\alpha+1}{t} \frac{d}{dt},$$

be the Bessel differential operator. The normalized Bessel function $j_\alpha(z)$ satisfies $B_\alpha j_\alpha(\lambda t) = -\lambda^2 j_\alpha(\lambda t)$ and is given by

$$(2.2) \quad j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1) (z/2)^{2k}}{k! \Gamma(k+\alpha+1)},$$

where $J_\alpha(z)$ is the Bessel function of order α . In particular, $j_{-1/2}(z) = \cos z$ and $j_{1/2}(z) = z^{-1} \sin z$. Moreover, the normalized Bessel function is the even entire function of exponential type 1, satisfying $j_\alpha(z) = \prod_{k=1}^{\infty} (1 - \frac{z^2}{q_{\alpha,k}^2})$, where $q_{\alpha,1} < q_{\alpha,2} < \dots$ are positive zeros of J_α .

The known formulas for Bessel functions imply

$$(2.3) \quad \frac{d}{dz} j_\alpha(z) = -\frac{z}{2(\alpha+1)} j_{\alpha+1}(z) = \frac{2\alpha}{z} (j_{\alpha-1}(z) - j_\alpha(z)),$$

$$(2.4) \quad \frac{d}{dz} (z^{2\alpha+2} j_{\alpha+1}(\lambda z)) = 2(\alpha+1) z^{2\alpha+1} j_\alpha(\lambda z),$$

and

$$(2.5) \quad \int_0^z j_\alpha(at) j_\alpha(bt) t^{2\alpha+1} dt = \frac{z^{2\alpha+2} \{a^2 j_{\alpha+1}(az) j_\alpha(bz) - b^2 j_\alpha(az) j_{\alpha+1}(bz)\}}{2(\alpha+1)(a^2 - b^2)}.$$

For $\lambda \in \mathbb{R}$, we have

$$(2.6) \quad |j_\alpha(\lambda)| \leq j_\alpha(0) = 1,$$

and for $|z| \rightarrow \infty$, $\operatorname{Re} z \geq 0$,

$$(2.7) \quad z^{\alpha+1/2} j_\alpha(z) = \frac{2^{\alpha+1/2} \Gamma(\alpha+1)}{\Gamma(1/2)} \left(\cos\left(z - \frac{\pi(2\alpha+1)}{4}\right) + O(|z|^{-1} e^{|\operatorname{Im} z|}) \right).$$

For $\alpha > -1/2$, we also have Poisson's integral representation

$$(2.8) \quad j_\alpha(\lambda) = c_\alpha \int_0^1 (1-t^2)^{\alpha-1/2} \cos(\lambda t) dt, \quad c_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+1/2)}.$$

Then using

$$(-1)^m \left(\cos \lambda - \sum_{k=0}^{m-1} \frac{(-1)^k \lambda^{2k}}{(2k)!} \right) \geq 0, \quad m \in \mathbb{N}, \quad \lambda \geq 0,$$

Poisson's representation gives

$$(2.9) \quad \psi_m(\lambda) = (-1)^m \left(j_\alpha(\lambda) - \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(\alpha+1) (\lambda/2)^{2k}}{k! \Gamma(k+\alpha+1)} \right) \geq 0.$$

Define

$$(2.10) \quad d\nu_\alpha(t) = b_\alpha t^{2\alpha+1} dt, \quad t \in \mathbb{R}_+, \quad b_\alpha^{-1} = 2^\alpha \Gamma(\alpha+1).$$

The Hankel transform is given by

$$\mathcal{H}_\alpha(f)(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) d\nu_\alpha(t), \quad \lambda \in \mathbb{R}_+.$$

It is an unitary operator in $L^2(\mathbb{R}_+, d\nu_\alpha)$ and $\mathcal{H}_\alpha^{-1} = \mathcal{H}_\alpha$.

If $f \in L^1(\mathbb{R}_+, d\nu_\alpha) \cap C(\mathbb{R}_+)$ and $\mathcal{H}_\alpha(f) \in L^1(\mathbb{R}_+, d\nu_\alpha)$, then, for any $t \in \mathbb{R}_+$, one has the inversion formula

$$(2.11) \quad f(t) = \int_0^\infty \mathcal{H}_\alpha(f)(\lambda) j_\alpha(\lambda t) d\nu_\alpha(\lambda).$$

We also recall the homogeneity property $\mathcal{H}_\alpha(f_a)(\lambda) = a^{-2\alpha-2} \mathcal{H}_\alpha(f)(\lambda/a)$, where $f_a(t) = f(at)$, $a > 0$. Note that the Hankel transform is a particular case of the one-dimensional Dunkl transform associated with the reflection group \mathbb{Z}_2 [35], see Section 6.

Let $\mathcal{B}_\alpha^\tau(\mathbb{R}_+)$ be the class of even entire functions f of exponential type at most $\tau > 0$ such that the restriction of f to \mathbb{R}_+ belongs to $L^1(\mathbb{R}_+, d\nu_\alpha)$. For such functions one has $|f(z)| \leq \|f\|_{C(\mathbb{R}_+)} e^{\tau|\operatorname{Im} z|}$, $\forall z \in \mathbb{C}$. Furthermore, the Paley-Wiener theorem states that $f \in \mathcal{B}_\alpha^\tau(\mathbb{R}_+)$ if and only if $f \in L^1(\mathbb{R}_+, d\nu_\alpha) \cap C(\mathbb{R}_+)$ and $\operatorname{supp} \mathcal{H}_\alpha(f) \subset [0, \tau]$ (see [27, Sect. 5], [2, Sect. 5], and [23]).

The following result ([25, 16], see also [22]) provides the Gauss and Radau (with multiple nodes) quadrature formulas for $\mathcal{B}_\alpha^\tau(\mathbb{R}_+)$ functions.

Lemma 2.1. *For any function $f \in \mathcal{B}_\alpha^\tau(\mathbb{R}_+)$ one has*

$$(2.12) \quad \left(\frac{\tau}{2}\right)^{2\alpha+2} \int_0^\infty f(\lambda) d\nu_\alpha(\lambda) = \sum_{k=1}^\infty \gamma_k f\left(\frac{2q_{\alpha,k}}{\tau}\right)$$

$$(2.13) \quad = \sum_{l=0}^{r-1} \alpha_{l,r} f^{(2l)}(0) + \sum_{k=1}^\infty \gamma_{k,r} f\left(\frac{2q_{\alpha+r,k}}{\tau}\right), \quad r \in \mathbb{N}.$$

The series in (2.12) and (2.13) converge absolutely and the weights γ_k , $\gamma_{k,r}$, $\alpha_{r-1,r}$ are positive.

Remark 2.1. (1) Formula (2.13) was formulated in [16] under the more restrictive condition $f(\lambda) = O(\lambda^{-\delta})$, $\lambda \rightarrow +\infty$, $\delta > 2\alpha + 2$. However, (2.12) was obtained for any $f \in L^1(\mathbb{R}_+, d\nu_\alpha)$ [25, 22]. It is easy to see that (2.13) follows from (2.12). Indeed, assuming $\tau = 2$, one applies (2.12) with $d\nu_{\alpha+r}$, $r \geq 1$ to the function

$$g(\lambda) = \lambda^{-2r} \left(f(\lambda) - j_{\alpha+r}^2(\lambda) \sum_{j=0}^{r-1} (f j_{\alpha+r}^{-2})^{(2j)}(0) \frac{\lambda^{2j}}{(2j)!} \right) \in \mathcal{B}_{\alpha+r}^2(\mathbb{R}_+).$$

Straightforward calculations give (2.13).

(2) One has $\alpha_{r-1,r} = c_{\alpha,r} \int_0^\infty j_{\alpha+r}^2(\lambda) d\nu_{\alpha+r-1}(\lambda) > 0$ with some $c_{\alpha,r} > 0$, see [16].

To construct extremizers for Problem C, we will use the generalized translation operator T_α^t given by, for $x, t \in \mathbb{R}_+$,

$$(2.14) \quad T_\alpha^t f(x) = \begin{cases} \frac{1}{2} (f(x+t) + f(|x-t|)), & \alpha = -1/2, \\ c_\alpha \int_0^\pi f(\sqrt{x^2 + t^2 - 2xt \cos \theta}) \sin^{2\alpha} \theta d\theta, & \alpha > -1/2, \end{cases}$$

where c_α is from (2.8) (see, e.g., [29, 24]). The translation operator is positive self-adjoint operator, $T_\alpha^t f(x) \in C(\mathbb{R}_+ \times \mathbb{R}_+)$ whenever $f \in C(\mathbb{R}_+)$, and T_α^t extends to the space $L^p(\mathbb{R}_+, d\nu_\alpha)$, $1 \leq p \leq \infty$. It is known that $T_\alpha^t j_\alpha(\lambda x) = j_\alpha(\lambda t) j_\alpha(\lambda x)$, which implies

$$(2.15) \quad \mathcal{H}_\alpha(T_\alpha^t f)(\lambda) = j_\alpha(t\lambda) \mathcal{H}_\alpha(f)(\lambda).$$

Moreover, $\text{supp } T_\alpha^t f(x) \subset [0, a+t]$ if $\text{supp } f \subset [0, a]$.

By means of the operator T_α^t we define the positive convolution operator

$$(f_1 *_\alpha f_2)(x) = \int_0^\infty T_\alpha^t f_1(x) f_2(t) d\nu_\alpha(t),$$

which satisfies $\mathcal{H}_\alpha(f_1 *_\alpha f_2) = \mathcal{H}_\alpha(f_1) \mathcal{H}_\alpha(f_2)$ and $\text{supp } (f_1 *_\alpha f_2) \subset [0, a_1 + a_2]$ if $\text{supp } f_i \subset [0, a_i]$.

Following Levitan [29, § 11], an even function is called positive definite with respect to the Hankel transform \mathcal{H}_α if for each N

$$\sum_{i,j=1}^N c_i \overline{c_j} T_\alpha^{x_i} f(x_j) \geq 0, \quad \forall c_1, \dots, c_N \in \mathbb{C}, \quad \forall x_1, \dots, x_N \in \mathbb{R}_+,$$

or, equivalently, the matrix $(T_\alpha^{x_i} f(x_j))_{i,j=1}^N$ is positive semidefinite. By Bochner-type theorem [29, Theorem 12.1], the condition that a continuous function f is positive definite is equivalent to the fact that f is the Hankel transform of a measure σ ,

$$f(\lambda) = \int_0^\infty j_\alpha(\lambda t) d\sigma(t),$$

where σ is a non-decreasing function of bounded variation. In particular, if $f \in L^1(\mathbb{R}_+, d\nu_\alpha)$, then $d\sigma = \mathcal{H}_\alpha(f) d\nu_\alpha$ and $\mathcal{H}_\alpha(f) \geq 0$.

Moreover, it is easy to see that if f is positive definite with respect to \mathcal{H}_β , then it is the same with respect to \mathcal{H}_α for $\alpha < \beta$, since

$$(2.16) \quad \mathcal{H}_\alpha(f)(t) = \frac{1}{2^{\beta-\alpha-1} \Gamma(\beta-\alpha)} \int_t^\infty s(s^2 - t^2)^{\beta-\alpha-1} \mathcal{H}_\beta(f)(s) ds, \quad t \in \mathbb{R}_+.$$

The latter follows from Sonine's first integral for the Bessel functions:

$$(2.17) \quad j_\beta(\lambda) = \frac{b_\beta^{-1}}{2^{\beta-\alpha-1} \Gamma(\beta-\alpha)} \int_0^1 (1-t^2)^{\beta-\alpha-1} j_\alpha(\lambda t) d\nu_\alpha(t),$$

where b_β is defined in (2.10).

Special attention will be paid below to the positive definite functions $j_{\alpha+1}(\lambda)$ and $j_{\alpha+1}^2(\lambda)$. By (2.17), we have

$$j_{\alpha+1}(\lambda) = b_{\alpha+1}^{-1} \int_0^1 j_{\alpha}(\lambda t) d\nu_{\alpha}(t) = b_{\alpha+1}^{-1} \mathcal{H}_{\alpha}(\chi_{[0,1]})(\lambda),$$

where $\chi_I(t)$ is the characteristic function of an interval I . Thus,

$$(2.18) \quad j_{\alpha+1}^2(\lambda) = b_{\alpha+1}^{-2} \mathcal{H}_{\alpha}(\chi_{[0,1]} *_{\alpha} \chi_{[0,1]})(\lambda)$$

and $\text{supp } \mathcal{H}_{\alpha}(j_{\alpha+1}^2) \subset [0, 2]$.

We will also use the following two lemmas.

