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Properties of moduli of smoothness in $L_p(\mathbb{R}^d)$

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ABSTRACT. In this paper, we discuss various basic properties of moduli of smoothness of functions from $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$. In particular, complete versions of Jackson-, Marchaud-, and Ulyanov-type inequalities are given for the whole range of p . Moreover, equivalences between moduli of smoothness and the corresponding K -functionals and the realization concept are proved.

1. Introduction

1.1. Goal of the paper. The subject of this paper is to collect the main properties of moduli of smoothness in $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$. Moduli of smoothness are known to be a very useful concept in many areas of analysis and the PDE's. Several basic properties are known for a long time. For example, a version of the inequality for moduli of smoothness of different orders – the so-called Marchaud's inequality – was obtained already in 1927 [43]. On the other hand, some fundamental characteristics of $\omega_r(f, \delta)_p$ for the whole range of p and r have remained unknown until now. In particular, the celebrated Jackson inequality is unknown for the case $0 < p < 1$ in the multidimensional case for non-periodic functions. Moreover, inequalities between moduli of smoothness in different metrics (the so-called Ulyanov-type inequalities), i.e., $\omega_r(f, \delta)_q$ vs. $\omega_k(f, \delta)_p$ for $0 < p < q \leq \infty$, can be found in the literature only in the weak form [7, 11, 12, 17].

In this paper, we not only survey some basic well-known properties such as Marchaud's inequality but also obtain the complete versions of Jackson and Ulyanov inequalities. For this we use needed Nikolskii–Stechkin and Hardy–Littlewood–Nikolskii type inequalities for entire functions of exponential type.

We also derive equivalences between moduli of smoothness and the following characteristics: the corresponding K -functionals, the realization concept, and the average moduli of smoothness. All of them turn out to be important tools in approximation theory and functional analysis. Here, again, the results are well known for $1 < p < \infty$ and only partially known outside this range.

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Throughout the paper, for $F, G \geq 0$, we use the notation $F \lesssim G$ for the estimate $F \leq CG$, where C is a positive constant independent of the essential variables in F and G (usually, f , δ , and σ). If $F \lesssim G \lesssim F$, we write $F \asymp G$.

Moreover, for $1 \leq p \leq \infty$, p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$. For any real number a , $[a]$ is the largest integer not greater than a and $a_+ = \max\{a, 0\}$.

The rest of the paper is organized as follows. In Subsection 1.2 we give the definitions of the moduli of smoothness of integer and fractional order. In Subsection 1.3 we present the main properties of moduli of smoothness and give some historical comments. In Section 2 we obtain two important inequalities for entire functions: the Nikol'skii–Stechking–Boas-type inequality and the Hardy–Littlewood–Nikol'skii inequality. Section 3 contains the proofs of the main results of the paper.

1.2. Definition of moduli of smoothness. As usual, the r -th order modulus of smoothness of a function $f \in L_p(\mathbb{R}^d)$ is defined by

$$(1.1) \quad \omega_r(f, \delta)_p := \sup_{|h| \leq \delta} \|\Delta_h^r f\|_p,$$

where

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^r = \Delta_h \Delta_h^{r-1}, \quad h \in \mathbb{R}^d, \quad d \geq 1,$$

and $|h| := (\sum_{j=1}^d |h_j|^2)^{1/2}$. Throughout the paper, by $\|f\|_p$ we mean the (quasi-)norm $\|f\|_p = \|f\|_{L_p(\mathbb{R}^d)}$, $0 < p \leq \infty$.

We also recall the definition of the modulus of smoothness $\omega_\alpha(f, \delta)_p$ of fractional order $\alpha > 0$:

$$(1.2) \quad \omega_\alpha(f, \delta)_p := \sup_{|h| \leq \delta} \|\Delta_h^\alpha f\|_p,$$

where

$$(1.3) \quad \Delta_h^\alpha f(x) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\alpha}{\nu} f(x + (\alpha - \nu)h), \quad h \in \mathbb{R}^d,$$

and $\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}$, $\binom{\alpha}{0} = 1$ (see [6, 54, 60]). It is clear that for integer α definition (1.2) coincides with the classical definition (1.1). Note that in the case of $0 < p < 1$, since

$$(1.4) \quad \sum_{\nu=0}^{\infty} \left| \binom{\alpha}{\nu} \right|^p < \infty \quad \text{for } \alpha \in \mathbb{N} \cup ((1/p - 1), \infty),$$

it is natural to assume that $\alpha > (1/p - 1)_+$ while defining the fractional modulus of smoothness in L_p .

Some basic properties of moduli of smoothness of integer order can be found in [2], [11], [64], and [69].

1.3. Main properties of moduli of smoothness in $L_p(\mathbb{R}^d)$. Below we present Properties 1–17 of (fractional) moduli of smoothness. Historical comments are given after each statement.

PROPERTY 1. For $f, f_1, f_2 \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, we have:

- (a) $\omega_\alpha(f, \delta)_p$ is a non-negative non-decreasing function of δ such that $\lim_{\delta \rightarrow 0+} \omega_\alpha(f, \delta)_p = 0$;
- (b) $\omega_\alpha(f_1 + f_2, \delta)_p \leq 2^{(\frac{1}{p}-1)+} (\omega_\alpha(f_1, \delta)_p + \omega_\alpha(f_2, \delta)_p)$;
- (c) $\omega_\alpha(f, \delta)_p \leq C(\alpha, p) \|f\|_p$;
- (d) $\|f\|_p \leq \lim_{\delta \rightarrow \infty} \omega_\alpha(f, \delta)_p \leq C(\alpha, p) \|f\|_p$ if $0 < p < \infty$.

PROPERTY 2. Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, $\lambda > 0$, and $\delta > 0$. Then

$$(1.5) \quad \omega_\alpha(f, \lambda\delta)_p \lesssim (1 + \lambda)^{\alpha + d(\frac{1}{p}-1)+} \omega_\alpha(f, \delta)_p.$$

Equivalently, for any $0 < h < \delta$, one has

$$(1.6) \quad \frac{\omega_\alpha(f, \delta)_p}{\delta^{\alpha + d(\frac{1}{p}-1)+}} \lesssim \frac{\omega_\alpha(f, h)_p}{h^{\alpha + d(\frac{1}{p}-1)+}}.$$

If $1 \leq p \leq \infty$ inequality (1.5) trivially follows from the equivalence between moduli of smoothness and K -functionals (see (1.36)). If $0 < p < 1$ only the periodic analogue of (1.5) was known. For $d = 1$ this was obtained in [50] (for integer α) and in [53] (for positive α). For $d \in \mathbb{N}$ and $\alpha > 0$ see [36].

In the following three properties, we deal with moduli of smoothness of integer order.

PROPERTY 3. Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, and $r \in \mathbb{N}$. Then, for any $\delta > 0$, we have

$$(1.7) \quad \omega_r(f, \delta)_p \asymp \sum_{k_1 + \dots + k_d = r} \omega_{k_1, \dots, k_d}(f, \delta)_p,$$

where $\omega_{k_1, \dots, k_d}(f, \delta)_p$ is the mixed modulus of smoothness, that is,

$$\omega_{k_1, \dots, k_d}(f, \delta)_p = \sup_{|h| \leq \delta} \|\Delta_{e_1 h_1}^{k_1} \dots \Delta_{e_d h_d}^{k_d} f\|_p$$

(here $\{e_j\}_{j=1}^d$ is the standard basis in $\mathbb{R}^d$). Moreover, if $1 < p < \infty$, then

$$(1.8) \quad \omega_r(f, \delta)_p \asymp \sum_{j=1}^d \omega_r^{(j)}(f, \delta)_p,$$

where $\omega_r^{(j)}(f, \delta)_p$ is the partial modulus of smoothness, that is,

$$(1.9) \quad \omega_r^{(j)}(f, \delta)_p = \sup_{|h| \leq \delta} \|\Delta_{e_j h}^r f\|_p, \quad j = 1, \dots, d.$$

For equivalence (1.7) in the case $1 \leq p \leq \infty$ see [2, p. 338]. For periodic functions equivalence (1.8) was given in [65]. Note also that the mixed moduli of smoothness were studied in, e.g., [13, 46, 48, 49].

Recall that the homogeneous Sobolev norm is given by $\|f\|_{\dot{W}_p^r} = \sum_{\nu_1 + \dots + \nu_d = r} \|D^\nu f\|_p$, where as usual $D^\nu f = \frac{\partial^\nu}{\partial^{r_1} x_1 \dots \partial^{r_d} x_d} f$.

PROPERTY 4. Let $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$, and $r \in \mathbb{N}$. Then

$$(1.10) \quad \sup_{h>0} \frac{\omega_r(f, h)_p}{h^r} \asymp \|f\|_{\dot{W}_p^r}.$$

This property can be found in, e.g., [39] and [44], see also [8].