Lemma 2.2 ([21]). *Let $\alpha \geq -1/2$. There exists an even entire function $\omega_{\alpha}(z)$, $z = x + iy$, of exponential type 2, positive for $x > 0$, and such that*

$$\omega_{\alpha}(x) \asymp x^{2\alpha+1}, \quad x \rightarrow +\infty, \quad |\omega_{\alpha}(iy)| \asymp y^{2\alpha+1} e^{2y}, \quad y \rightarrow +\infty,$$

where $F_1 \asymp F_2$ means that $C^{-1}F_1 \leq F_2 \leq CF_1$, $C > 0$. One can take $\omega_{\alpha}(z) = z^{2m+2} j_{\nu}(z+i) j_{\nu}(z-i)$, where $\alpha = m - \nu$, $m \in \mathbb{Z}_+$, and $\nu \in [-1/2, 1/2]$.

Lemma 2.3. *Let F be an even entire function of exponential type $\tau > 0$ bounded on \mathbb{R} . Let Ω be an even entire function of finite exponential type, all the zeroes of Ω be zeros of F , and, for some $m \in \mathbb{Z}_+$,*

$$\liminf_{y \rightarrow +\infty} e^{-\tau y} y^{2m} |\Omega(iy)| > 0.$$

Then the function $F(z)/\Omega(z)$ is an even polynomial of degree at most $2m$.

Lemma 2.3 is an easy consequence of Akhiezer's result [28, Appendix VII.10].

3. LOGAN PROBLEM FOR THE HANKEL TRANSFORM

Let $\alpha \geq -1/2$ and $m \in \mathbb{Z}_+$. In this section we solve the generalized Logan problem (with parameter m) for the Hankel transform \mathcal{H}_{α} in $(\mathbb{R}_+, d\nu_{\alpha}(\lambda))$. This is the crucial step to prove Theorems 1.1 and 1.2.

Consider the class $\mathcal{E}_{\alpha}(\mathbb{R}_+)$ of real-valued even entire functions f of finite exponential type such that

$$(3.1) \quad f(\lambda) = \int_0^{\tau(f)} j_{\alpha}(\lambda t) d\sigma(t),$$

where σ is a function of bounded variation.

Let $\lambda(f) = \sup\{\lambda > 0: f(\lambda) > 0\}$. For $m \in \mathbb{Z}_+$, denote by $\mathcal{E}_{\alpha,m}(\mathbb{R}_+)$ the subclass of functions $f \in \mathcal{E}_{\alpha}(\mathbb{R}_+)$ such that $\lambda((-1)^m f) < \infty$ and, if $m \geq 1$, $f \in L^1(\mathbb{R}_+, \lambda^{2m-2} d\nu_{\alpha})$ and for $k = 0, \dots, m-1$

$$(3.2) \quad B_{\alpha}^k \mathcal{H}_{\alpha}(f)(0) = (-1)^k \int_0^{\infty} \lambda^{2k} f(\lambda) d\nu_{\alpha}(\lambda) = 0.$$

We will see that this class is not empty, in particular, $f_{\alpha,m}(\lambda) = j_{\alpha}(\lambda) g_{\alpha,m}(\lambda) \in \mathcal{E}_{\alpha,m}(\mathbb{R}_+)$, see (1.3) and (1.4). Due to (2.15), for the Hankel transforms of functions $f_{\alpha,m}$ and $g_{\alpha,m}$ one has

$$\mathcal{H}_{\alpha}(f_{\alpha,m}) = T_{\alpha}^1 \mathcal{H}_{\alpha}(g_{\alpha,m}).$$

Theorem 3.1. (i) *Let $f \in \mathcal{E}_{\alpha,m}(\mathbb{R}_+) \setminus \{0\}$ be given by (3.1) such that σ is non-decreasing in some neighborhood of the origin. Then*

$$(3.3) \quad f \in L^1(\mathbb{R}_+, \lambda^{2m} d\nu_{\alpha}), \quad (-1)^m \int_0^{\infty} \lambda^{2m} f(\lambda) d\nu_{\alpha}(\lambda) \geq 0,$$

and

$$(3.4) \quad 2q_{\alpha,m+1} \leq \lambda((-1)^m f)\tau(f).$$

Moreover, inequality (3.4) is sharp and the function $f_{\alpha,m}$ is the unique extremizer up to a positive constant.

(ii) The functions $g_{\alpha,m}$ and $f_{\alpha,m}$ are positive definite and

$$(3.5) \quad \int_0^\infty \lambda^{2m} f_{\alpha,m}(\lambda) d\nu_\alpha(\lambda) = 0.$$

(iii) There holds $g_{\alpha,m} = \mathcal{H}_\alpha(p_{\alpha,m}\chi_{[0,1]})$, where $p_{\alpha,m}(t)$ is decreasing on $[0,1]$ and has a zero of multiplicity $2m+1$ at $t=1$.

Proof. The proof is divided into several steps. Since the class $\mathcal{E}_{\alpha,m}(\mathbb{R}_+)$ and the quantity $\lambda((-1)^m f)\tau(f)$ are invariant under dilations, we let for convenience $\tau(f) = 2$. We also denote $q_k = q_{\alpha,k}$ for $k \geq 1$.

Proof of (3.3). Let $m = 0$. The embedding $\mathcal{E}_{\alpha,0}(\mathbb{R}_+) \subset L^1(\mathbb{R}_+, d\nu_\alpha)$ can be shown using the method of Logan, see [32, Lemma].

We consider the positive definite kernel $\varphi_\varepsilon(x) = j_{\alpha+1}^2(\varepsilon|x|/2)$, $\varepsilon > 0$. By (2.7), (2.6), and (2.18), $\varphi_\varepsilon \in C(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, d\nu_\alpha)$, $\|\varphi_\varepsilon\|_{C(\mathbb{R}_+)} = \varphi_\varepsilon(0) = 1$, and $\text{supp } \mathcal{H}_\alpha(\varphi_\varepsilon) \subset [0, \varepsilon]$. Since $d\sigma \geq 0$ in some neighborhood of the origin, then for sufficiently small ε we have

$$\begin{aligned} 0 &\leq \int_0^\varepsilon \mathcal{H}_\alpha(\varphi_\varepsilon)(t) d\sigma(t) = \int_0^\infty \mathcal{H}_\alpha(\varphi_\varepsilon)(t) d\sigma(t) = \int_0^\infty f(\lambda)\varphi_\varepsilon(\lambda) d\nu_\alpha(\lambda) \\ &= \int_0^{\lambda(f)} f(\lambda)\varphi_\varepsilon(\lambda) d\nu_\alpha(\lambda) - \int_{\lambda(f)}^\infty |f(\lambda)|\varphi_\varepsilon(\lambda) d\nu_\alpha(\lambda), \end{aligned}$$

which implies

$$\int_{\lambda(f)}^\infty |f(\lambda)|\varphi_\varepsilon(\lambda) d\nu_\alpha(\lambda) \leq \int_0^{\lambda(f)} f(\lambda)\varphi_\varepsilon(\lambda) d\nu_\alpha(\lambda) \leq \int_0^{\lambda(f)} |f(\lambda)| d\nu_\alpha.$$

Letting $\varepsilon \rightarrow 0$, Fatou's lemma yields

$$\int_{\lambda(f)}^\infty |f(\lambda)| d\nu_\alpha(\lambda) \leq \int_0^{\lambda(f)} |f(\lambda)| d\nu_\alpha(\lambda) < \infty,$$

which implies $f \in L^1(\mathbb{R}_+, d\nu_\alpha)$.

Let $m \geq 1$. We have $f \in L^1(\mathbb{R}_+, d\nu_\alpha)$ and $d\sigma(t) = \mathcal{H}_\alpha(f)(t) dt$, where $\mathcal{H}_\alpha(f)(t)$ is continuous and nonnegative in some neighborhood of the origin. Moreover, $(-1)^{m+1}f(\lambda) = |f(\lambda)|$ for $\lambda \geq \lambda((-1)^m f)$.

Consider

$$\rho_\varepsilon(\lambda) = \frac{(2m)! \psi_m(\varepsilon\lambda)}{\varepsilon^{2m} \psi_m^{(2m)}(0)},$$

where $\psi_m(\lambda)$ is given in (2.9). We have

$$(3.6) \quad \psi_m^{(2m)}(0) > 0, \quad \rho_\varepsilon(\lambda) \geq 0, \quad \lim_{\varepsilon \rightarrow 0} \rho_\varepsilon(\lambda) = \lambda^{2m}, \quad \lambda \in \mathbb{R}_+.$$

In light of

$$|\rho_\varepsilon(\lambda) - \lambda^{2m}| \leq \frac{(2m)!}{\psi_m^{(2m)}(0)} \varepsilon^2 e^{\lambda^2/4}$$

we derive that $\rho_\varepsilon(\lambda)$ converges uniformly to λ^{2m} on any finite interval $[0, b]$ as $\varepsilon \rightarrow 0$.

Taking into account (2.9), (3.6), the orthogonality condition (3.2), and nonnegativity of $\mathcal{H}_\alpha(f)$ near the origin, we obtain

$$(3.7) \quad \begin{aligned} (-1)^m \int_0^\infty \rho_\varepsilon(\lambda) f(\lambda) d\nu_\alpha(\lambda) &= \frac{(2m)!}{\varepsilon^{2m} \psi_m^{(2m)}(0)} \int_0^\infty f(\lambda) j_\alpha(\varepsilon\lambda) d\nu_\alpha(\lambda) \\ &= \frac{(2m)!}{\varepsilon^{2m} \psi_m^{(2m)}(0)} \mathcal{H}_\alpha(f)(\varepsilon) \geq 0. \end{aligned}$$

Thus,

$$(3.8) \quad (-1)^{m+1} \int_{\lambda((-1)^m f)}^\infty \rho_\varepsilon(\lambda) f(\lambda) d\nu_\alpha(\lambda) \leq (-1)^m \int_0^{\lambda((-1)^m f)} \rho_\varepsilon(\lambda) f(\lambda) d\nu_\alpha(\lambda).$$

Using (3.7), (3.6), and Fatou's lemma we arrive at

$$\begin{aligned} (-1)^{m+1} \int_{\lambda((-1)^m f)}^\infty \lambda^{2m} f(\lambda) d\nu_\alpha(\lambda) &= (-1)^{m+1} \int_{\lambda((-1)^m f)}^\infty \lim_{\varepsilon \rightarrow 0} \rho_\varepsilon(\lambda) f(\lambda) d\nu_\alpha(\lambda) \\ &\leq \liminf_{\varepsilon \rightarrow 0} (-1)^{m+1} \int_{\lambda((-1)^m f)}^\infty \rho_\varepsilon(\lambda) f(\lambda) d\nu_\alpha(\lambda). \end{aligned}$$

In light of (3.8), we continue as follows

$$\begin{aligned} &\leq \liminf_{\varepsilon \rightarrow 0} (-1)^m \int_0^{\lambda((-1)^m f)} \rho_\varepsilon(\lambda) f(\lambda) d\nu_\alpha(\lambda) \\ &= (-1)^m \int_0^{\lambda((-1)^m f)} \lim_{\varepsilon \rightarrow 0} \rho_\varepsilon(\lambda) f(\lambda) d\nu_\alpha(\lambda) = (-1)^m \int_0^{\lambda((-1)^m f)} \lambda^{2m} f(\lambda) d\nu_\alpha(\lambda) < \infty, \end{aligned}$$

which gives (3.3).

Proof of (3.4). Let $f \in \mathcal{E}_{\alpha,m}(\mathbb{R}_+)$. We will prove that $q_{m+1} \leq \lambda((-1)^m f)$. Assume the converse, i.e., $\lambda((-1)^m f) < q_{m+1}$. We have $(-1)^m f(\lambda) \leq 0$ for $\lambda \geq q_{m+1}$. By (3.3) we have $\lambda^{2m} f \in \mathcal{B}_\alpha^2(\mathbb{R}_+)$. Then using Gauss' quadrature formula (2.12) and (3.2), we get

$$(3.9) \quad \begin{aligned} 0 &\leq (-1)^m \int_0^\infty \lambda^{2m} f(\lambda) d\nu_\alpha(\lambda) = (-1)^m \int_0^\infty \prod_{k=1}^m (\lambda^2 - q_k^2) f(\lambda) d\nu_\alpha(\lambda) \\ &= (-1)^m \sum_{s=m+1}^\infty \gamma_s f(q_s) \prod_{k=1}^m (q_s^2 - q_k^2) \leq 0. \end{aligned}$$

Therefore, q_s , $s \geq m+1$, are zeros of multiplicity 2 for f . Similarly, applying Gauss' quadrature formula for f , we obtain

$$(3.10) \quad 0 = \int_0^\infty \prod_{\substack{k=1 \\ k \neq s}}^m (\lambda^2 - q_k^2) f(\lambda) d\nu_\alpha(\lambda) = \gamma_s \prod_{\substack{k=1 \\ k \neq s}}^m (q_s^2 - q_k^2) f(q_s), \quad s = 1, \dots, m.$$

Therefore, q_s , $s = 1, \dots, m$, are zeros of f .