PROPERTY 5. Let $0 < p, q, s \leq \infty$, $1/p + 1/q = 1/s$, and $r \in \mathbb{N}$. Then for any $f \in L_p(\mathbb{R}^d)$ and $g \in L_q(\mathbb{R}^d)$, we have

$$(1.11) \quad \omega_r(fg, \delta)_s \leq \sum_{k=0}^r \binom{r}{k} \omega_k(f, \delta)_p \cdot \omega_{r-k}(g, \delta)_q,$$

where $\omega_0(f, \delta)_p = \|f\|_p$ and $\omega_0(g, \delta)_q = \|g\|_q$.

For the proof of inequality (1.11) see [28]; see also [69, 4.6.12] for applications of this inequality.

PROPERTY 6. Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, $0 < q < \infty$, and $r \in \mathbb{N}$. Then, for any $\delta > 0$,

$$(1.12) \quad \omega_r(f, \delta)_p \asymp \left(\delta^{-d} \int_{|h| \leq \delta} \|\Delta_h^r f\|_p^q dh \right)^{1/q}$$

and if, additionally, $q \leq p$, then

$$(1.13) \quad \omega_r(f, \delta)_p \asymp \left\| \left(\delta^{-d} \int_{|h| \leq \delta} |\Delta_h^r f(\cdot)|^q dh \right)^{1/q} \right\|_p.$$

Moreover, if $1 < p < \infty$ and $\alpha > 0$, then, for any $\delta > 0$,

$$(1.14) \quad \omega_\alpha(f, \delta)_p \asymp \delta^{-d} \int_{|h| \leq \delta} \|\Delta_h^\alpha f\|_p dh$$

and

$$(1.15) \quad \omega_{2r}(f, \delta)_p \asymp \left\| \delta^{-d} \int_{|h| \leq \delta} \Delta_h^{2r} f(\cdot - rh) dh \right\|_p.$$

If $d = 1$, then equivalence (1.14) holds also for any $0 < p \leq \infty$ and $\alpha > (1/p - 1)_+$.

In the case $1 \leq p \leq \infty$, the first equivalence (1.12) is well known, see, e.g., [11, Ch. 6, § 5] or [30, Appendix A]; see also [13]. For periodic functions $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $\alpha \in ((1/p - 1)_+, \infty)$, equivalence (1.14) was derived in [33]. Equivalence (1.15) for functions on $\mathbb{T}^d$ was obtained in [69, 8.2.9].

PROPERTY 7. (MARCHAUD INEQUALITY). Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, and $\gamma > 0$ be such that $\alpha + \gamma \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then, for any $\delta \in (0, 1)$, we have

$$(1.16) \quad \omega_\alpha(f, \delta)_p \lesssim \delta^\alpha \left(\int_\delta^1 \left(\frac{\omega_{\gamma+\alpha}(f, t)_p}{t^\alpha} \right)^\theta \frac{dt}{t} + \|f\|_p^\theta \right)^{\frac{1}{\theta}},$$

where

$$(1.17) \quad \theta = \theta(p) = \begin{cases} \min(p, 2), & p < \infty; \\ 1, & p = \infty. \end{cases}$$

Equivalently,

$$(1.18) \quad \omega_\alpha(f, \delta)_p \lesssim \delta^\alpha \left(\int_\delta^\infty \left(\frac{\omega_{\gamma+\alpha}(f, t)_p}{t^\alpha} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}}.$$

The Marchaud inequality for the moduli of smoothness of integer order is the classical result in approximation theory (see, e.g., [11, p. 48] and [14]). The case $1 < p < \infty$ for fractional moduli was handled in [67, Theorem 2.1].

PROPERTY 8. (REVERSE MARCHAUD INEQUALITY). *Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, and $\alpha, \beta \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then*

$$(1.19) \quad \omega_{\alpha+\beta}(f, \delta)_p \lesssim \omega_\beta(f, \delta)_p.$$

Moreover, if $1 < p < \infty$ and $\alpha, \beta > 0$, then, for any $\delta \in (0, 1)$, we have

$$(1.20) \quad \delta^\alpha \left(\int_\delta^1 \left(\frac{\omega_{\alpha+\beta}(f, t)_p}{t^\alpha} \right)^\tau \frac{dt}{t} \right)^{\frac{1}{\tau}} \lesssim \omega_\beta(f, \delta)_p,$$

where $\tau = \max(p, 2)$.

Inequality (1.19) easily follows from (1.4). Inequalities of type (1.20) in the one-dimensional periodic case were first obtained by Timan in [66]. The general case was developed in [10].

PROPERTY 9. (SHARP ULYANOV INEQUALITY). *Let $f \in L_p(\mathbb{R}^d)$, $0 < p < q \leq \infty$, $\alpha \in \mathbb{N} \cup ((1 - 1/q)_+, \infty)$, and $\gamma \geq 0$ be such that $\alpha + \gamma \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then, for any $\delta \in (0, 1)$, we have*

$$(1.21) \quad \omega_\alpha(f, \delta)_q \lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+\gamma}(f, t)_p}{t^\gamma} \eta\left(\frac{1}{t}\right) \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \delta^\alpha \|f\|_p,$$

where

$$q_1 := \begin{cases} q, & q < \infty; \\ 1, & q = \infty \end{cases}$$

and

1) if $0 < p \leq 1$ and $p < q \leq \infty$, then

$$(1.22) \quad \eta(t) := \begin{cases} t^{d(\frac{1}{p}-1)}, & \gamma > d\left(1 - \frac{1}{q}\right)_+; \\ t^{d(\frac{1}{p}-1)}, & \gamma = d\left(1 - \frac{1}{q}\right)_+ \geq 1, d \geq 2, \text{ and } \alpha + \gamma \in \mathbb{N}; \\ t^{d(\frac{1}{p}-1)} \ln^{\frac{1}{q-1}}(t+1), & \gamma = d\left(1 - \frac{1}{q}\right)_+ \geq 1, d \geq 2, \text{ and } \alpha + \gamma \notin \mathbb{N}; \\ t^{d(\frac{1}{p}-1)} \ln^{\frac{1}{q}}(t+1), & 0 < \gamma = d\left(1 - \frac{1}{q}\right)_+ = 1 \text{ and } d = 1; \\ t^{d(\frac{1}{p}-1)} \ln^{\frac{1}{q}}(t+1), & 0 < \gamma = d\left(1 - \frac{1}{q}\right)_+ < 1; \\ t^{d(\frac{1}{p}-\frac{1}{q})-\gamma}, & 0 < \gamma < d\left(1 - \frac{1}{q}\right)_+; \\ t^{d(\frac{1}{p}-\frac{1}{q})}, & \gamma = 0, \end{cases}$$

2) if $1 < p < q \leq \infty$, then

$$(1.23) \quad \eta(t) := \begin{cases} 1, & \gamma \geq d(\frac{1}{p} - \frac{1}{q}), \quad q < \infty; \\ 1, & \gamma > \frac{d}{p}, \quad q = \infty; \\ \ln^{\frac{1}{p}}(t+1), & \gamma = \frac{d}{p}, \quad q = \infty; \\ t^{d(\frac{1}{p}-\frac{1}{q})-\gamma}, & 0 \leq \gamma < d(\frac{1}{p} - \frac{1}{q}). \end{cases}$$

Moreover, the term $\delta^\alpha \|f\|_p$ in (1.21) can be dropped if any of the following conditions holds:

$$\begin{cases} \gamma = 0, & 0 < p < q \leq 1; \\ 0 \leq \gamma < d(1 - \frac{1}{q}), & 0 < p \leq 1 < q \leq \infty; \\ \gamma = 1, & d = 1, 0 < p \leq 1 < q = \infty; \\ \gamma = d(1 - \frac{1}{q}) \geq 1, & 0 < p \leq 1 < q \leq \infty, d \geq 2, \alpha + \gamma \in \mathbb{N}; \\ 0 \leq \gamma \leq d(\frac{1}{p} - \frac{1}{q}), & 1 < p < q < \infty; \\ 0 \leq \gamma < \frac{d}{p}, & 1 < p < q = \infty. \end{cases}$$

In the case $\gamma = 0$, inequality (1.21) is the classical Ulyanov inequality of different metrics [70] given by

$$(1.24) \quad \omega_\alpha(f, \delta)_q \lesssim \left(\int_0^\delta \left(\frac{\omega_\alpha(f, t)_p}{t^\gamma} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad \gamma = d(1/p - 1/q), 0 < p < q \leq \infty.$$

The detailed historical review can be found in [36]. Let us only note that the comprehensive study of (1.24) was given in [17]. The sharp Ulyanov inequality in the form

$$\omega_\alpha(f, \delta)_q \lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+\gamma}(f, t)_p}{t^\gamma} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad \gamma = d(1/p - 1/q), 1 < p < q < \infty,$$

was first derived in [55] and [67] (see [63] for the limiting cases $p = 1$ and/or $q = \infty$ for functions on $\mathbb{T}$). The periodic analogue of (1.21) for any $\gamma > 0$ and $0 < p < q \leq \infty$ has been recently obtained in [36]. For various moduli of smoothness sharp Ulyanov inequalities were also established in [20, 21, 22].