Take the function $\omega_\alpha(\lambda)$ from Lemma 2.2 and consider the following even functions of exponential type 4:

$$F(\lambda) = \omega_\alpha(\lambda) f(\lambda), \quad \Omega(\lambda) = \frac{\omega_\alpha(\lambda) j_\alpha^2(\lambda)}{\prod_{k=1}^m (1 - \lambda^2/q_k^2)}.$$

Note that $F \in L^1(\mathbb{R})$ since $f \in L^1(\mathbb{R}_+, \lambda^{2\alpha+1} d\lambda)$ and $\omega_\alpha(\lambda) \asymp \lambda^{2\alpha+1}$, $\lambda \rightarrow +\infty$. Then F is bounded on \mathbb{R} .

From (2.7) and Lemma 2.2 we have $|\Omega(iy)| \asymp y^{-2m} e^{4y}$ as $y \rightarrow +\infty$. Since all zeros of $\Omega(\lambda)$ are also zeros of $F(\lambda)$, taking into account Lemma 2.3, we obtain

$$f(\lambda) = \frac{j_\alpha^2(\lambda) \sum_{k=0}^m c_k \lambda^{2k}}{\prod_{k=1}^m (1 - \lambda^2/q_k^2)},$$

where $c_k \neq 0$ for some k . Note that $j_\alpha(\lambda) \notin L^2(\mathbb{R}_+, d\nu_\alpha)$, see (2.7). This contradicts $f \in L^1(\mathbb{R}_+, \lambda^{2m} d\nu_\alpha)$. Hence, $\lambda((-1)^m f) \geq q_{m+1}$ and $\lambda((-1)^m f)\tau(f) \geq 2q_{m+1}$.

Now we consider the function $f_{\alpha,m}$ given by (1.3). Note that in virtue of the estimate $f_{\alpha,m}(\lambda) = O(\lambda^{-2\alpha-2m-3})$ as $\lambda \rightarrow \infty$ we have $f_{\alpha,m} \in L^1(\mathbb{R}_+, \lambda^{2m} d\nu_\alpha)$. Moreover, $\tau(f_{\alpha,m}) = 2$ and $\lambda((-1)^m f_{\alpha,m}) = q_{\alpha,m+1}$. Part (i) is proved.

To verify part (ii), we first note that Gauss' quadrature formula implies (3.5). To show the positive definiteness of $f_{\alpha,m}$, it is enough to prove that $g_{\alpha,m}$ is positive definite.

Positive definiteness of the function $g_{\alpha,m}$. This result has been recently obtained by Cohn and de Courcy-Ireland [12] for $\alpha = d/2 - 1$, $d \in \mathbb{N}$. We prove the same statement for any α . For this, we calculate the Hankel transform of $g_{\alpha,m}$ and show that it is nonnegative.

For fixed $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, consider the polynomial

$$\omega_k(\lambda) = \omega(\lambda, \lambda_1, \dots, \lambda_k) = \prod_{i=1}^k (\lambda_i - \lambda), \quad \lambda \in \mathbb{R}.$$

Then

$$\frac{1}{\omega_k(\lambda)} = \sum_{i=1}^k \frac{1}{\omega'_k(\lambda_i)(\lambda_i - \lambda)}.$$

Setting $\lambda_i = q_i^2$, we have

$$(3.11) \quad \frac{1}{\prod_{i=1}^k (1 - \lambda^2/q_i^2)} = \prod_{i=1}^k q_i^2 \frac{1}{\omega_k(\lambda^2)} = \prod_{i=1}^k q_i^2 \sum_{i=1}^k \frac{1}{\omega'_k(q_i^2)(q_i^2 - \lambda^2)} = \sum_{i=1}^k \frac{A_i}{q_i^2 - \lambda^2},$$

where

$$(3.12) \quad \omega'_k(q_i^2) = \prod_{\substack{j=1 \\ j \neq i}}^k (q_j^2 - q_i^2), \quad A_i = \frac{\prod_{j=1}^k q_j^2}{\omega'_k(q_i^2)}.$$

Note that

$$(3.13) \quad \text{sign } A_i = (-1)^{i-1}.$$

Setting

$$\varphi_i(t) = j_\alpha(q_i t), \quad i = 1, \dots, m+1,$$

we remark that $\varphi_i(t)$ are eigenfunctions and q_i^2 are eigenvalues of the following Sturm–Liouville problem on $[0, 1]$:

$$(3.14) \quad (t^{2\alpha+1} u')' + \lambda^2 t^{2\alpha+1} u = 0, \quad u'(0) = 0, \quad u(1) = 0.$$

It follows from (2.5), (2.3), and $j_\alpha(q_i) = 0$ that

$$\int_0^\infty \varphi_i(t) \chi_{[0,1]}(t) j_\alpha(\lambda t) t^{2\alpha+1} dt = \int_0^1 j_\alpha(q_i t) j_\alpha(\lambda t) t^{2\alpha+1} dt = -\frac{\varphi'_i(1) j_\alpha(\lambda)}{q_i^2 - \lambda^2},$$

or, equivalently,

$$(3.15) \quad \mathcal{H}_\alpha \left(-b_\alpha^{-1} \frac{\varphi_i \chi_{[0,1]}}{\varphi'_i(1)} \right) (\lambda) = \frac{j_\alpha(\lambda)}{q_i^2 - \lambda^2}.$$

Note that

$$(3.16) \quad \text{sign } \varphi'_i(1) = (-1)^i.$$

Consider the following polynomial in eigenfunctions $\varphi_i(t)$:

$$(3.17) \quad p_{\alpha,m}(t) = -b_\alpha^{-1} \sum_{i=1}^{m+1} \frac{A_i}{\varphi'_i(1)} \varphi_i(t) = \sum_{i=1}^{m+1} B_i \varphi_i(t).$$

Due to (3.13), (3.14), and (3.16), we have that $B_i > 0$, $p_{\alpha,m}(0) > 0$, and $p_{\alpha,m}(1) = 0$. Moreover, in virtue of (3.11) and (3.15),

$$(3.18) \quad g_{\alpha,m}(\lambda) = \frac{j_\alpha(\lambda)}{\prod_{i=1}^{m+1} (1 - \lambda^2/q_i^2)} = \mathcal{H}_\alpha(p_{\alpha,m} \chi_{[0,1]})(\lambda).$$

From this, it is enough to show that $p_{\alpha,m}(t) \geq 0$ on $[0, 1]$.

Define the Vandermonde determinant $\Delta(\lambda_1, \dots, \lambda_k) = \prod_{1 \leq j < i \leq k} (\lambda_i - \lambda_j)$, then

$$\frac{\Delta(\lambda_1, \dots, \lambda_k)}{\omega'_k(\lambda_i)} = (-1)^{i-1} \Delta(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_k).$$

In virtue of (3.11) and (3.12), we have

$$(3.19) \quad p_{\alpha,m}(t) = -c \sum_{i=1}^{m+1} (-1)^{i-1} \Delta(q_1^2, \dots, q_{i-1}^2, q_{i+1}^2, \dots, q_{m+1}^2) \frac{\varphi_i(t)}{\varphi'_i(1)} \\ = -c \begin{vmatrix} \frac{\varphi_1(t)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}(t)}{\varphi'_{m+1}(1)} \\ 1 & \cdots & 1 \\ q_1^2 & \cdots & q_{m+1}^2 \\ \vdots & \cdots & \vdots \\ q_1^{2m-2} & \cdots & q_{m+1}^{2m-2} \end{vmatrix}, \quad c = \frac{b_\alpha^{-1} \prod_{j=1}^{m+1} q_j^2}{\Delta(q_1^2, \dots, q_{m+1}^2)} > 0.$$

Here and in what follows if $m = 0$ we deal with only the $(1, 1)$ entries of the matrices.

Let us show that

$$(3.20) \quad \begin{vmatrix} \frac{\varphi_1(t)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}(t)}{\varphi'_{m+1}(1)} \\ 1 & \cdots & 1 \\ q_1^2 & \cdots & q_{m+1}^2 \\ \vdots & \cdots & \vdots \\ q_1^{2m-2} & \cdots & q_{m+1}^{2m-2} \end{vmatrix} = (-1)^{\frac{m(m-1)}{2}} \begin{vmatrix} \frac{\varphi_1(t)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}(t)}{\varphi'_{m+1}(1)} \\ 1 & \cdots & 1 \\ \frac{\varphi_1^{(3)}(1)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}^{(3)}(1)}{\varphi'_{m+1}(1)} \\ \vdots & \cdots & \vdots \\ \frac{\varphi_1^{(2m-1)}(1)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}^{(2m-1)}(1)}{\varphi'_{m+1}(1)} \end{vmatrix}.$$

By (3.14), we derive

$$t\varphi_i''(t) + (2\alpha + 1)\varphi_i'(t) + q_i^2 t\varphi_i(t) = 0.$$

Therefore,

$$t\varphi_i^{(s+2)}(t) + s\varphi_i^{(s+1)}(t) + (2\alpha + 1)\varphi_i^{(s+1)}(t) + q_i^2 t\varphi_i^{(s)}(t) + s q_i^2 \varphi_i^{(s-1)}(t) = 0,$$

which implies for $t = 1$

$$\varphi_i^{(s+2)}(1) = -(s + 2\alpha + 1)\varphi_i^{(s+1)}(1) - q_i^2 \varphi_i^{(s)}(1) - s q_i^2 \varphi_i^{(s-1)}(1), \quad \varphi_i^{(0)}(1) = 0.$$

By induction we then obtain for $k = 0, 1, \dots$

$$\varphi_i^{(2k+1)}(1) = \varphi_i'(1) \sum_{j=0}^k a_{kj}(\alpha) q_i^{2j}, \quad \varphi_i^{(2k+2)}(1) = \varphi_i'(1) \sum_{j=0}^k b_{kj}(\alpha) q_i^{2j},$$

where $a_{kj}(\alpha)$, $b_{kj}(\alpha)$ are polynomials in α with coefficients not depending on q_i and moreover $a_{kk}(\alpha) = (-1)^k$. This implies for $k = 1, 2, \dots$

$$(3.21) \quad \frac{\varphi_i^{(2k)}(1)}{\varphi_i'(1)} = \sum_{s=1}^k c_{0s}(\alpha) \frac{\varphi_i^{(2s-1)}(1)}{\varphi_i'(1)},$$

$$(3.22) \quad \frac{\varphi_i^{(2k+1)}(1)}{\varphi_i'(1)} = \sum_{s=1}^k c_{1s}(\alpha) \frac{\varphi_i^{(2s-1)}(1)}{\varphi_i'(1)} + (-1)^k q_i^{2k},$$

where $c_{0s}(\alpha)$ and $c_{1s}(\alpha)$ do not depend on q_i . Then (3.22) implies (3.20) since

$$\begin{vmatrix} \frac{\varphi_1(t)}{\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}(t)}{\varphi_{m+1}'(1)} \\ 1 & \cdots & 1 \\ \frac{\varphi_1^{(3)}(1)}{\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}^{(3)}(1)}{\varphi_{m+1}'(1)} \\ \vdots & \ddots & \vdots \\ \frac{\varphi_1^{(2m-1)}(1)}{\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}^{(2m-1)}(1)}{\varphi_{m+1}'(1)} \end{vmatrix} = \begin{vmatrix} \frac{\varphi_1(t)}{\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}(t)}{\varphi_{m+1}'(1)} \\ 1 & \cdots & 1 \\ -q_1^2 & \cdots & -q_{m+1}^2 \\ \vdots & \ddots & \vdots \\ (-1)^{m-1} q_1^{2m-2} & \cdots & (-1)^{m-1} q_{m+1}^{2m-2} \end{vmatrix}.$$

Further, taking into account (3.19) and (3.20), we derive

$$(3.23) \quad p_{\alpha,m}^{(1)}(1) = p_{\alpha,m}^{(3)}(1) = \dots = p_{\alpha,m}^{(2m-1)}(1) = 0.$$

Therefore, by (3.17) and (3.21), we obtain for $k = 0, \dots, m$ that

$$\begin{aligned} p_{\alpha,m}^{(2k)}(1) &= -b_{\alpha}^{-1} \sum_{i=1}^{m+1} A_i \frac{\varphi_i^{(2k)}(1)}{\varphi_i'(1)} = -b_{\alpha}^{-1} \sum_{i=1}^{m+1} A_i \sum_{s=1}^k c_{0s}(\alpha) \frac{\varphi_i^{(2s-1)}(1)}{\varphi_i'(1)} \\ &= -b_{\alpha}^{-1} \sum_{s=1}^k c_{0s}(\alpha) \sum_{i=1}^{m+1} A_i \frac{\varphi_i^{(2s-1)}(1)}{\varphi_i'(1)} = \sum_{s=1}^k c_{0s}(\alpha) p_{\alpha,m}^{(2s-1)}(1) = 0. \end{aligned}$$

Together with (3.23), this implies that the zero $t = 1$ of the polynomial $p_{\alpha,m}(t)$ has multiplicity $2m+1$. Then taking into account (3.18), the same also holds for $\mathcal{H}_{\alpha}(g_{\alpha,m})(t)$.