Note that in the proof of Property 9 given in Section 3, we obtain slightly stronger inequalities than those given in (1.21) including some important corollaries.

Now we concern with the Kolyada-type inequality, which is another improvement of the classical Ulyanov inequality (1.24) along with the sharp Ulyanov inequality (1.21).

PROPERTY 10. (KOLYADA INEQUALITY). *Let $f \in L_p(\mathbb{R}^d)$, $1 < p < q < \infty$, $\theta = d(1/p - 1/q)$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, $\alpha > \theta$. Then*

$$(1.25) \quad \delta^{\alpha-\theta} \left(\int_{\delta}^{\infty} \left(\frac{\omega_{\alpha}(f, t)_q}{t^{\alpha-\theta}} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \lesssim \left(\int_0^{\delta} \left(\frac{\omega_{\alpha}(f, t)_p}{t^{\theta}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

The periodic analogue of (1.25) was obtained by Kolyada [38]. He also derived such inequalities for analytic Hardy spaces on the disc in the case $0 < p < q < \infty$. For functions on $\mathbb{R}^d$, inequality (1.25) was proved for Lebesgue spaces in [67] (see also [23]) and for Hardy spaces in [36]. Note that (1.25) is not valid in $L_p(\mathbb{R}^d)$ spaces for $p = 1$, $d = 1$ but is true for $p = 1$, $d \geq 2$.

PROPERTY 11. *Let $f \in L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ and $r, m \in \mathbb{N}$. Then*

$$(1.26) \quad \delta^{-m} \omega_{r+m}(f, \delta)_p \lesssim \sup_{|\beta|=m} \omega_r(D^{\beta} f, \delta)_p \lesssim \int_0^{\delta} u^{-m} \omega_{r+m}(f, u)_p \frac{du}{u}.$$

Moreover, if $1 < p < \infty$, then

$$(1.27) \quad \sup_{|\beta|=m} \omega_r(D^{\beta} f, \delta)_p \lesssim \left(\int_0^{\delta} (u^{-m} \omega_{r+m}(f, u)_p)^{\theta} \frac{du}{u} \right)^{1/\theta},$$

where $\theta = \min(p, 2)$. If, in addition, f is such that $\frac{\partial^m f}{\partial x_j^m} \in L_p(\mathbb{R}^d)$ for $j = 1, \dots, d$, then,

$$(1.28) \quad \left(\int_0^{\delta} (u^{-m} \omega_{r+m}(f, u)_p)^{\tau} \frac{du}{u} \right)^{1/\tau} \lesssim \sup_{j=1, \dots, d} \omega_r \left(\frac{\partial^m f}{\partial x_j^m}, \delta \right)_p,$$

where $\tau = \max(p, 2)$.

For inequalities (1.26) see [2, p. 342–343]. Inequality (1.27) was proved in [67, Theorem 2.3]. Inequality (1.28) has been recently obtained in [19, Theorem 12.2]. See also [18], [34], and [35] for periodic analogues of (1.26) and (1.27) in the case $0 < p < 1$.

To formulate the next properties, we introduce some notations. By $\mathcal{B}_{\sigma,p} = \mathcal{B}_{\sigma,p}(\mathbb{R}^d)$, $\sigma > 0$, $0 < p \leq \infty$, we denote the Bernstein space of entire functions of exponential type σ (e.f.e.t.). That is, $f \in \mathcal{B}_{\sigma,p}$ if $f \in L_p(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ and $\text{supp } \mathcal{F}(f) \subset B_{\sigma} = \{x \in \mathbb{R}^d : |x| < \sigma\}$. Here and in what follows, the Fourier transform of $f \in L_1(\mathbb{R}^d)$ is given by

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} dx.$$

In the case $0 < p < 1$, we assume that f belongs to $\mathcal{S}'(\mathbb{R}^d)$, the space of all tempered distributions on $\mathbb{R}^d$.

Let $E_\sigma(f)_p$ be the best approximation of $f \in L_p(\mathbb{R}^d)$ by e.f.e.t. $P \in \mathcal{B}_{\sigma,p}$, i.e.,

$$E_\sigma(f)_p = \inf_{P \in \mathcal{B}_{\sigma,p}} \|f - P\|_p.$$

We also set $E_0(f)_p = \|f\|_p$ for $p < \infty$ and $E_0(f)_\infty = \inf_{c \in \mathbb{C}} \|f - c\|_\infty$.

PROPERTY 12. (JACKSON INEQUALITY). *Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, $\sigma > 0$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then*

$$(1.29) \quad E_\sigma(f)_p \lesssim \omega_\alpha \left(f, \frac{1}{\sigma} \right)_p.$$

Moreover, if $1 < p < \infty$ and $\alpha > 0$, then for any $\sigma \geq 1$ we have

$$(1.30) \quad \frac{1}{\sigma^\alpha} \left(\sum_{k=1}^{[\sigma]} (k+1)^{\alpha\tau-1} E_k(f)_p^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_\alpha \left(f, \frac{1}{\sigma} \right)_p,$$

where $\tau = \max(p, 2)$.

For Jackson's inequality (1.29) in the case $1 \leq p \leq \infty$ see, e.g., [64, p. 279]. In the case $0 < p < 1$, $\alpha \in \mathbb{N}$, and $d = 1$, this inequality was derived in [61] (see also [5]). Sharp Jackson inequality (1.30) was obtained in [10]. It is worth mentioning that this inequality is equivalent to (1.20), see [10]. Some historical remarks on Jackson's inequality for periodic functions can be found in [27].

PROPERTY 13. (INVERSE APPROXIMATION THEOREM). *Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, and $\sigma \geq 1$. Then we have*

$$(1.31) \quad \omega_\alpha \left(f, \frac{1}{\sigma} \right)_p \lesssim \frac{1}{\sigma^\alpha} \left(\sum_{k=0}^{[\sigma]} (k+1)^{\alpha\theta-1} E_k(f)_p^\theta \right)^{\frac{1}{\theta}},$$

where $\theta = \min(p, 2)$ if $p < \infty$ and $\theta = 1$ if $p = \infty$.

In the case $1 \leq p \leq \infty$, $\alpha \in \mathbb{N}$, inequality (1.31) is well known (see, e.g., [11, Ch. 7], [9], [64] and the references therein). In the case $0 < p < 1$, $d = 1$, and $\alpha \in \mathbb{N}$, inequality (1.31) was obtained in [61]. In other cases, the result seems to be new (cf. [15]). The proof is based on the corresponding Bernstein inequality.

PROPERTY 14. *Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. The following conditions are equivalent:*

(i) *for some $\beta > \alpha + (1/p - 1)_+$ we have*

$$\omega_\alpha(f, \delta)_p \asymp \omega_\beta(f, \delta)_p \quad \text{for all } \delta \in (0, 1),$$

(ii) *there holds*

$$\omega_\alpha \left(f, \frac{1}{\sigma} \right)_p \asymp E_\sigma(f)_p \quad \text{for all } \sigma \geq 1.$$

In the case $1 \leq p \leq \infty$ and $\alpha \in \mathbb{N}$, this property was obtained in [51] (the case $d = 1$) and in [24] (the case $d \geq 1$). In the case $0 < p < 1$, Property 14 is known only for functions $f \in L_p(\mathbb{T})$ and $f \in L_p[-1, 1]$, see, e.g., [31] and [34].

In what follows, we will use the directional derivative of f of order $\alpha > 0$ along a vector $\zeta \in \mathbb{R}^d$ given by

$$D_\zeta^\alpha f(x) = \mathcal{F}^{-1} \left((i\xi, \zeta)^\alpha \widehat{f}(\xi) \right) (x).$$

PROPERTY 15. *Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then*

$$(1.32) \quad 2^{-n\alpha} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_{2^n}\|_p \lesssim \omega_\alpha(f, 2^{-n})_p \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_{2^k}\|_p,$$

where $P_{2^k} \in \mathcal{B}_{2^k, p}$, $k \in \mathbb{N}$, are such that $\|f - P_{2^k}\|_p = E_{2^k}(f)_p$.