Let us show that $p_{\alpha,m}(t)$ does not have zeros on $[0, 1)$ and hence $p_{\alpha,m}(t) > 0$ on $[0, 1)$. This yields that $g_{\alpha,m}$ is the positive definite function.

We use the facts that $\{\varphi_i(t)\}_{i=1}^{m+1}$ for any $m \in \mathbb{Z}_+$ is the Chebyshev system on the interval $(0, 1)$ (see Theorem 7.1 below) and any polynomial $\sum_{i=1}^{m+1} c_i \varphi_i(t)$ on $(0, 1)$ has at most m zeros, counting multiplicity.

We now consider the polynomial

$$(3.24) \quad p(t, \varepsilon) = \begin{vmatrix} \frac{\varphi_1(t)}{\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}(t)}{\varphi_{m+1}'(1)} \\ \frac{\varphi_1(1-\varepsilon)}{(-\varepsilon)\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}(1-\varepsilon)}{(-\varepsilon)\varphi_{m+1}'(1)} \\ \frac{\varphi_1(1-2\varepsilon)}{(-2\varepsilon)^3\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}(1-2\varepsilon)}{(-2\varepsilon)^3\varphi_{m+1}'(1)} \\ \vdots & \ddots & \vdots \\ \frac{\varphi_1(1-m\varepsilon)}{(-m\varepsilon)^{2m-1}\varphi_1'(1)} & \cdots & \frac{\varphi_{m+1}(1-m\varepsilon)}{(-m\varepsilon)^{2m-1}\varphi_{m+1}'(1)} \end{vmatrix}.$$

If $m = 0$, it is positive on $(0, 1)$ and, if $m \geq 1$, for any $0 < \varepsilon < 1/m$, it has m zeros at the points $t_j = 1 - j\varepsilon$, $j = 1, \dots, m$. Letting $\varepsilon \rightarrow 0+$, we note that the polynomial $\lim_{\varepsilon \rightarrow 0+} p(t, \varepsilon)$ does not have zeros on $(0, 1)$. Let us show that

$$(3.25) \quad \lim_{\varepsilon \rightarrow 0+} p(t, \varepsilon) = cp_{\alpha,m}(t)$$

with some $c \neq 0$. This implies that the polynomial $p_{\alpha,m}(t)$ is positive on $[0, 1)$.

To show (3.25), by Taylor's theorem, we have

$$\frac{\varphi_i(1-j\varepsilon)}{(-j\varepsilon)^{2j-1}\varphi'_i(1)} = \sum_{s=1}^{2j-2} \frac{\varphi_i^{(s)}(1)}{s!(-j\varepsilon)^{2j-1-s}\varphi'_i(1)} + \frac{\varphi_i^{(2j-1)}(1) + o(1)}{(2j-1)!\varphi'_i(1)}, \quad \varepsilon \rightarrow 0,$$

for $j = 1, \dots, m-1$. Using formulas (3.21) and (3.22) and progressively subtracting the row j from the row $j-1$ in the determinant (3.24), we arrive at

$$p(t, \varepsilon) = \frac{1}{\prod_{j=1}^{m-1} (2j-1)!} \begin{vmatrix} \frac{\varphi_1(t)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}(t)}{\varphi'_{m+1}(1)} \\ 1 + o(1) & \cdots & 1 + o(1) \\ \frac{\varphi_1^{(3)}(1) + o(1)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}^{(3)}(1) + o(1)}{\varphi'_{m+1}(1)} \\ \vdots & \ddots & \vdots \\ \frac{\varphi_1^{(2m-1)}(1) + o(1)}{\varphi'_1(1)} & \cdots & \frac{\varphi_{m+1}^{(2m-1)}(1) + o(1)}{\varphi'_{m+1}(1)} \end{vmatrix}.$$

Then, taking into account (3.19) and (3.20), we have (3.25).

Monotonicity of $p_{\alpha, m}$. The polynomial $p(t, \varepsilon)$ vanishes at $m+1$ points: $t_j = 1 - j\varepsilon$, $j = 1, \dots, m$, and $t_m = 1$, thus its derivative $p'(t, \varepsilon)$ has m zeros on the interval $(1-\varepsilon, 1)$.

In virtue of (2.3),

$$\varphi'_i(t) = -\frac{q_i^2 t}{2(\alpha+1)} j_{\alpha+1}(q_i t), \quad t \in [0, 1].$$

This and Theorem 7.1 imply that $\{\varphi'_i(t)\}_{i=1}^{m+1}$ is the Chebyshev system on $(0, 1)$. Therefore, $p'(t, \varepsilon)$ does not have zeros on $(0, 1-\varepsilon]$. Then for $\varepsilon \rightarrow 0+$ we derive that $p'_{\alpha, m}(t)$ does not have zeros on $(0, 1)$. Since $p_{\alpha, m}(0) > 0$ and $p_{\alpha, m}(1) = 0$, then $p'_{\alpha, m}(t) < 0$ on $(0, 1)$. Thus, $p_{\alpha, m}(t)$ is decreasing on the interval $[0, 1]$. This completes the proof of part (iii).

Uniqueness of the extremizer $f_{\alpha, m}$. As above, we will use Lemmas 2.2 and 2.3. Let $f(\lambda)$ be an extremizer and $\lambda((-1)^m f) = q_{m+1}$. Consider the functions

$$F(\lambda) = \omega_\alpha(\lambda)f(\lambda), \quad \Omega(\lambda) = \omega_\alpha(\lambda)f_{\alpha, m}(\lambda),$$

where $f_{\alpha, m}$ is defined in (1.3) and ω_α is from Lemma 2.2.

Note that all zeros of $\Omega(\lambda)$ are also zeros of $F(\lambda)$. Indeed, we have $(-1)^m f(\lambda) \leq 0$ for $\lambda \geq q_{m+1}$ and $f(q_{m+1}) = 0$ (otherwise $\lambda((-1)^m f) < q_{m+1}$, which is a contradiction). This and (3.9) imply that the points q_s , $s \geq m+2$, are double zeros of f . By (3.10), we also have that $f(q_s) = 0$ for $s = 1, \dots, m$ and therefore the function f has zeros (at least, of order one) at the points q_s , $s = 1, \dots, m+1$.

Using asymptotic relations given in Lemma 2.2, we derive that $F(\lambda)$ is the entire function of exponential type, integrable on real line and therefore bounded. Taking into account (2.7) and Lemma 2.2, we get

$$|\Omega(iy)| \asymp y^{-2m-2} e^{4y}, \quad y \rightarrow +\infty.$$

Now using Lemma 2.3, we arrive at $f(\lambda) = \psi(\lambda)f_{\alpha, m}(\lambda)$, where $\psi(\lambda)$ is an even polynomial of degree at most $2m+2$. Note that the degree cannot be $2s$, $s = 1, \dots, m+1$, since in this case (2.7) implies that $f \notin L^1(\mathbb{R}_+, \lambda^{2m} d\nu_\alpha)$. Thus, $f(\lambda) = cf_{\alpha, m}(\lambda)$, $c > 0$. \square

4. UNCERTAINTY PRINCIPLE FOR BANDLIMITED FUNCTIONS ON \mathbb{R}_+

Let as above $\lambda(f) = \sup\{\lambda > 0: f(\lambda) > 0\}$, $\mathcal{E}_\alpha(\mathbb{R}_+)$ be the class of real-valued even bandlimited functions $f \in C(\mathbb{R}_+)$, $\tau(f)$ be the type of a bandlimited function f , and B_α denote the operator (2.1).

Following the proof of Theorem 3.1, we obtain the following uncertainty principle for bandlimited functions on \mathbb{R}_+ .

Theorem 4.1. *Let $\alpha \geq -1/2$ and $m, s \in \mathbb{Z}_+$.*

(i) *One has*

$$\inf \lambda((-1)^m f) \tau(f) = 2q_{\alpha+s+1, m+1},$$

where the infimum is taken over all nontrivial functions $f \in \mathcal{E}_\alpha(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, \lambda^{2m} d\nu_\alpha)$ such that

$$(4.1) \quad \begin{cases} B_\alpha^k \mathcal{H}_\alpha(f)(0) = 0, & k = 0, \dots, m-1, \\ B_\alpha^l f(0) = 0, & l = 0, \dots, s-1, \end{cases}$$

and

$$(4.2) \quad B_\alpha^m \mathcal{H}_\alpha(f)(0) \geq 0, \quad B_\alpha^s f(0) \leq 0.$$

Moreover, the function $\lambda^{2s+2} f_{\alpha+s+1, m}(\lambda)$ is the unique extremizer up to a positive constant, which additionally satisfies $B_\alpha^m \mathcal{H}_\alpha(f)(0) = B_\alpha^s f(0) = 0$.

(ii) *One has*

$$\inf \lambda((-1)^m f) \tau(f) = 2q_{\alpha+s, m+1},$$

where the infimum is taken over all nontrivial functions $f \in \mathcal{E}_\alpha(\mathbb{R}_+) \cap L^1(\mathbb{R}_+, \lambda^{2m+2s} d\nu_\alpha)$ such that

$$(4.3) \quad B_\alpha^k \mathcal{H}_\alpha(f)(0) = 0, \quad k = s, \dots, m+s-1, \quad B_\alpha^{m+s} \mathcal{H}_\alpha(f)(0) \geq 0.$$

Moreover, the function $f_{\alpha+s, m}(\lambda)$ is the unique extremizer up to a positive constant, which additionally satisfies $B_\alpha^{m+s} \mathcal{H}_\alpha(f)(0) = 0$.

Proof. Part (i). Let f be an admissible function. Without loss of generality we can assume that $\tau(f) = 2$. Unlike the proof of Theorem 3.1 we will use the Radau quadrature formula (2.13) with $r = s+1$.

First, we show that $f^{(2l)}(0) = 0$ for $0 \leq l \leq s-1$ and $f^{(2s)}(0) \leq 0$. Indeed, we have $B_\alpha \lambda^{2j} = 2j(2\alpha + 2j) \lambda^{2j-2}$, and therefore for $j, l \in \mathbb{Z}_+$, by induction, we obtain for the l -th power of B_α that $B_\alpha^l \lambda^{2j} = c_{\alpha, j, l} \lambda^{2j-2l}$, where $c_{\alpha, j, l} > 0$ for $j \geq l$ and $c_{\alpha, j, l} = 0$ otherwise. This and Taylor's expansion of f imply

$$B_\alpha^l f(0) = \frac{c_{\alpha, l, l}}{(2l)!} f^{(2l)}(0).$$

Second, let $\lambda((-1)^m f) < q'_{m+1}$, where for simplicity we put $q'_k = q_{\alpha+s+1, k}$, $k \geq 1$. Recall that $q_{\alpha, k}$ are zeros of the Bessel function $j_\alpha(\lambda)$. Applying (2.13) to $g(\lambda) = (-1)^m \prod_{k=1}^m (\lambda^2 - q_k'^2) f(\lambda)$ (note that $g \in \mathcal{B}_\alpha^2(\mathbb{R}_+)$), we derive

$$\int_0^\infty g(\lambda) d\nu_\alpha(\lambda) = \sum_{l=0}^s \alpha_{l, s+1} g^{(2l)}(0) + \sum_{k=1}^\infty \gamma_{k, s+1} g(q'_k).$$

On the other hand, we have

$$\int_0^\infty g(\lambda) d\nu_\alpha(\lambda) = (-1)^m \int_0^\infty \lambda^{2m} f(\lambda) d\nu_\alpha(\lambda) = B_\alpha^m \mathcal{H}_\alpha(f)(0) \geq 0$$

and

$$g^{(2j)}(0) = 0, \quad j = 0, \dots, s-1, \quad g^{(2s)}(0) = f^{(2s)}(0) \prod_{k=1}^m q_k'^2 \leq 0.$$

Therefore,

$$0 \leq \alpha_{s,s+1} g^{(2s)}(0) + \sum_{k=m+1}^{\infty} \gamma_{k,s+1} g(q_k') \leq 0,$$

where we have used that $\gamma_{k,s+1} > 0$ and the fact that $g(\lambda) \leq 0$ for $\lambda \geq \lambda((-1)^m f)$. Thus, f has double zeros at the points q_k' , $k \geq m+1$, and the zero of order $2s+2$ at the origin.

Further, applying formula (2.13) for $j = 1, \dots, m$ to the functions $\prod_{\substack{k=1 \\ k \neq j}}^m (\lambda^2 - q_k'^2) f(\lambda)$, we conclude that the function f has at least simple zeros at the points q_j , $1 \leq j \leq m$. Then as in the proof of Theorem 3.1, using Lemmas 2.2, 2.3 and the fact that $\lambda^{2s+2} j_{\alpha+s+1}^2(\lambda) \notin L^1(\mathbb{R}_+, d\nu_\alpha)$, we derive that

$$f(\lambda) = \frac{\lambda^{2s+2} j_{\alpha+s+1}^2(\lambda) \sum_{k=0}^m c_k \lambda^{2k}}{\prod_{k=1}^m (1 - \lambda^2/q_k'^2)} \notin L^1(\mathbb{R}_+, \lambda^{2m} d\nu_\alpha).$$

Hence, following arguments similar to those used to show (3.4), we obtain that $\lambda((-1)^m f) \geq q_{m+1}'$. In fact, we have that $\lambda((-1)^m f) = q_{m+1}'$ for

$$(4.4) \quad f(\lambda) = \frac{\lambda^{2s+2} j_{\alpha+s+1}^2(\lambda)}{\prod_{k=1}^{m+1} (1 - \lambda^2/q_k'^2)} \in L^1(\mathbb{R}_+, \lambda^{2m} d\nu_\alpha).$$

Moreover, f is a unique extremizer up to a positive constant (similarly to the proof of the uniqueness of $f_{\alpha,m}$ in Theorem 3.1).

Using (2.13) and $f^{(2s)}(0) = 0$, we also have $B_\alpha^m \mathcal{H}_\alpha(f)(0) = B_\alpha^s f(0) = 0$.

Part (ii). The case $s = 0$ follows from Theorem 3.1 since to prove estimate (3.4), we only used condition (3.3).

Let $s \geq 1$. We observe that for any admissible function f , that is, satisfying condition (4.3), the function $g(\lambda) = \lambda^{2s} f(\lambda)$ satisfies conditions (4.1) and (4.2) with the parameter $s-1$ in place of s . At the same time, we have $\lambda((-1)^m f) \tau(f) = \lambda((-1)^m g) \tau(g)$. Hence, using the fact that $c \lambda^{2s} f_{\alpha+s,m}(\lambda)$ is the unique extremizer in part (i), we conclude that $c f_{\alpha+s,m}(\lambda)$ is the unique extremizer in problem (ii). □

5. NUMBER OF ZEROS OF POSITIVE DEFINITE FUNCTION ON \mathbb{R}_+

It was proved in [33] that if a function from the class (1.1) has n zeros on the interval $[0, L]$, then $L \geq \frac{\pi}{2} n$. Moreover,

$$F_n(x) = \left(\cos \frac{x}{n} \right)^n$$

is the unique extremal function. Note that the functions $F_n(\pi n(x - \frac{1}{2}))$ for $n = 1$ and 3 coincide, up to constants, with the cosine Fourier transform of f_0 and f_1 (see Introduction) on $[0, 1]$.

In this section we study a similar problem for the Hankel transform \mathcal{H}_α with $\alpha \geq -1/2$. We will use the approach which was developed in Section 3. The key argument in the proof is based on the properties of the polynomial $p_{\alpha,m}(t)$ defined in (3.17).

Let $N_I(f)$ be the number of zeros of f on I , counting multiplicity. We will say that $f \in \mathcal{E}_\alpha^+(\mathbb{R}_+)$ if $f(\lambda) = \int_0^1 j_\alpha(\lambda t) d\sigma(t)$ with a nonnegative bounded Stieltjes measure $d\sigma \neq 0$.

Theorem 5.1. *Let $\alpha \geq -1/2$, $n \in \mathbb{N}$, and*

$$L(f, n) = \inf \{L > 0 : N_{[0, L]}(f) \geq n\}.$$

Then

$$\inf_{f \in \mathcal{E}_\alpha^+(\mathbb{R}_+)} L(f, n) \leq \theta_{\alpha, n} = \begin{cases} q_{\alpha, m+1}, & n = 2m + 1, \\ q_{\alpha+1, m+1}, & n = 2m + 2. \end{cases}$$

Moreover, there exists a function $F_{\alpha, n} \in \mathcal{E}_\alpha^+(\mathbb{R}_+)$ such that $L(F_{\alpha, n}, n) = \theta_{\alpha, n}$.

Remark 5.1. (1) For $\alpha = -1/2$, we have $q_{-1/2, m+1} = \frac{\pi}{2}(2m+1)$, $q_{1/2, m+1} = \pi(m+1)$, and, therefore, $\theta_{-1/2, n} = \frac{\pi}{2}n$. Hence, we arrive at the mentioned above result [33]

$$\inf_{f \in \mathcal{E}_{-1/2}^+(\mathbb{R}_+)} L(f, n) = \frac{\pi}{2}n,$$

where the extremal function $F_{-1/2, n}(\lambda) = (\cos \frac{\lambda}{n})^n$ has on $[0, \frac{\pi}{2}n]$ the unique zero $\lambda = \frac{\pi}{2}n$ of multiplicity n .

(2) We will show that the function $F_{\alpha, n}(\lambda)$ has on $[0, \theta_{\alpha, n}]$ the unique zero $\lambda = \theta_{\alpha, n}$ of multiplicity n . Moreover, one has for $\lambda \in [0, \theta_{\alpha, n}]$

$$F_{\alpha, n}(\lambda) = \begin{cases} p_{\alpha, m}(\lambda/q_{\alpha, m+1}), & n = 2m + 1, \\ \int_{\lambda/q_{\alpha+1, m+1}}^1 sp_{\alpha+1, m}(s) ds, & n = 2m + 2. \end{cases}$$

Proof. Let $n = 2m + 1$. Consider the polynomial (see (3.17))

$$p_{\alpha, m}(t) = \sum_{i=1}^{m+1} B_i j_\alpha(q_i t), \quad t \in \mathbb{R}_+,$$

where $q_i = q_{\alpha, i}$. It has positive coefficients B_i and the unique zero $t = 1$ of multiplicity $2m + 1$ on the interval $[0, 1]$ (see Theorem 3.1 (iii)). This and (3.18) imply that the function

$$F_{\alpha, n}(\lambda) = \sum_{i=1}^{m+1} B_i j_\alpha\left(\frac{q_i}{q_{m+1}} \lambda\right), \quad \lambda \in \mathbb{R}_+,$$

is the positive definite entire function of exponential type 1 such that $\lambda = q_{m+1}$ is a unique zero of multiplicity $2m + 1$ on the interval $[0, q_{m+1}]$. Therefore, $L(F_{\alpha, n}, 2m + 1) \leq q_{m+1}$.

Assume that $n = 2m + 2$. Consider the polynomial of type (3.17), with respect to the parameter $\alpha + 1$:

$$p_{\alpha+1, m}(t) = \sum_{i=1}^{m+1} B'_i j_{\alpha+1}(q'_i t), \quad t \in \mathbb{R}_+,$$

where $q'_i = q_{\alpha+1, i}$. As above, $B'_i > 0$ and

$$(5.1) \quad B'_i = -b_{\alpha+1}^{-1} \frac{A'_i}{\frac{d}{dt} j_{\alpha+1}(q'_i t)|_{t=1}}, \quad \sum_{i=1}^{m+1} \frac{A'_i}{q_i'^2 - \lambda^2} = \frac{1}{\prod_{i=1}^{m+1} (1 - \lambda^2/q_i'^2)}.$$

Set

$$P(t) = \int_t^1 sp_{\alpha+1, m}(s) ds = 2(\alpha + 1) \sum_{i=1}^{m+1} \frac{B'_i}{q_i'^2} (j_\alpha(q'_i t) - j_\alpha(q'_i)),$$

where we have used (2.3).

In virtue of (2.4), $\frac{d}{dt} j_{\alpha+1}(q'_i t)|_{t=1} = 2(\alpha + 1)j_\alpha(q'_i)$ and therefore the polynomial $p_{\alpha+1, m}$ is positive and decreasing on $[0, 1)$ and it has zero of multiplicity $2m + 1$ at $t = 1$. Then

it is clear that the polynomial $P(t)$ is positive and decreasing on $[0, 1)$ and it has zero of multiplicity $2m + 2$ at $t = 1$.

Moreover, $P(t)$ can be represented as follows

$$P(t) = B_0'' + \sum_{i=1}^{m+1} B_i'' j_\alpha(q_i' t),$$

where $B_i'' > 0$ for $i \geq 1$ and, by (5.1),

$$B_0'' = -2(\alpha + 1) \sum_{i=1}^{m+1} \frac{B_i'}{q_i'^2} j_\alpha(q_i') = b_{\alpha+1}^{-1} \sum_{i=1}^{m+1} \frac{A_i'}{q_i'^2} = b_{\alpha+1}^{-1} > 0.$$

We finish the proof defining

$$F_{\alpha,n}(\lambda) = B_0'' + \sum_{i=1}^{m+1} B_i'' j_\alpha\left(\frac{q_i'}{q_{m+1}'} \lambda\right), \quad \lambda \in \mathbb{R}_+,$$

which is a positive definite entire function of exponential type 1, having the unique zero $\lambda = q_{m+1}'$ of multiplicity $2m + 2$ on $[0, q_{m+1}']$. Therefore, $L(F_{\alpha,n}, 2m + 2) \leq q_{m+1}'$. \square

6. GENERALIZED LOGAN PROBLEM FOR DUNKL AND FOURIER TRANSFORMS

In this section we solve the Logan problem for the Dunkl transform. We remark that in this case we will use the function $f_{\alpha,m}$ defined by (1.3) for any $\alpha \geq -1/2$ unlike the case of Fourier transform where we deal with only $\alpha = d/2 - 1$.

Basic facts on Dunkl harmonic analysis can be found in, e.g., [35]. Let a finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ be a root system, $G(R) \subset O(d)$ be a finite reflection group, generated by reflections $\{\sigma_a : a \in R\}$, where σ_a is a reflection with respect to hyperplane $\langle a, x \rangle = 0$, and $\kappa : R \rightarrow \mathbb{R}_+$ be a G -invariant multiplicity function. The Dunkl weight is given by

$$v_\kappa(x) = \prod_{a \in R_+} |\langle a, x \rangle|^{2\kappa(a)},$$

where R_+ positive subsystem of R .

Let $E_\kappa(x, y)$ be the symmetric Dunkl kernel associated with G and κ and $e_\kappa(x, y) = E_\kappa(x, iy)$ be the generalized exponential function. It is known that

$$e_\kappa(x, y) = \int_{\mathbb{R}^d} e^{i\langle \xi, y \rangle} d\mu_x^\kappa(\xi),$$

where μ_x^κ is a probability Borel measure supported on the convex hull of the G -orbit of $x \in \mathbb{R}^d$. Moreover, one has $(-\Delta_\kappa)^r e_\kappa(\cdot, y) = |y|^{2r} e_\kappa(\cdot, y)$, $r \in \mathbb{Z}_+$, where Δ_κ is the Dunkl Laplacian.

Denote

$$\alpha_\kappa = \frac{d}{2} - 1 + \sum_{a \in R_+} \kappa(a).$$

We will need the following Fischer-type decomposition for the Dunkl Laplacian: any even polynomial $P(x)$, $x \in \mathbb{R}^d$, of degree at most $2r$ can be represented by

$$P(x) = \sum_{m=0}^r \sum_{j=0}^m |x|^{2m-2j} H_{m,2j}(x),$$

where $H_{m,2j}$ are even κ -harmonic homogeneous polynomials of degree $2j$, i.e., $\Delta_\kappa H_{m,2j} = 0$ (see [14, Sec. 5.1]). Such polynomials satisfy

$$\Delta_\kappa |x|^{2i} H_{m,2j}(x) = 2i(2i + 4j + 2\alpha_\kappa) |x|^{2i-2} H_{m,2j}(x)$$

(see [14, Lemma 5.1.9]), which implies

$$(6.1) \quad \Delta_\kappa^l |x|^{2i} H_{m,2j}(x) = c_{ijl} |x|^{2i-2l} H_{m,2j}(x), \quad c_{ijl} = 0 \quad \text{for } i < l.$$

The Dunkl transform is defined as follows

$$\mathcal{F}_\kappa(f)(y) = c_\kappa \int_{\mathbb{R}^d} f(x) \overline{e_\kappa(x, y)} v_\kappa(x) dx, \quad y \in \mathbb{R}^d,$$

where $c_\kappa^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2} v_\kappa(x) dx$ is the Macdonald–Mehta–Selberg integral. It is a unitary operator in $L^2(\mathbb{R}^d, d\mu_\kappa)$ such that $\mathcal{F}_\kappa^{-1}(f)(x) = \mathcal{F}_\kappa(f)(-x)$.

In the non-weighted case ($\kappa = 0$) we have $d\mu_0(x) = (2\pi)^{-d/2} dx$, $e_0(x, y) = e^{i\langle x, y \rangle}$, $\Delta_0 = \Delta$, and \mathcal{F}_0 is the Fourier transform.