Moreover, if $1 < p < \infty$, then

$$(1.33) \quad \left(\sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|(-\Delta)^{\alpha/2} P_{2^k}\|_p^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_\alpha(f, 2^{-n})_p$$

$$\lesssim \left(\sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|(-\Delta)^{\alpha/2} P_{2^k}\|_p^\theta \right)^{\frac{1}{\theta}},$$

where $\tau = \max(2, p)$ and $\theta = \min(2, p)$.

Concerning the existence of P_{2^k} in Property 15 see, e.g., [64, Theorem 2.6.3]. The above inequalities (1.32) and (1.33) were obtained in [37].

REMARK 1. *Note that in the case $1 < p < \infty$, the best approximants P_{2^k} in inequalities (1.33) can be replaced by other methods of approximation such as the ℓ_q -Fourier means with $q = 1, \infty$, the de la Vallée Poussin means, or the Riesz spherical means. See details in [37].*

Now we deal with equivalence results for moduli of smoothness in terms of K -functionals and their realizations. Let us recall that for any $f \in L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, and $r \in \mathbb{N}$, we have (see [29] and [2, Ch.5])

$$(1.34) \quad \omega_r(f, \delta)_p \asymp K(f, \delta; L_p(\mathbb{R}^d), \dot{W}_p^r(\mathbb{R}^d)) := \inf_g \{ \|f - g\|_p + \delta^r \|g\|_{\dot{W}_p^r} \}.$$

Moreover, in the case $1 < p < \infty$ and $\alpha > 0$ we have (see [71])

$$(1.35) \quad \omega_\alpha(f, \delta)_p \asymp K(f, \delta; L_p(\mathbb{R}^d), \dot{H}_p^\alpha(\mathbb{R}^d)) := \inf_g \{ \|f - g\|_p + \delta^\alpha \|(-\Delta)^{\alpha/2} g\|_p \}.$$

Here the Sobolev and the Riesz potential spaces are given by

$$\dot{W}_p^r(\mathbb{R}^d) := \{f \in \dot{\mathcal{S}}'(\mathbb{R}^d) : \|f\|_{\dot{W}_p^r} = \sum_{|\nu|_{\ell_1}=r} \|D^\nu f\|_p < \infty\}$$

and

$$\dot{H}_p^\alpha(\mathbb{R}^d) := \{f \in \dot{\mathcal{S}}'(\mathbb{R}^d) : \|f\|_{\dot{H}_p^\alpha} = \|(-\Delta)^{\alpha/2} f\|_p < \infty\}$$

respectively, where $\dot{\mathcal{S}}'(\mathbb{R}^d)$ is the space of all continuous functionals on $\dot{\mathcal{S}}(\mathbb{R}^n)$ given by

$$\dot{\mathcal{S}}(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : (D^\nu \widehat{\varphi})(0) = 0 \text{ for all } \nu \in \mathbb{N}^n \cup \{0\}\}.$$

To obtain analogues of the above equivalences for all $0 < p \leq \infty$ and $\alpha > 0$, we introduce the following K -functional with respect to the directional derivative:

$$\mathcal{K}_\alpha(f, \delta)_p = \inf_g \{ \|f - g\|_p + \delta^\alpha \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha g\|_p \}.$$

PROPERTY 16. *Let $f \in L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, and $\alpha > 0$. Then, for any $\delta \in (0, 1)$, we have*

$$(1.36) \quad \omega_\alpha(f, \delta)_p \asymp \mathcal{K}_\alpha(f, \delta)_p.$$

This equivalence fails for $0 < p < 1$ since in this case $\mathcal{K}_\alpha(f, \delta)_p \equiv 0$ (see [16]). A suitable substitute for the K -functional for $p < 1$ is the realization concept given by

$$\mathcal{R}_\alpha(f, \delta)_p = \inf_{P \in \mathcal{B}_{1/\delta, p}} \left\{ \|f - P\|_p + \delta^\alpha \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha P\|_p \right\}.$$

PROPERTY 17. *Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then, for any $\delta \in (0, 1)$, we have*

$$(1.37) \quad \omega_\alpha(f, \delta)_p \asymp \mathcal{R}_\alpha(f, \delta)_p.$$

Moreover, if $P \in \mathcal{B}_{1/\delta, p}$ is such that $\|f - P\|_p \lesssim E_{1/\delta}(f)_p$, then

$$(1.38) \quad \omega_\alpha(f, \delta)_p \asymp \|f - P\|_p + \delta^\alpha \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha P\|_p.$$

In particular, we have for $r \in \mathbb{N}$

$$(1.39) \quad \omega_r(f, \delta)_p \asymp \inf_{P \in \mathcal{B}_{1/\delta, p}} \left\{ \|f - P\|_p + \delta^r \|P\|_{\dot{W}_p^r} \right\}.$$

Moreover, if $P \in \mathcal{B}_{1/\delta, p}$ is such that $\|f - P\|_p \lesssim E_{1/\delta}(f)_p$, then

$$\omega_r(f, \delta)_p \asymp \|f - P\|_p + \delta^r \|P\|_{\dot{W}_p^r}.$$

In the case $1 < p < \infty$, equivalence (1.37) is a combination of results from [26] and (1.35). For the case $0 < p \leq 1$, $d = 1$, and $\alpha \in \mathbb{N}$ see [16].

REMARK 2. *Note that Properties 1–17 except Property 1(d) hold for periodic functions from $L_p(\mathbb{T}^d)$. Observe that for $f \in L_p(\mathbb{T}^d)$, $0 < p \leq \infty$, the term $\|f\|_p$ in (1.16) and (1.21) can be dropped.*

2. Useful inequalities for entire functions

2.1. Polynomial inequalities of Nikol'skii–Stechkin–Boas–types. The results of this subsection are crucial to verify Properties 2, 13, 15, and 17.

LEMMA 1. *Let $0 < p \leq \infty$, $\alpha, \sigma > 0$, and $\zeta \in \mathbb{R}^d$, $0 < |\zeta| \leq 1/\sigma$. Then, for any $P_\sigma \in \mathcal{B}_{\sigma, p}$, we have*

$$(2.1) \quad \|D_\zeta^\alpha P_\sigma\|_p \asymp \|\Delta_\zeta^\alpha P_\sigma\|_p,$$

where the constants in this equivalence depend only on p , α , and d .

PROOF. Inequalities (2.1) are proved in [40] using certain estimates for maximal functions. Alternatively, (2.1) can be shown repeating step-by-step the proof of the corresponding result for trigonometric polynomials given in [36, Theorem 3.1]. $\square$

For periodic functions $f \in L_p(\mathbb{T})$, $1 \leq p \leq \infty$, equivalence (2.1) is the known result by Nikol'skii [45], Stechkin [56], and Boas [3]. Detailed historical observations can be found in [36, Section 3].

REMARK 3. (i) *Under conditions of Lemma 1, we have for any $P_\sigma \in \mathcal{B}_{\sigma,p}$ and $\delta \in (0, \sigma]$ that*

$$(2.2) \quad \omega_\alpha(P_\sigma, \delta)_p \asymp \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|\Delta_{\zeta\delta}^\alpha P_\sigma\|_p.$$

Indeed, applying equivalence (2.1) twice, we have

$$\begin{aligned} \omega_\alpha(P_\sigma, \delta)_p &= \sup_{0 < h \leq \delta} \sup_{|\zeta|=1} \|\Delta_{\zeta h}^\alpha P_\sigma\|_p \lesssim \sup_{0 < h \leq \delta} \sup_{|\zeta|=1} \|D_{\zeta h}^\alpha P_\sigma\|_p \\ &= \sup_{0 < h \leq \delta} h^\alpha \sup_{|\zeta|=1} \|D_\zeta^\alpha P_\sigma\|_p = \sup_{|\zeta|=1} \|D_{\zeta\delta}^\alpha P_\sigma\|_p \lesssim \sup_{|\zeta|=1} \|\Delta_{\zeta\delta}^\alpha P_\sigma\|_p. \end{aligned}$$

The inverse estimate is clear.

(ii) *Note that in the one-dimensional case, the above equivalences (2.1) and (2.2) have the following form:*

$$\frac{\omega_\alpha(P_\sigma, \delta)_{L_p(\mathbb{R})}}{\delta^\alpha} \asymp \frac{\|\Delta_h^\alpha P_\sigma\|_{L_p(\mathbb{R})}}{h^\alpha} \asymp \|P_\sigma^{(\alpha)}\|_{L_p(\mathbb{R})}$$

for any $\delta, h \in (0, 1/\sigma]$.

Using Lemma 1 and Remark 3, we obtain the following result.