Let $f \in C(\mathbb{R}^d)$ be such that

$$(6.2) \quad f(x) = \int_{\mathbb{R}^d} e_\kappa(x, y) d\mu(y)$$

with a finite nonnegative Borel measure μ . We call such functions positive definite with respect to the Dunkl transform, if μ is nonnegative. For $\kappa = 0$, by Bochner's theorem, we arrive at the usual concept of positive definiteness.

Denote by $\mathcal{E}_\kappa(\mathbb{R}^d)$ the class of all even real-valued continuous bandlimited functions f of form (6.2) with the compactly supported measure μ . As usual, $\tau(f)$ is the exponential (spherical) type of f if $\text{supp } \mu \subset B_{\tau(f)}^d$ (cf. [26]). Recall that $\lambda(f) = \sup\{|x| > 0 : f(x) > 0\}$.

We are now in a position to formulate the complete solution of the generalized Logan problem as well as the uncertainty principle relations for the Dunkl transform.

Theorem 6.1. *Let $d \geq \mathbb{N}$ and $m, s \in \mathbb{Z}_+$.*

(i) *One has*

$$\inf \lambda((-1)^m f) \tau(f) = 2q_{\alpha_\kappa, m+1},$$

where the infimum is taken over all nontrivial functions $f \in \mathcal{E}_\kappa(\mathbb{R}^d)$ such that the measure μ in (6.2) is nonnegative in some neighborhood of the origin and, if $m \geq 1$, $f \in L^1(\mathbb{R}^d, |x|^{2m-2} v_\kappa(x) dx)$ and the condition

$$\Delta_\kappa^j \mathcal{F}_\kappa(f)(0) = 0, \quad j = 0, \dots, m-1,$$

is fulfilled. Moreover, the positive definite radial function $f_{\alpha_\kappa, m}(|x|)$ is the unique extremizer up to a positive constant. This function satisfies $f \in L^1(\mathbb{R}^d, |x|^{2m} v_\kappa(x) dx)$ and $\Delta_\kappa^m \mathcal{F}_\kappa(f)(0) = 0$.

(ii) *One has*

$$\inf \lambda((-1)^m f) \tau(f) = 2q_{\alpha_\kappa + s + 1, m+1},$$

where the infimum is taken over all nontrivial functions $f \in \mathcal{E}_\kappa(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, |x|^{2m} v_\kappa(x) dx)$ such that

$$(6.3) \quad \begin{cases} \Delta_\kappa^j \mathcal{F}_\kappa(f)(0) = 0, & j = 0, \dots, m-1, \\ \Delta_\kappa^l f(0) = 0, & l = 0, \dots, s-1, \end{cases}$$

and

$$\Delta_\kappa^m \mathcal{F}_\kappa(f)(0) \geq 0, \quad \Delta_\kappa^s f(0) \leq 0.$$

Moreover, each extremizer has the form $r(x) f_{\alpha_\kappa + s + 1, m}(|x|)$ and satisfies the condition $\Delta_\kappa^m \mathcal{F}_\kappa(f) = \Delta_\kappa^s f(0) = 0$. Here

$$r(x) = \sum_{j=0}^{s+1} |x|^{2s+2-2j} h_{2j}(x) \geq 0, \quad |x| \geq q_{\alpha_\kappa + s, m+1},$$

where $h_{2j}(x)$ are even κ -harmonic polynomials of order at most $2j$ such that $h_0 > 0$, $h_{2j}(0) = 0$, $j = 1, \dots, s+1$.

(iii) One has

$$\inf \lambda((-1)^m f) \tau(f) = 2q_{\alpha_\kappa+s, m+1},$$

where the infimum is taken over all nontrivial functions $f \in \mathcal{E}_\kappa(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, |x|^{2m+2s} v_\kappa(x) dx)$ such that

$$\Delta_\kappa^j \mathcal{F}_\kappa(f)(0) = 0, \quad j = s, \dots, m+s-1, \quad \Delta_\kappa^{m+s} \mathcal{F}_\kappa(f)(0) \geq 0.$$

The function $f_{\alpha_\kappa+s, m}(|x|)$ is the unique extremizer up to a positive constant. Moreover, this function satisfies $\Delta_\kappa^{m+s} \mathcal{F}_\kappa(f)(0) = 0$.

Remark 6.1. (1) For $s = 0$, the class of admissible functions in part (iii) of Theorem 6.1 contains admissible functions from part (i).

(2) For $\kappa = 0$, part (i) implies Theorem 1.1, part (ii) implies Theorem 1.2, and part (iii) implies Remark 1.2.

(3) In part (ii), if a polynomial $r(x)$ is nonnegative on \mathbb{R}^d , then it is an even homogeneous polynomial of order $2s+2$.

Proof. Our main idea is to reduce the proof of Theorem 6.1 to the case of Hankel transform of radial functions. Using polar coordinates, we have

$$\begin{aligned} c_\kappa \int_{\mathbb{R}^d} f(x) v_\kappa(x) dx &= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(\lambda x') c_\kappa v_\kappa(x') d\omega_\kappa(x') \lambda^{2\alpha_\kappa+1} d\lambda \\ &= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(\lambda x') d\omega_\kappa(x') d\nu_{\alpha_\kappa}(\lambda), \end{aligned}$$

where $d\nu_{\alpha_\kappa}$ is given by (2.10), $\mathbb{S}^{d-1} = \{x' \in \mathbb{R}^d : |x'| = 1\}$ is the Euclidean sphere, and $d\omega_\kappa(x') = b_{\alpha_\kappa}^{-1} c_\kappa v_\kappa(x') dx'$ is a probability measure on \mathbb{S}^{d-1} [36, Sec. 2.2]. In particular, for a radial function $f(x) = f_0(|x|)$ one has

$$(6.4) \quad \mathcal{F}_\kappa(f)(0) = c_\kappa \int_{\mathbb{R}^d} f(x) v_\kappa(x) dx = \int_0^\infty f_0(\lambda) d\nu_{\alpha_\kappa}(\lambda).$$

Let now $f \in \mathcal{E}_\kappa(\mathbb{R}^d)$ be a function of type τ , written $f(x) = \int_{B_\tau^d} e_\kappa(x, y) d\mu(y)$. We consider its radial part $f_0(\lambda) = \int_{\mathbb{S}^{d-1}} f(\lambda x') d\omega_\kappa(x')$. Due to the well-known formula [36, Corollary 2.5]

$$\int_{\mathbb{S}^{d-1}} e_\kappa(\lambda x', y) d\omega_\kappa(x') = j_{\alpha_\kappa}(\lambda|y|), \quad y \in \mathbb{R}^d,$$

we conclude that f_0 can be represented by

$$(6.5) \quad f_0(\lambda) = \int_{B_\tau^d} j_{\alpha_\kappa}(\lambda|y|) d\mu(y) = \int_0^\tau j_{\alpha_\kappa}(\lambda t) d\sigma(t),$$

where σ is a function of bounded variation. It is also clear that if $d\mu$ in (6.5) is non-negative in some neighborhood of the origin (or everywhere), then $d\sigma$ satisfies the same property.

In light of (6.4) and (6.5), we derive that

$$\begin{aligned} (6.6) \quad B_{\alpha_\kappa}^r \mathcal{H}_{\alpha_\kappa}(f_0)(0) &= \Delta_\kappa^r \mathcal{F}_\kappa(f)(0) = (-1)^r c_\kappa \int_{\mathbb{R}^d} |x|^{2r} f(x) v_\kappa(x) dx, \\ B_{\alpha_\kappa}^r f_0(0) &= \Delta_\kappa^r f_0(0) = (-1)^r \int_{\mathbb{R}^d} |y|^{2r} d\mu(y). \end{aligned}$$

In virtue of these relationships we note that if a function f is admissible in any of problems (i)–(iii) in Theorem 6.1, then its radial part $f_0(|x|)$ is also admissible in

the same problem and $\lambda((-1)^m f_0)\tau(f_0) \leq \lambda((-1)^m f)\tau(f)$. Hence, the corresponding infimums are attained on radial functions.

Formulas (6.5) and (6.6) also imply that radial extremizers in problems (i)–(iii) coincide with extremizers in Theorems 3.1 and 4.1 for Hankel transforms. Thus, the functions $f_{\alpha_\kappa, m}(|x|)$, $|x|^{2s+2}f_{\alpha_\kappa+s+1, m}(|x|)$, and $f_{\alpha_\kappa+s, m}(|x|)$ are extremizers for problems (i), (ii), and (iii), respectively.

Note that for any admissible function f from part (i), taking into account Theorem 3.1, we have that $\Delta_\kappa^m \mathcal{F}_\kappa(f)(0) = B_{\alpha_\kappa}^m \mathcal{H}_{\alpha_\kappa}(f_0)(0) \geq 0$. This implies part (1) of Remark 6.1.

It is left to prove the uniqueness of extremizers in problems (i)–(iii).

Part (ii). Let $\tau = 2$, $q'_j = q_{\alpha_\kappa+s+1, j}$, and f be an extremizer. Then $(-1)^{m+1}f(x) \geq 0$ for $|x| \geq q'_{m+1}$ and its radial part is

$$(6.7) \quad f_0(\lambda) = c\lambda^{2s+2}f_{\alpha_\kappa+s+1, m}(\lambda), \quad c > 0.$$

Therefore, $\int_{\mathbb{S}^{d-1}} f(q'_j x') d\omega_\kappa(x') = 0$ for $j \geq m+1$, which gives $f(x) = 0$ if $|x| = q'_j$, $j \geq m+1$. Moreover, $f \in L^1(\mathbb{R}^d, |x|^{2m} v_\kappa(x) dx)$, since, in light of (4.4),

$$c_\kappa \int_{|x| \geq q'_{m+1}} |x|^{2m} |f(x)| v_\kappa(x) dx = (-1)^{m+1} c \int_{q'_{m+1}}^\infty \lambda^{2m+2s+2} f_{\alpha_\kappa+s+1, m}(\lambda) d\nu_{\alpha_\kappa}(\lambda) < \infty.$$

Denote $f(x) = f(\lambda x') = f_{x'}(\lambda)$, where $\lambda = |x|$, $x' = x/|x|$. Since f is even and $f_{x'}(\lambda) = \int_{B_\tau^d} e_\kappa(\lambda x', y) d\mu(y)$, then $f_{x'}$ is the even entire function of exponential type τ bounded on \mathbb{R} . By Fubini's theorem, $f_{x'} \in L^1(\mathbb{R}_+, \lambda^{2m} d\nu_{\alpha_\kappa})$.

The function $f_{x'}(\lambda)$ keeps its sign for $\lambda \geq q'_{m+1}$ and $f_{x'}(q'_j) = 0$ for $j \geq m+1$. Hence, q'_j are double zeros for $j \geq m+2$. Therefore, we have

$$(6.8) \quad f_{x'}(\lambda) = r_{x'}(\lambda) f_{\alpha_\kappa+s+1, m}(\lambda)$$

with some even entire function $r_{x'}(\lambda)$ of exponential type. Similar to the proof of uniqueness of extremizer $f_{\alpha, m}$ in Theorem 3.1, using Lemmas 2.2 and 2.3, we obtain that $r_{x'}(\lambda)$ is an even polynomial of degree at most $2s+2$ (otherwise $f_{x'} \notin L^1(\mathbb{R}_+, \lambda^{2m} d\nu_{\alpha_\kappa})$).

Thus, by (6.8), we have

$$(6.9) \quad f(x) = r(x) f_{\alpha_\kappa+s+1, m}(|x|),$$

where $r(x) = \sum_{k=0}^{s+1} c_k(x') |x|^{2k}$. Taylor's expansions are given by

$$f(x) = \sum_{l=0}^{\infty} A_l(x') |x|^{2l}, \quad f_{\alpha_\kappa+s+1, m}(|x|) = \sum_{j=0}^{\infty} a_j |x|^{2j},$$

where $A_l(x')$ are homogeneous polynomials of degree $2l$, $A_0(x') = A_0$, and $a_0 = 1$. Therefore, we arrive at the linear system

$$\sum_{j=0}^l c_j(x') a_{l-j} = A_l(x'), \quad l = 0, 1, \dots, s+1,$$

in variables $c_j(x')$. We derive that

$$c_0 = A_0, \quad c_j(x') = A_j(x') + \sum_{i=0}^{j-1} b_{ij} A_i(x'), \quad j = 1, \dots, l.$$

Thus, $c_j(x') = A_j(x') + \sum_{i=0}^{j-1} b_{ij} A_i(x') |x|^{2j-2i}$ are homogeneous polynomials of degree $2j$, $j = 1, \dots, l$, and then $r(x)$ is an even polynomial of degree $2s+2$.

Now we find under which conditions on r the function f is an extremizer. Since $\lambda((-1)^m f) = q'_{m+1}$, we necessarily have

$$r(x) \geq 0, \quad |x| \geq q'_{m+1}.$$

We write $r(x) = \sum_{k=0}^{s+1} r_{2k}(x)$, where $r_{2k}(x)$ are homogeneous polynomials of degree $2k$. By (6.9) and (6.7),

$$f_0(\lambda) = f_{\alpha_\kappa+s+1,m}(\lambda) \sum_{k=0}^{s+1} \int_{\mathbb{S}^{d-1}} \lambda^{2k} r_{2k}(x') d\omega_\kappa(x') = c\lambda^{2s+2} f_{\alpha_\kappa+s+1,m}(\lambda).$$

This implies

$$(6.10) \quad \int_{\mathbb{S}^{d-1}} r_{2k}(x') d\omega_\kappa(x') = 0, \quad k = 0, 1, \dots, s, \quad \int_{\mathbb{S}^{d-1}} r_{2s+2}(x') d\omega_\kappa(x') > 0.$$

In particular, $r_0 = 0$. Furthermore, (6.10), the Fisher-type decomposition

$$r(x) = \sum_{j=0}^{s+1} |x|^{2s+2-2j} h_{2j}(x)$$

with $h_{2j}(x)$ being even κ -harmonic polynomials of order at most $2j$, and the fact that $h_{2j}(0) = \int_{\mathbb{S}^{d-1}} h_{2j}(x) d\omega_\kappa(x')$ imply that

$$(6.11) \quad h_0 > 0, \quad h_{2j}(0) = 0, \quad j = 1, \dots, s+1.$$

It is enough to verify that the function $f(x) = r(x)f_{\alpha_\kappa+s+1,m}(|x|)$ is an extremizer.

Let us show (6.3). By Theorem 4.1, for $k = 0, 1, \dots, m$ we have

$$\begin{aligned} c_\kappa \int_{\mathbb{R}^d} |x|^{2k} f(x) v_\kappa(x) dx &= \sum_{j=0}^{s+1} \int_0^\infty \lambda^{2k+2s+2-2j} f_{\alpha_\kappa+s+1,m}(\lambda) d\nu_{\alpha_\kappa}(\lambda) \int_{\mathbb{S}^{d-1}} h_{2j}(x) v_\kappa(x') d\omega_\kappa(x') \\ &= \sum_{j=0}^{s+1} h_{2j}(0) \int_0^\infty \lambda^{2k+2s+2-2j} f_{\alpha_\kappa+s+1,m}(\lambda) d\nu_{\alpha_\kappa}(\lambda) \\ &= h_0(0) \int_0^\infty \lambda^{2k+2s+2} f_{\alpha_\kappa+s+1,m}(\lambda) d\nu_{\alpha_\kappa}(\lambda) = 0. \end{aligned}$$

Since

$$f(x) = \sum_{j=0}^{s+1} |x|^{2s+2-2j} h_{2j}(x) \sum_{k=0}^\infty c_k |x|^{2k} = \sum_{k=0}^\infty c_k \sum_{j=0}^{s+1} |x|^{2s+2-2j+2k} h_{2j}(x),$$

both (6.1) and (6.11) imply that $\Delta_\kappa^l f(0) = 0$ for $l = 0, 1, \dots, s$. Thus, condition (6.3) holds and moreover, $\Delta_\kappa^m \mathcal{F}_\kappa(f) = \Delta_\kappa^s f(0) = 0$ is valid.

Finally, let us show that if $r(x) \geq 0$ on \mathbb{R}^d , then $r(x)$ is homogeneous polynomials of degree $2s+2$. Assume that $r(x) = \lambda^{k_0} \sum_{k=k_0}^{s+1} \lambda^{k-k_0} r_{2k}(x')$, $x = \lambda x'$, where $1 \leq k_0 \leq s$ and $r_{2k_0}(x) \neq 0$ (recall that $r_0 = 0$). Using $\int_{\mathbb{S}^{d-1}} r_{2k_0}(x') d\omega_\kappa(x') = 0$, we derive $r(\lambda x'_0) < 0$ for some $x'_0 \in \mathbb{S}^{d-1}$ and sufficiently small $\lambda > 0$. This contradiction implies that $r(x) = r_{2s+2}(x)$.

Parts (i) and (iii) with $s = 0$. Similar reasonings as above imply that any extremizer has the form $cf_{\alpha_\kappa,m}(|x|)$ with $c > 0$.

Part (iii) with $s \geq 1$. As in the proof of Theorem 4.1, we reduce the question about uniqueness of an extremizer f in part (iii) to similar problem in part (ii) with $s - 1$ in place of s . Thus, we arrive at the function $cf_{\alpha_\kappa+s,m}(|x|)$, $c > 0$. \square

7. CHEBYSHEV SYSTEMS OF NORMALIZED BESSEL FUNCTIONS

Recall that $N_I(f)$ stands for the number of zeros of f on I , counting multiplicity. A family of real-valued functions $\{\varphi_k(t)\}$ defined on an interval $I \subset \mathbb{R}$ is a Chebyshev system (T-system) if for any $n \in \mathbb{N}$ and any nontrivial linear combination

$$P(t) = \sum_{k=1}^n A_k \varphi_k(t),$$

there holds $N_I(P) \leq n - 1$, see, e.g., [1, Chap. II].

As above we assume that $\alpha \geq -1/2$, $q_k = q_{\alpha,k}$, and $q'_k = q_{\alpha+1,k}$ for $k \in \mathbb{N}$. The main result of this section is the following theorem.

Theorem 7.1. (i) *The families of the Bessel functions*

$$(7.1) \quad \{j_\alpha(q_k t)\}_{k=1}^\infty, \quad \{1, j_\alpha(q'_k t)\}_{k=1}^\infty$$

form Chebyshev systems on $[0, 1)$ and $[0, 1]$, respectively.

(ii) *The families of the Bessel functions*

$$\{j_{\alpha+1}(q_k t)\}_{k=1}^\infty, \quad \{j_\alpha(q'_k t) - j_\alpha(q'_k)\}_{k=1}^\infty$$

form Chebyshev systems on $(0, 1)$.

For $\alpha = -1/2$ this theorem becomes the well-known result for trigonometric systems, which has many applications in approximation theory (see [1, Chap. II]). For $\alpha > -1/2$ this result seems to be new.

We will use the following Sturm's theorem on zeros of linear combinations of eigenfunctions of Sturm–Liouville problem. This result is not widely known in the literature, see the discussion in [5].

Theorem 7.2 (Sturm, 1836; Liouville, 1836). *Let $\{V_k\}_{k=1}^\infty$ be the system of eigenfunctions associated to eigenvalues $\rho_1 < \rho_2 < \dots$ of the following Sturm–Liouville problem on the interval $[a, b]$:*

$$(7.2) \quad (KV')' + (\rho G - L)V = 0, \quad (KV' - hV)(a) = 0, \quad (KV' + HV)(b) = 0,$$

where $G, K, L \in C[a, b]$, $K \in C^1(a, b)$, $K, G > 0$ on (a, b) , $h, H \in [0, \infty]$ and ρ denotes the spectral parameter.

Then for any nontrivial real polynomial of the form

$$P = \sum_{k=m}^n A_k V_k, \quad m, n \in \mathbb{N}, \quad m \leq n,$$

we have

$$m - 1 \leq N_{(a,b)}(P) \leq n - 1.$$

In particular, every k -th eigenfunction V_k has exactly $k - 1$ simple zeros in (a, b) .

For trigonometric system this result is known as the Sturm–Hurwitz theorem (see, e.g., [3]).

Note that in the proof given by Liouville (see [5]) it is enough to assume that $K, G > 0$ only on the interval (a, b) . This allows us to include the singular case, that is, when K

and G may have zeros at the endpoints of $[a, b]$. In particular, we may deal with the Sturm–Liouville problem for Bessel functions.

Proof of Theorem 7.1. We will use the fact that, by Theorem 7.2, the system of eigenfunctions $\{V_k\}_{k=1}^\infty$ is the Chebyshev system. We note that (7.1) are the families of eigenfunctions for the (singular for $\alpha > -1/2$) Sturm–Liouville problem (see [29])

$$(7.3) \quad \begin{aligned} (t^{2\alpha+1}u'(t))' + \lambda^2 t^{2\alpha+1}u(t) &= 0, \quad t \in [0, 1], \\ u'(0) &= 0, \quad \cos \theta u(1) + \sin \theta u'(1) = 0, \end{aligned}$$

where $\theta \in [0, \pi/2]$ and λ^2 is the spectral parameter. Here for the family $\{j_\alpha(q_k t)\}_{k=1}^\infty$, we assume the Dirichlet conditions $\theta = 0$ and $u(1) = 0$ and, for $\{1, j_\alpha(q'_k t)\}_{k=1}^\infty$, the Neumann conditions $\theta = \pi/2$ and $u'(1) = 0$.

In virtue of (2.3), we have

$$\cos \theta j_\alpha(\lambda) - \sin \theta \frac{\lambda^2}{2(\alpha+1)} j_{\alpha+1}(\lambda) = 0,$$

or, equivalently,

$$(7.4) \quad \cos \theta J_\alpha(\lambda) - \sin \theta \lambda J_{\alpha+1}(\lambda) = A J_\alpha(\lambda) + B \lambda J'_\alpha(\lambda) = 0,$$

where $A = \cos \theta - \alpha \sin \theta$, $B = \sin \theta$. Since $A/B + \alpha = \tan \theta \geq 0$, $\alpha > -1$, we have that equation (7.4) has only real roots (see [6, Sec. 7.9]). Due to evenness, it is enough to consider only nonnegative zeros, which we denote by $0 \leq r_1 < r_2 < \dots$. Then the eigenvalues and the eigenfunctions of the Sturm–Liouville problem (7.3) are r_k^2 and $j_\alpha(r_k t)$, $k \in \mathbb{N}$, respectively. In particular, we have $r_k = q_k$ for $\theta = 0$ and $r_k = q'_{k-1}$ for $\theta = \pi/2$, where we put $q'_0 = 0$.

The Sturm–Liouville problem (7.3) is a particular case of the problem (7.2); take $K = G = w$, $L = 0$, $r = \lambda^2$, $h = 0$, and $H = \cot \theta$. Then the statement of part (i) is valid for the interval $(0, 1)$. In order to include the endpoints, we first prove part (ii).

Let us show that the family $\{j_{\alpha+1}(q_k t)\}_{k=1}^\infty$ is the Chebyshev system on $(0, 1)$. Assume that the polynomial $P(t) = \sum_{k=1}^n A_k j_{\alpha+1}(q_k t)$ has n zeros on $(0, 1)$. We consider $F(t) = t^{2\alpha+2} P(t)$. It has at least $n+1$ zeros including $t = 0$. By Rolle's theorem, for a smooth real function f one has $N_{(a,b)}(f') \geq N_{(a,b)}(f) - 1$ (see [5]). Thus, P' has at least n zeros on $(0, 1)$. In light of (2.4), we obtain

$$F'(t) = 2(\alpha+1)t^{2\alpha+1} \sum_{k=1}^n A_k j_\alpha(q_k t).$$

This contradicts the fact that $\{j_\alpha(q_k t)\}_{k=1}^\infty$ is the Chebyshev system on $(0, 1)$.

To prove that $\{j_\alpha(q'_k t) - j_\alpha(q'_k)\}_{k=1}^\infty$ is the Chebyshev system on $(0, 1)$, assume that $P(t) = \sum_{k=1}^n A_k (j_\alpha(q'_k t) - j_\alpha(q'_k))$ has n zeros on $(0, 1)$. Taking into account the zero $t = 1$, its derivative (see (2.3))

$$P'(t) = -\frac{t}{2\alpha+2} \sum_{k=1}^n A_k q_k^2 j_{\alpha+1}(q'_k t)$$

has at least n zeros on $(0, 1)$. This contradicts the fact that $\{j_{\alpha+1}(q_{\alpha+1,k} t)\}_{k=1}^\infty$ is the Chebyshev system on $(0, 1)$.

Now we are in a position to show that the first system in (7.1) is Chebyshev on $[0, 1)$. Note that if $P(t) = \sum_{k=1}^n A_k j_\alpha(q_k t)$ has n zeros on $[0, 1)$, then always $P(0) = 0$. Moreover, $P(1) = 0$. Therefore, P' has at least n zeros on $(0, 1)$, which is impossible since $P'(t) = -\frac{t}{2\alpha+2} \sum_{k=1}^n A_k q_k^2 j_{\alpha+1}(q_k t)$ and $j_{\alpha+1}(q_k t)$ is the Chebyshev system on $(0, 1)$.