COROLLARY 2. *Let $0 < p \leq \infty$, $\alpha, \sigma > 0$, and $0 < \delta \leq 1/\sigma$. Then, for any $P_\sigma \in \mathcal{B}_{\sigma,p}$, we have*

$$(2.3) \quad \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha P_\sigma\|_p \asymp \delta^{-\alpha} \omega_\alpha(P_\sigma, \delta)_p.$$

In particular, this implies

$$\sigma^\alpha \omega_\alpha(P_\sigma, 1/\sigma)_p \asymp \delta^{-\alpha} \omega_\alpha(P_\sigma, \delta)_p.$$

Further, using (2.3) and (1.4), we obtain the following Bernstein type inequality for fractional directional derivatives of entire functions.

COROLLARY 3. *Let $0 < p \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, and $\sigma > 0$. Then, for any $P_\sigma \in \mathcal{B}_{\sigma,p}$, we have*

$$(2.4) \quad \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha P_\sigma\|_p \lesssim \sigma^\alpha \|P_\sigma\|_p.$$

Inequality (2.4) is the classical Bernstein inequality for $d = 1$ (see, e.g., [12, Ch. 4]). For the fractional α and $d = 1$ see [42] and [1]. For $1 \leq p \leq \infty$, $\alpha > 0$, and $d \in \mathbb{N}$, inequality (2.4) can be obtained from [71, Theorem 3].

For moduli of smoothness of integer order, we have the following Nikol'skii–Stechkin–Boas result.

COROLLARY 4. *Let $0 < p \leq \infty$, $r \in \mathbb{N}$, $\sigma > 0$, and $0 < \delta \leq 1/\sigma$. Then, for any $P_\sigma \in \mathcal{B}_{\sigma,p}$, we have*

$$\|P_\sigma\|_{\dot{W}_p^r} \asymp \delta^{-r} \omega_r(P_\sigma, \delta)_p.$$

PROOF. The proof is the same as the proof of Theorem 3.2 in [36]. It is based on Lemma 1, Corollary 2, and equivalence (1.7). $\square$

We will also need the following equivalence between the fractional Laplacian and the directional derivatives.

COROLLARY 5. *Let $1 < p < \infty$ and $\alpha > 0$. Then, for any $P \in \cup_{\sigma>0} \mathcal{B}_{\sigma,p}$, we have*

$$\sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha P\|_p \asymp \|(-\Delta)^{\alpha/2} P\|_p.$$

Moreover, if $0 < p \leq \infty$ and $r \in \mathbb{N}$, then

$$(2.5) \quad \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^r P\|_p \asymp \|P\|_{\dot{W}_p^r}.$$

PROOF. The first part follows from (2.3) and the equivalence $\delta^{-\alpha} \omega_\alpha(P, \delta)_p \asymp \|(-\Delta)^{\alpha/2} P\|_p$ for $P \in \mathcal{B}_{1/\delta,p}$, $1/\delta = \sigma$ (see [71]). Equivalence (2.5) follows from Corollaries 2 and 4. $\square$

Note that for $1 \leq p \leq \infty$ equivalence (2.5) can be extended to any $f \in W_p^r(\mathbb{R}^d)$, $r \in \mathbb{N}$. Namely we have

$$\|f\|_{\dot{W}_p^r} \asymp \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^r f\|_p.$$

2.2. Hardy–Littlewood–Nikol’skii inequalities for entire functions. For applications, in particular, to obtain the sharp Ulyanov inequality for moduli of smoothness (Property 9), it is important to derive the Hardy–Littlewood–Nikol’skii inequalities for directional derivatives. As a simple example of such inequalities, we mention the following estimate:

$$\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_q \lesssim \eta(\sigma) \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p, \quad P_\sigma \in \mathcal{B}_{\sigma,p}.$$

For $\eta(\sigma) = 1$, $\gamma = d(1/p - 1/q)$, and $1 < p < q < \infty$, this corresponds to the Hardy–Littlewood inequality for fractional integrals. For $\eta(\sigma) = \sigma^{d(\frac{1}{p} - \frac{1}{q})}$, $\gamma = 0$, and $0 < p < q \leq \infty$, this coincides with the Nikolskii inequality (see [17])

$$(2.6) \quad \|P_\sigma\|_q \lesssim \sigma^{d(\frac{1}{p} - \frac{1}{q})} \|P_\sigma\|_p, \quad P_\sigma \in \mathcal{B}_{\sigma,p}.$$

In this subsection, we obtain similar inequalities for the previously unknown cases $0 < p \leq 1$ and $q = \infty$.

Let us recall the definition of the Besov space $B_{p,q}^s(\mathbb{R}^d)$ (see, e.g., [68]). We consider the Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\text{supp } \varphi \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}$, $\varphi(\xi) > 0$ for $1/2 < |\xi| < 2$, $\varphi(\xi) = 1$ for $5/4 < |\xi| < 7/4$, and

$$\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1 \quad \text{if } \xi \neq 0.$$

We also introduce the functions φ_k and ψ by means of the relations

$$\mathcal{F}\varphi_k(\xi) = \varphi(2^{-k}\xi) \quad \text{and} \quad \mathcal{F}\psi(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi).$$

We say that $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the (non-homogeneous) Besov space $B_{p,q}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, if

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d)} = \|\psi * f\|_p + \left(\sum_{k=1}^{\infty} 2^{sqk} \|\varphi_k * f\|_p^q \right)^{1/q} < \infty$$

(with the usual modification in the case $q = \infty$).

LEMMA 6. *Let $0 < p \leq 1$, $1 < q < \infty$, $\alpha > 0$, $\gamma = d(1 - 1/q)$, $\sigma \geq 1$, and $\alpha + \gamma \neq 2k + 1$, $k \in \mathbb{Z}_+$. Then, for any $P_\sigma \in \mathcal{B}_{\sigma,p}$, we have*

$$\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_q \lesssim \sigma^{d(\frac{1}{p}-1)} \ln^{\frac{1}{q}}(\sigma + 1) \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p + \|P_\sigma\|_q.$$

PROOF. Let $n \in \mathbb{N}$ be such that $2^{n-1} \leq \sigma < 2^n$. We have

$$P_\sigma = P_\sigma * \psi + \sum_{k=1}^{n+1} P_\sigma * \varphi_k.$$

By Corollary 5 and the embedding $B_{1,q}^{d(1-1/q)}(\mathbb{R}^d) \subset L_q(\mathbb{R}^d)$ (see, e.g., [68, 2.7.1]), we derive

$$\begin{aligned} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha (P_\sigma - P_\sigma * \psi)\|_q &\lesssim \|(-\Delta)^{\alpha/2} (P_\sigma - P_\sigma * \psi)\|_q \\ (2.7) \quad &\lesssim \left(\sum_{k=1}^{n+1} 2^{d(q-1)k} \|(-\Delta)^{\alpha/2} P_\sigma * \varphi_k\|_1^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{k=1}^{n+1} 2^{d(q-1)k - \gamma q k} \|\mathcal{D}^{\alpha+\gamma} P_\sigma\|_1^q \right)^{\frac{1}{q}}, \end{aligned}$$

where

$$\mathcal{D}^{\alpha+\gamma} P_\sigma(x) := \sum_{j=1}^d \left(\frac{\partial}{\partial x_j} \right)^{\alpha+\gamma} P_\sigma(x) = \mathcal{F}^{-1} \left(w(\xi) \widehat{P_\sigma}(\xi) \right)(x),$$

and $w(\xi) := (i\xi_1)^{\alpha+\gamma} + \dots + (i\xi_d)^{\alpha+\gamma}$. In the last inequality in (2.7), we applied the fact that the function

$$g(\xi) = \frac{|\xi|^\alpha \varphi(\xi)}{(i\xi_1)^{\alpha+\gamma} + \dots + (i\xi_d)^{\alpha+\gamma}} = \frac{|\xi|^\alpha \varphi(\xi)}{w(\xi)}$$

is the Fourier multiplier in $L_1(\mathbb{R}^d)$, i.e., $\|\mathcal{F}^{-1}(gf)\|_1 \lesssim \|f\|_1$ for any $f \in L_1(\mathbb{R}^d)$. To verify this, one can use Theorem 6.8 in [41] stating that if $g \in L_1(\mathbb{R}^d)$ and $D^\nu g \in L_p(\mathbb{R}^d)$

for some $1 < p \leq 2$ and all $\nu \in \{0, 1\}^d$, $\nu \neq 0$, then g is an L_1 -multiplier. Here we observe that

$$w(\xi) = \cos \frac{(\alpha + \gamma)\pi}{2} (|\xi_1|^{\alpha+\gamma} + \dots + |\xi_d|^{\alpha+\gamma}) \\ + i \left(\sin \frac{(\alpha + \gamma)\pi \operatorname{sign} \xi_1}{2} |\xi_1|^{\alpha+\gamma} + \dots + \sin \frac{(\alpha + \gamma)\pi \operatorname{sign} \xi_d}{2} |\xi_d|^{\alpha+\gamma} \right),$$

which implies that $\operatorname{Re} w(\xi) \neq 0$ for $\xi \neq 0$ and $\alpha + \gamma \neq 2k + 1$, $k \in \mathbb{Z}_+$.

Next, using Nikolskii's inequality, we derive from (2.7) that

$$(2.8) \quad \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha (P_\sigma - P_\sigma * \psi)\|_q \lesssim \left(\sum_{k=1}^{n+1} \|\mathcal{D}^{\alpha+\gamma} P_\sigma\|_1^q \right)^{\frac{1}{q}} \\ \asymp n^{\frac{1}{q}} \|\mathcal{D}^{\alpha+\gamma} P_\sigma\|_1 \\ \lesssim 2^{d(\frac{1}{p}-1)n} n^{\frac{1}{q}} \|\mathcal{D}^{\alpha+\gamma} P_\sigma\|_p \\ \lesssim \sigma^{d(\frac{1}{p}-1)} \ln^{\frac{1}{q}}(\sigma + 1) \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p.$$

Finally, taking into account Bernstein's inequality (2.4) and Young's convolution inequality, we get

$$\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha (P_\sigma * \psi)\|_q \lesssim \|P_\sigma * \psi\|_q \lesssim \|P_\sigma\|_q.$$

This and (2.8) imply the required estimate. $\square$

LEMMA 7. Let $P \in \mathcal{B}_{\sigma,1}$, $\sigma > 0$.

(i) For $1/q^* = (d-1)/d$, we have

$$(2.9) \quad \|P\|_{q^*} \leq \frac{1}{d} \sum_{j=1}^d \left\| \frac{\partial P}{\partial x_j} \right\|_1.$$

(ii) We have

$$(2.10) \quad \|P\|_\infty \leq \|P\|_{\dot{W}_1^d}.$$

PROOF. Inequality (2.9) is given in [57, pp. 129–130]. To prove (2.10), we use the equality

$$P(x) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_d} \frac{\partial^d}{\partial x_1 \dots \partial x_d} P(t_1, \dots, t_d) dt_1 \dots dt_d$$

noting that by Theorem 3.2.5 in [46] we have $\lim_{|x| \rightarrow \infty} P(x) = 0$. $\square$

Now, we obtain the sharp version of the Hardy-Littlewood-Nikolskii inequality in the case $\alpha + \gamma \in \mathbb{N}$ and $\gamma \geq 1$, cf. Lemma 6.

LEMMA 8. Let $0 < p \leq 1$, $1 < q \leq \infty$, $d \geq 2$, $\alpha > 0$, $\gamma = d(1 - 1/q) \geq 1$, and $\alpha + \gamma \in \mathbb{N}$. Then, for any $P_\sigma \in \mathcal{B}_{\sigma,p}$, we have

$$(2.11) \quad \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_q \lesssim \sigma^{d(\frac{1}{p}-1)} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p.$$

PROOF. By Nikol'skii's inequality, we have

$$\|P_\sigma\|_{\dot{W}_1^{\alpha+\gamma}} \lesssim \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_1 \lesssim \sigma^{d(\frac{1}{p}-1)} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p$$

and, therefore, by Corollary 5,

$$(2.12) \quad \frac{\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_q}{\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p} \lesssim \sigma^{d(\frac{1}{p}-1)} \begin{cases} \frac{\|(-\Delta)^{\alpha/2} P_\sigma\|_q}{\|P_\sigma\|_{\dot{W}_1^{\alpha+\gamma}}}, & 1 < q < \infty; \\ \frac{\|P_\sigma\|_{\dot{W}_q^\alpha}}{\|P_\sigma\|_{\dot{W}_1^{\alpha+\gamma}}}, & q = \infty. \end{cases}$$

First, we consider the case $1 < q < \infty$. Choose $q^* \in (1, q]$ such that

$$\gamma - 1 = d \left(\frac{1}{q^*} - \frac{1}{q} \right).$$

Then, using the Hardy-Littlewood inequality for fractional integrals and Lemma 7 (i), we have

$$\begin{aligned} \|(-\Delta)^{\alpha/2} P_\sigma\|_q &\lesssim \|(-\Delta)^{(\alpha+\gamma-1)/2} P_\sigma\|_{q^*} \lesssim \|P_\sigma\|_{\dot{W}_{q^*}^{\alpha+\gamma-1}} \\ &\lesssim \|P_\sigma\|_{\dot{W}_1^{\alpha+\gamma}}, \end{aligned}$$

which together with (2.12) implies (2.11).

To obtain (2.11) in the case $q = \infty$, it is enough to note that by (2.10), we have

$$\|P_\sigma\|_{\dot{W}_\infty^\alpha} \lesssim \|P_\sigma\|_{\dot{W}_1^{\alpha+d}},$$

which completes the proof. $\square$

LEMMA 9. Let $1 < p < \infty$, $\alpha > 0$, $\gamma = d/p$, and $\sigma \geq 1$. Then, for any $P_\sigma \in \mathcal{B}_{\sigma,p}$, we have

$$(2.13) \quad \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_\infty \lesssim \ln^{\frac{1}{p'}}(\sigma + 1) \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p + \|P_\sigma\|_p.$$

PROOF. The proof is based on the Brézis–Wainger-type inequality (see [4]), which states that for every $f \in H_q^l(\mathbb{R}^d)$, $1 \leq q \leq \infty$, $1 < p < \infty$, and $l > d/q$, one has

$$(2.14) \quad \|f\|_\infty \lesssim \left(1 + \ln^{\frac{1}{p'}}(1 + \|f\|_{H_q^l}) \right)$$

provided that $\|f\|_{H_p^k} \leq 1$ with $k = d/p$. Here, $H_p^s(\mathbb{R}^d)$ is the fractional Sobolev space, i.e., $\|f\|_{H_p^s} = \|(I - \Delta)^{s/2} f\|_p < \infty$.

We use inequality (2.14) with $f = \frac{P_\sigma}{\|P_\sigma\|_{H_p^\gamma}}$, $\gamma = k = d/p$, $q = p$, and $l = d/p + 1$. Then the Bernstein inequality implies that

$$\|P_\sigma\|_\infty \lesssim \left(1 + \ln^{\frac{1}{p'}} \left(1 + \frac{\|P_\sigma\|_{H_p^{\gamma+1}}}{\|P_\sigma\|_{H_p^\gamma}} \right) \right) \|P_\sigma\|_{H_p^\gamma} \lesssim \ln^{\frac{1}{p'}}(1 + \sigma) \|P_\sigma\|_{H_p^\gamma}.$$

Next, taking into account that $\|P_\sigma\|_{H_p^\gamma} \asymp \|(-\Delta)^{\gamma/2} P_\sigma\|_p + \|P_\sigma\|_p$, Corollary 5, and the Bernstein inequality, we derive

$$(2.15) \quad \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_\infty \lesssim \ln^{\frac{1}{p'}} (1 + \sigma) \left(\|(-\Delta)^{(\gamma+\alpha)/2} P_\sigma\|_p + \|(-\Delta)^{\alpha/2} P_\sigma\|_p \right) \\ \lesssim \ln^{\frac{1}{p'}} (1 + \sigma) \left(\|(-\Delta)^{(\gamma+\alpha)/2} P_\sigma\|_p + \sigma^\alpha \|P_\sigma\|_p \right).$$

Finally, using the substitution $P_\sigma(\cdot) \rightarrow P_\sigma(\varepsilon \cdot)$ with $\varepsilon = \sigma^{p\alpha+\delta}$, $\delta > 0$, by homogeneity we see that inequality (2.15) can be written in the following form:

$$\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_\infty \lesssim \ln^{\frac{1}{p'}} (1 + \sigma) \|(-\Delta)^{(\gamma+\alpha)/2} P_\sigma\|_p + \|P_\sigma\|_p,$$

which by Corollary 5 implies (2.13). $\square$

3. Proofs of Properties 1–17

Before presenting the proofs of Properties 1–17, we discuss their relationship. First, we would like to mention that in the proofs of some results (namely, Properties 2, 6, 9, and 16) we use the equivalence of moduli of smoothness and realizations of K -functionals given in Property 17. Second, to prove Properties 7, 13, 15, and 17, the Jackson inequality given in Property 12 is applied. At the same time, the proof of Property 12 is independent on the mentioned above Properties 2, 6, 7, 9, 13, 15, 16, and 17. The proof of Property 17 is based on the Stechkin-Nikolskii inequality given in Corollary 2, and Properties 1 and 12.