Similarly, if $P(t) = \sum_{k=0}^{n-1} A_k j_\alpha(q'_k t)$ (we assume $q'_0 = 0$) has n zeros on $[0, 1]$, then one of the endpoints is a zero. Then $P'(t) = -\frac{t}{2\alpha+2} \sum_{k=1}^{n-1} A_k q_k'^2 j_{\alpha+1}(q'_k t)$ has at least $n-1$ zeros in $(0, 1)$, which is impossible for Chebyshev system $\{j_{\alpha+1}(q_{\alpha+1,k} t)\}_{k=1}^\infty$. \square

8. AN ALTERNATIVE PROOF OF POSITIVE DEFINITENESS OF THE FUNCTION $g_{\alpha,m}$

In [12], the positive definiteness of the function $g_{d/2-1,m}$ given by (1.4) was proved based on the use of classical translation operator in \mathbb{R}^d . This causes the restriction $\alpha = d/2 - 1$. Another approach to see that $g_{\alpha,m}$ is positive definite, is to employ Bochner's theorem and show that the Hankel transform of $g_{\alpha,m}$ is nonnegative, which is equivalent to fact that the matrix of the generalized translations $(T_\alpha^{x_i} f(x_j))_{i,j=1}^N$ is positive definite, see Section 2. Here we follow this approach and ideas from [12].

Let $R_n^{(\alpha)}(\theta) = \frac{P_n^{(\alpha,\alpha)}(\theta)}{P_n^{(\alpha,\alpha)}(1)}$ be the normalized Jacobi polynomial and $-1 < r_n < \dots < r_1 < 1$ be its zeros, see, e.g., [37]. Define the generalized translation operator on $[-1, 1]$ as follows

$$(8.1) \quad \tau^\theta f(\rho) = c_\alpha \int_0^\pi f(\sqrt{1-\theta^2}\sqrt{1-\rho^2} + 2\theta\rho \cos \varphi) \sin^{2\alpha} \varphi d\varphi,$$

where c_α is defined in (2.8). We remark that $\tau^\theta R_n^{(\alpha)}(\rho) = R_n^{(\alpha)}(\theta) R_n^{(\alpha)}(\rho)$.

Consider the polynomial $p_{n-k}(\theta) = \frac{R_n^{(\alpha)}(\theta)}{(\theta-r_1)\dots(\theta-r_k)}$. It was shown in [10] that

$$p_{n-k}(\theta) = \sum_{s=0}^{n-k} a_s R_s^{(\alpha)}(\theta), \quad a_s \geq 0, \quad i = 0, \dots, n-k.$$

This implies that for any choice of $\theta_1, \dots, \theta_N \subset [-1, 1]$ the matrix $(\tau^{\theta_i} p_{n-k}(\theta_j))_{i,j=1}^N$ is positive semidefinite, i.e.,

$$\begin{aligned} \sum_{i,j=1}^N c_i \bar{c}_j \tau^{\theta_i} p_{n-k}(\theta_j) &= \sum_{s=0}^{n-k} a_s \sum_{i,j=1}^N c_i \bar{c}_j \tau^{\theta_i} R_s^{(\alpha)}(\theta_j) \\ &= \sum_{s=0}^{n-k} a_s \sum_{i,j=1}^N c_i \bar{c}_j R_s^{(\alpha)}(\theta_i) R_s^{(\alpha)}(\theta_j) = \sum_{s=0}^{n-k} a_s \left| \sum_{i=1}^N c_i R_s^{(\alpha)}(\theta_i) \right|^2 \geq 0. \end{aligned}$$

Recall again that $q_i = q_{\alpha,i}$ are zeros of $j_\alpha(y)$ and $g_k(y) = \frac{j_\alpha(y)}{(q_1^2 - y^2) \dots (q_k^2 - y^2)}$. We note (see [37, Sec. 8.1]) that

$$\lim_{n \rightarrow \infty} R_n^{(\alpha)} \left(1 - \frac{y^2}{2n^2} + o\left(\frac{1}{n^2}\right) \right) = j_\alpha(y)$$

uniformly in $y \in [0, L]$ for any positive L . Since ([37, Sec. 8.1])

$$r_i = 1 - \frac{q_i^2}{2n^2} + o\left(\frac{1}{n^2}\right),$$

then setting $\theta = 1 - y^2/(2n^2) + o(1/n^2)$, we obtain

$$\lim_{n \rightarrow \infty} (2n^2)^k (\theta - r_1) \dots (\theta - r_k) = (q_1^2 - y^2) \dots (q_k^2 - y^2)$$

uniformly in $y \in [0, L]$.

Let us show that there holds

$$(8.2) \quad \lim_{n \rightarrow \infty} (2n^2)^{-k} p_{n-k} \left(1 - \frac{y^2}{2n^2} + o\left(\frac{1}{n^2}\right) \right) = g_k(y)$$

uniformly in $y \in [0, L]$. This is true on any interval without arbitrarily small neighborhoods of points q_i , $i = 1, \dots, k$. Without loss of generality, it is enough to consider a

small neighborhood of q_1 , since $(q_2^2 - y^2) \cdots (q_k^2 - y^2)$ is bounded away from zero in this neighborhood.

Using (2.2) implies

$$\begin{aligned} \frac{j_\alpha(y)}{q_1^2 - y^2} &= \frac{j_\alpha(y) - j_\alpha(q_1)}{q_1^2 - y^2} = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu \Gamma(\alpha+1)}{4^\nu \nu! \Gamma(\nu + \alpha + 1)} \frac{y^{2\nu} - q_1^{2\nu}}{q_1^2 - y^2} \\ &= \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} \Gamma(\alpha+1)}{4^\nu \nu! \Gamma(\nu + \alpha + 1)} \sum_{s=0}^{\nu-1} y^{2s} q_1^{2(\nu-1-s)} \\ &= \sum_{s=0}^{\infty} \left(\frac{y^2}{q_1^2}\right)^s \sum_{k=s}^{\infty} \frac{(-1)^k \Gamma(\alpha+1) q_1^{2k}}{4^{k+1} (k+1)! \Gamma(k + \alpha + 2)} \\ &= \frac{1}{4} \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(s+l+2) \Gamma(s+l+\alpha+2)} \left(-\frac{y^2}{4}\right)^s \left(-\frac{q_1^2}{4}\right)^l. \end{aligned}$$

Similarly, if $\theta = 1 - y^2/(2n^2) + o(1/n^2)$, then [37, Sec. 4.21]

$$\begin{aligned} \binom{n+\alpha}{n} \frac{R_n^{(\alpha)}(\theta)}{\theta - r_1} &= \sum_{\nu=1}^n \frac{\Gamma(n+\nu+2\alpha+1) \Gamma(n+\alpha+1) ((\theta-1)^\nu - (r_1-1)^\nu)}{2^\nu \nu! \Gamma(n-\nu+1) \Gamma(n+2\alpha+1) \Gamma(\nu+\alpha+1) (\theta-r_1)} \\ &= \sum_{\nu=1}^n \frac{\Gamma(n+\nu+2\alpha+1) \Gamma(n+\alpha+1)}{2^\nu \nu! \Gamma(n-\nu+1) \Gamma(n+2\alpha+1) \Gamma(\nu+\alpha+1)} \sum_{s=0}^{\nu-1} (\theta-1)^s (r_1-1)^{\nu-s-1} \\ &= \sum_{s=0}^{n-1} \sum_{\nu=s}^{n-1} \frac{\Gamma(n+\nu+2\alpha+2) \Gamma(n+\alpha+1) (\theta-1)^s (r_1-1)^{\nu-s}}{2^{\nu+1} (\nu+1)! \Gamma(n-\nu) \Gamma(n+2\alpha+1) \Gamma(\nu+\alpha+2)} \\ &= \frac{1}{2} \sum_{s=0}^{n-1} \sum_{l=0}^{n-1-s} \frac{\Gamma(n+s+l+2\alpha+2) \Gamma(n+\alpha+1) (\theta-1)^s (r_1-1)^l}{2^{s+l} \Gamma(s+l+2) \Gamma(n-s-l) \Gamma(n+2\alpha+1) \Gamma(s+l+\alpha+2)} \\ &= \frac{1}{2} \sum_{s=0}^{n-1} \sum_{l=0}^{n-1-s} \frac{\Gamma(n+s+l+2\alpha+2) \Gamma(n+\alpha+1) (-y^2/4)^s (-q_1^2/4)^l (1+o(1/n^2))}{n^{2(s+l)} \Gamma(s+l+2) \Gamma(n-s-l) \Gamma(n+2\alpha+1) \Gamma(s+l+\alpha+2)}. \end{aligned}$$

Since

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b}, \quad \binom{n+\alpha}{n} \sim \frac{n^\alpha}{\Gamma(\alpha+1)}, \quad n \rightarrow \infty,$$

then, for fixed s and l ,

$$\begin{aligned} &\frac{\Gamma(\alpha+1) \Gamma(n+s+l+2\alpha+2) \Gamma(n+\alpha+1) (1+o(1/n^2)) (-y^2/4)^s (-q_1^2/4)^l}{4n^{2(s+l+1)+\alpha} \Gamma(n-s-l) \Gamma(n+2\alpha+1) \Gamma(s+l+2) \Gamma(s+l+\alpha+2)} \\ (8.3) \quad &\sim \frac{\Gamma(\alpha+1) (-y^2/4)^s (-q_1^2/4)^l}{4\Gamma(s+l+2) \Gamma(s+l+\alpha+2)}, \quad n \rightarrow \infty, \end{aligned}$$

and, for $\theta = 1 - y^2/(2n^2) + o(1/n^2)$, we have, uniformly on $y \in [0, L]$,

$$\lim_{n \rightarrow \infty} 2^{-1} n^{\alpha-2} \frac{R_n^{(\alpha)}(\theta)}{\theta - r_1} = \frac{j_\alpha(y)}{q_1^2 - y^2}.$$

We should explain how we take the limit under the sum. Since for any $n \geq 1$, $0 \leq s \leq n-1$, $0 \leq l \leq n-1-s$,

$$\frac{\Gamma(n+s+l+2\alpha+2)}{\Gamma(n+2\alpha+1)} \leq (2n+2\alpha)^{s+l+1},$$

$$\frac{\Gamma(n)}{\Gamma(n-s-l)} \leq n^{s+l}, \quad \frac{\Gamma(n+\alpha+1)}{\Gamma(n)} \leq C(\alpha)n^{\alpha+1},$$

then (8.3) can be bounded from above by

$$\begin{aligned} & \frac{\Gamma(\alpha+1)\Gamma(n+s+l+2\alpha+2)\Gamma(n+\alpha+1)|1+o(1/n^2)|(y^2/4)^s(q_1^2/4)^l}{4n^{2(s+l+1)+\alpha}\Gamma(n-s-l)\Gamma(n+2\alpha+1)\Gamma(s+l+2)\Gamma(s+l+\alpha+2)} \\ & \leq C_1(\alpha) \frac{(y^2/4)^s(q_1^2/4)^l}{\Gamma(s+l+2)\Gamma(s+l+\alpha+2)}. \end{aligned}$$

Moreover, the following series converges uniformly on any interval $[0, L]$, $q_1 \leq L$,

$$\begin{aligned} & \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \frac{(y^2/4)^s(q_1^2/4)^l}{\Gamma(s+l+2)\Gamma(s+l+\alpha+2)} \\ & \leq \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \frac{(L^2/4)^{s+l}}{\Gamma(s+l+2)\Gamma(s+l+\alpha+2)} \leq \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \frac{(L^2/4)^{s+l}}{(s+l+1)!} \\ & = \sum_{m=0}^{\infty} (m+1) \frac{(L^2/4)^m}{(m+1)!} = \sum_{m=0}^{\infty} \frac{(L^2/4)^m}{m!} = e^{L^2/4}. \end{aligned}$$

Thus, (8.2) is proved.

Let

$$x_i \in [0, \infty), \quad \frac{x_i}{n} \leq 1, \quad \theta_i = \sqrt{1 - \left(\frac{x_i}{n}\right)^2}, \quad i = 1, \dots, N.$$

For $i, j = 1, \dots, N$, there holds, uniformly on $\varphi \in [0, \pi]$ and for sufficiently large n ,

$$\sqrt{1 - \left(\frac{x_i}{n}\right)^2} \sqrt{1 - \left(\frac{x_j}{n}\right)^2} + 2 \frac{x_i x_j}{n^2} \cos \varphi = 1 - \frac{y_{ij}^2}{2n^2} + o\left(\frac{1}{n^2}\right),$$

where

$$y_{ij} = \sqrt{x_i^2 + x_j^2 - 2x_i x_j \cos \varphi}.$$

Therefore, by (8.2) and the definitions of the generalized translation operator (2.14) and (8.1), for any i, j ,

$$\lim_{n \rightarrow \infty} (2n^2)^{-k} \tau^{\theta_i} p_{n-k}(\theta_j) = T_{\alpha}^{x_i} g_k(x_j).$$

Since the matrix $(\tau^{\theta_i} p_{n-k}(\theta_j))$ is positive semidefinite, then the matrix $(T_{\alpha}^{x_i} g_k(x_j))$ is also positive semidefinite. Then, by Levitan's theorem, $\mathcal{H}_{\alpha}(g_k)(t) \geq 0$ and the functions g_k and (1.4) are positive definite.

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