Proof of Property 1. The proof of statements (a) (b), and (c) is standard. For details see also [6, 54, 60]. The proof of (d) repeats the proof of (3.6) in [44].

Proof of Property 2. To prove (1.5), we follow the ideas from [52, p. 194] and [36, Theorem 4.3]. We will consider only the case $0 < p < 1$. The case $1 \leq p \leq \infty$ follows from (1.36).

Denote $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{R}^d$, $\delta_j > 0$, $j = 1, \dots, d$, and

$$K_\delta(x) = \widehat{\varphi_\delta}(x),$$

where $\varphi_\delta(t) = v((\delta, t))$, $v \in C^\infty(\mathbb{R})$, $v(s) = 1$ for $|s| \leq 1/2$ and $v(s) = 0$ for $|s| > 1$.

We can assume that $\lambda > 1$. Fix $h > 0$ and suppose that $|\delta| \leq h$ and $P \in \mathcal{B}_{1/h, p}$. Using Property 1 and Lemma 1, we obtain

$$(3.1) \quad \begin{aligned} \|\Delta_{\lambda\delta}^\alpha f\|_p^p &\lesssim \|f - K_{\lambda\delta} * P\|_p^p + \|\Delta_{\lambda\delta}^\alpha (K_{\lambda\delta} * P)\|_p^p \\ &\lesssim \|f - P\|_p^p + \|P - K_{\lambda\delta} * P\|_p^p + \|D_{\lambda\delta}^\alpha (K_{\lambda\delta} * P)\|_p^p \\ &= \|f - P\|_p^p + \|P - K_{\lambda\delta} * P\|_p^p + \lambda^{\alpha p} \|K_{\lambda\delta} * D_\delta^\alpha P\|_p^p. \end{aligned}$$

Now, we will show that

$$(3.2) \quad \|P - K_{\lambda\delta} * P\|_p \lesssim \lambda^{\alpha+d(\frac{1}{p}-1)} \|D_\delta^\alpha P\|_p$$

and

$$(3.3) \quad \|K_{\lambda\delta} * D_\delta^\alpha P\|_p \lesssim \lambda^{d(\frac{1}{p}-1)} \|D_\delta^\alpha P\|_p.$$

It is easy to see that (3.2) is equivalent to the following inequality

$$\|A_{\lambda\delta} * P\|_p \lesssim \lambda^{d(\frac{1}{p}-1)} \|P\|_p \quad \text{for all } P \in \mathcal{B}_{1/h,p},$$

where

$$A_\delta(x) = \widehat{\psi_\delta}(x)$$

and

$$\psi_\delta(t) = \frac{(1 - \varphi_\delta(t))}{(it, \delta)^\alpha} v \left(2^{-1} \sqrt{(t_1 \delta_1)^2 + \cdots + (t_d \delta_d)^2} \right).$$

Using the properties of the convolution algebras in $L_p(\mathbb{R}^d)$, $0 < p < 1$, see, e.g., [59, p. 220], and denoting $\mathbf{1} = (1, \dots, 1)$, we have

$$\begin{aligned} \|A_{\lambda\delta} * P\|_p &\lesssim \left(\prod_{j=1}^d \delta_j \right)^{1-\frac{1}{p}} \|A_{\lambda\delta}\|_p \|P\|_p = \|\widehat{\psi_{\lambda\mathbf{1}}}\|_p \|P\|_p \\ &= \lambda^{d(\frac{1}{p}-1)} \|\widehat{\psi_{\mathbf{1}}}\|_p \|P\|_p \lesssim \lambda^{d(\frac{1}{p}-1)} \|P\|_p, \end{aligned}$$

that is, (3.2) is verified. Inequality (3.3) can be obtained similarly.

Combining (3.1)–(3.3), we obtain

$$\begin{aligned} \|\Delta_{\lambda\delta}^\alpha f\|_p &\lesssim \|f - P\|_p + \lambda^{\alpha+d(\frac{1}{p}-1)} \|D_\delta^\alpha P\|_p \\ &\lesssim \lambda^{\alpha+d(\frac{1}{p}-1)} \left(\|f - P\|_p + |\delta|^\alpha \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P\|_p \right). \end{aligned}$$

Finally, taking infimum over all $P \in \mathcal{B}_{1/h,p}$ and using (1.37), we get

$$\|\Delta_{\lambda\delta}^\alpha f\|_p \lesssim \lambda^{\alpha+d(\frac{1}{p}-1)} \mathcal{R}_\alpha(f, h)_p \lesssim \lambda^{\alpha+d(\frac{1}{p}-1)} \omega_\alpha(f, h)_p,$$

which yields (1.5). $\square$

Property 2 immediately implies the following result (see also [50] and [62]).

REMARK 4. Let $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then there exists a function φ such that $\varphi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, $\varphi(\delta)$ is nondecreasing, $\varphi(\delta)\delta^{-\alpha-d(1/p-1)_+}$ is nonincreasing, and

$$\varphi(\delta) \asymp \omega_\alpha(f, \delta)_p, \quad \delta > 0.$$

PROOF. Let us define $\varphi(t) := t^\beta \inf_{0 < \xi \leq t} \{\xi^{-\beta} \omega_\alpha(f, \xi)_p\}$ with $\beta = \alpha + d(1/p - 1)_+$. It is not difficult to see that φ satisfies the required properties. Clearly, $\varphi(\delta) \leq \omega_\alpha(f, \delta)_p$. At the same time by (1.6) we have

$$\omega_\alpha(f, t)_p \lesssim t^\beta \inf_{0 < \xi \leq t} \{\xi^{-\beta} \omega_\alpha(f, \xi)_p\} = \varphi(t).$$

$\square$

Proof of Property 3. In the case $1 \leq p \leq \infty$, the proof of (1.7) is given in [2, Lemma 4.11, p. 338] and is based on the representation of the total difference of a function via the sum of the mixed differences (Kemperman's formula). Following the same proof, we arrive at equivalence (1.7) in the case $0 < p < 1$ as well.

The proof of (1.8) can be obtained repeating the proof of Theorem 8 in [65]. $\square$

Proof of Property 4. The proof of (1.10) follows from the equivalence of the K -functional corresponding to the couple $(L_p, \dot{W}_p^r)$ and the modulus $\omega_r(f, \delta)_p$ and the closedness of the unit ball of $\dot{W}_p^r(\mathbb{R}^d)$ in $L_p(\mathbb{R}^d)$, see [39] and [8].

Proof of Property 5. The proof of (1.11) follows by applying the Hölder inequality and the following equality, which can be easily verified by induction:

$$\Delta_h^r(fg) = \sum_{k=0}^r \binom{r}{k} \Delta_h^k f(x) \Delta_h^{r-k} g(x + kh).$$

Proof of Property 6. In the case $\alpha \in \mathbb{N}$, the equivalence in (1.12) can be proved by standard way using the equality

$$(3.4) \quad \Delta_h^r f(x) = \sum_{k=1}^r (-1)^k \binom{r}{k} [\Delta_{kt}^r f(x + kh) - \Delta_{h+kt}^r f(x)],$$

which holds for any $t, h \in \mathbb{R}^d$, see, e.g. [11, Ch. 6, § 5].

Consider the equivalence in (1.13). We give the detailed proof of the estimate

$$(3.5) \quad \omega_r(f, \delta)_p \lesssim \left\| \left(\delta^{-d} \int_{|t| \leq \delta} |\Delta_t^r(\cdot)|^q dt \right)^{1/q} \right\|_p.$$

The inverse estimate is clear by Minkowski's inequality under the condition $q \leq p$.

Integrating (3.4), we have for any positive q and $\delta \geq |h|$ that

$$(3.6) \quad |\Delta_h^r f(x)|^q \lesssim \sum_{k=1}^r \delta^{-d} \int_{|t| \leq \delta} (|\Delta_{kt}^r f(x + kh)|^q + |\Delta_{h+kt}^r f(x)|^q) dt.$$

Therefore,

$$(3.7) \quad \begin{aligned} \|\Delta_h^r f\|_p &\lesssim \sum_{k=1}^r \left\| \left(\delta^{-d} \int_{|t| \leq \delta} |\Delta_{kt}^r f(x + kh)|^q dt \right)^{1/q} \right\|_p \\ &\quad + \sum_{k=1}^r \left\| \left(\delta^{-d} \int_{|t| \leq \delta} |\Delta_{h+kt}^r f(x)|^q dt \right)^{1/q} \right\|_p \\ &\lesssim \left\| \left(\delta^{-d} \int_{|t| \leq (r+1)\delta} |\Delta_t^r f(x)|^q dt \right)^{1/q} \right\|_p. \end{aligned}$$

Using Property 2, it is not difficult to see that (3.7) implies (3.5).

Now, we prove (1.14) for fractional moduli of smoothness. It is clear that it suffices to prove the estimate from above. Let $P \in \mathcal{B}_{\lambda_\alpha/\delta, p}$, where $\lambda_\alpha \in (0, 1)$ will be chosen later. By (1.35), we have

$$(3.8) \quad \omega_\alpha(f, \delta)_p \lesssim \|f - P\|_p + \delta^\alpha \|(-\Delta)^{\alpha/2} P\|_p.$$

Suppose that $\|f - P\|_p \lesssim E_{\lambda_\alpha/\delta}(f)_p$. Using Jackson's inequality (1.29), Property 2, and (1.12), we have for an integer $r > \alpha$ that

$$(3.9) \quad \begin{aligned} \|f - P\|_p &\lesssim \omega_r(f, \lambda_\alpha^{-1}\delta)_p \lesssim \omega_r(f, \delta)_p \\ &\lesssim \delta^{-d} \int_{|h| \leq \delta} \|\Delta_h^r f\|_p dh \lesssim \delta^{-d} \int_{|h| \leq \delta} \|\Delta_h^\alpha f\|_p dh. \end{aligned}$$

Let us show that

$$(3.10) \quad \delta^\alpha \|(-\Delta)^{\alpha/2} P\|_p \lesssim \left\| \delta^{-d} \int_{|h| \leq \delta} \Delta_h^\alpha P(\cdot - \alpha h) dh \right\|_p.$$

We will follow an idea from [69, 8.2.9], see also [32]. First, we note that (3.10) can be rewritten in the following equivalent form

$$\|\Lambda_{\delta, \alpha}(P)\|_p \lesssim \|P\|_p \quad \text{for any } P \in \mathcal{B}_{\lambda_\alpha/\delta, p},$$

where

$$\Lambda_{\delta, \alpha}(P)(x) = \mathcal{F}^{-1} \left(g(\delta \xi) \widehat{P}(\xi) \right) (x)$$

and

$$g(x) = \frac{|x|^{\alpha} v(\lambda_\alpha^{-1} x)}{\int_{|h| \leq 1} (1 - e^{i(h, x)})^\alpha dh},$$

$v \in C^\infty(\mathbb{R}^d)$, $v(x) = 1$ for $|x| \leq 1$ and $v(x) = 0$ for $|x| \geq 2$. Here and throughout to define the fractional exponents, we use the principal branch of the logarithm. Thus, to prove (3.10) it suffices to show that the function g is a Fourier multiplier in $L_p(\mathbb{R}^d)$. After some simple calculations, we derive that

$$(3.11) \quad \begin{aligned} \int_{|h| \leq 1} (1 - e^{i(h, x)})^\alpha dh &= c_d \int_0^1 r^{d-1} \int_{-1}^1 (1 - e^{iru|x|})^\alpha (1 - u^2)^{\frac{d-3}{2}} du dr \\ &= c_d 2^{\alpha+1} \int_0^1 r^{d-1} \int_0^1 \cos \frac{\alpha(ru|x| - \pi)}{2} \left(\sin \frac{ru|x|}{2} \right)^\alpha (1 - u^2)^{\frac{d-3}{2}} du dr, \end{aligned}$$

where $c_d = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$. Then, we can set $g(0) = \lim_{|x| \rightarrow 0} \frac{|x|^{\alpha} v(\lambda_\alpha^{-1} x)}{\int_{|h| \leq 1} (1 - e^{i(h, x)})^\alpha dh}$. It is also clear that we can always find positive λ_α such that the integral in (3.11) is strictly positive or negative for $0 < |x| \leq 2\lambda_\alpha$. Therefore, g is a smooth function on $\mathbb{R}^d \setminus \{0\}$ with a compact support, which implies that g is a Fourier multiplier in $L_p(\mathbb{R}^d)$ (see [25, 5.2.3]). This proves (3.10).

Next, combining (3.9) and (3.10) and using Minkowski's inequality, we derive

$$(3.12) \quad \begin{aligned} \delta^\alpha \|(-\Delta)^{\alpha/2} P\|_p &\lesssim \|f - P\|_p + \left\| \delta^{-d} \int_{|h| \leq \delta} \Delta_h^\alpha f(\cdot - \alpha h) dh \right\|_p \\ &\lesssim \delta^{-d} \int_{|h| \leq \delta} \|\Delta_h^\alpha f\|_p dh. \end{aligned}$$

Finally, collecting (3.8), (3.9), and (3.12), we prove the required estimate.

To prove (1.14) in the cases $d = 1$ and $\alpha > (1/p - 1)_+$, one should repeat step-by-step the proof of Theorem 2.1 from [33] with the help of the Jackson inequality (1.29)

and Lemma 1. Equivalence (1.15) was obtained in [69, 8.2.9] for periodic functions. The proof on $\mathbb{R}^d$ is similar using the fact that $\xi_j/|\xi|$ is the Fourier multiplier in $L_p(\mathbb{R}^d)$. $\square$

Proof of Property 7. The proof of (1.16) is a combination of the direct and inverse estimates (1.29) and (1.31) as well as the relation $\omega_\alpha(f, \delta)_p \asymp \omega_\alpha(f, 2\delta)_p$ (see (1.5)). Note that $E_0(f)_p \leq \|f\|_p$.

Inequality (1.18) follows from (1.16) by replacing $f \rightarrow f(\varepsilon \cdot)$ and $\delta \rightarrow \frac{\delta}{\varepsilon}$, changing the variables and letting $\varepsilon \rightarrow \infty$. On the other hand, (1.18) clearly implies (1.16) since $\omega_{\gamma+\alpha}(f, t)_p \lesssim \|f\|_p$. $\square$

Proof of Property 8. Inequality (1.19) follows from (1.4). Inequality (1.20) follows from [10] and (1.35). $\square$

Proof of Property 9. This property follows from Theorem 12 below taking $m = \gamma$. We start with the following lemma.

LEMMA 10. *Let $f \in L_p(\mathbb{R}^d)$, $0 < p < q \leq \infty$, $\alpha \in \mathbb{N} \cup ((1/q - 1)_+, \infty)$, and $\gamma, m > 0$ be such that $\alpha + \gamma, \alpha + m, m - \gamma \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$. Then, for all $\delta \in (0, 1)$, we have*

$$(3.13) \quad \omega_\alpha(f, \delta)_q \lesssim \delta^\alpha \left(\int_\delta^1 \left(\frac{\omega_{\alpha+\gamma}(f, t)_p}{t^{\alpha+d(\frac{1}{p}-\frac{1}{q})}} \right)^\theta \frac{dt}{t} + \|f\|_p^\theta \right)^{\frac{1}{\theta}} + \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}},$$

where

$$(3.14) \quad \theta = \begin{cases} \min(q, 2), & q < \infty; \\ 1, & q = \infty, \end{cases} \quad q_1 = \begin{cases} q, & q < \infty; \\ 1, & q = \infty. \end{cases}$$

PROOF. The proof is based on the Marchaud inequality given by (1.16) and the following Ulyanov type inequalities (see [17, Theorem 7.2])

$$(3.15) \quad \omega_{\alpha+m}(f, \delta)_q \lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}$$

and

$$(3.16) \quad \|f\|_q \lesssim \left(\int_0^1 \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \|f\|_p.$$

In the latter estimate, we have taken into account the fact that Jackson's inequality (1.29) holds for any $0 < p \leq \infty$. Using (1.16) and (3.15), we derive

$$\omega_\alpha(f, \delta)_q \lesssim \delta^\alpha \left(\int_\delta^1 u^{-\alpha\theta} \left(\int_0^u \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{\theta}{q_1}} \frac{du}{u} + \|f\|_q^\theta \right)^{\frac{1}{\theta}}.$$

It remains to apply the weighted Hardy inequality for averages (see, e.g., [2, p. 124]) and take into account that in light of (3.16), we have

$$\|f\|_q \lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \left(\delta^{\alpha\theta} \int_\delta^1 \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{\alpha+d(\frac{1}{p}-\frac{1}{q})}} \right)^\theta \frac{dt}{t} + \|f\|_p^\theta \right)^{\frac{1}{\theta}}.$$

□

COROLLARY 11. *Under the conditions of Lemma 10, we have*

$$(3.17) \quad \omega_\alpha(f, \delta)_q \lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \delta^\alpha \|f\|_p + \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^{\gamma+d(\frac{1}{p}-1)_+}} \begin{cases} 1, & \gamma > d\left(\frac{1}{p}-\frac{1}{q}\right) - d\left(\frac{1}{p}-1\right)_+; \\ \ln^{1/\theta}\left(\frac{1}{\delta}+1\right), & \gamma = d\left(\frac{1}{p}-\frac{1}{q}\right) - d\left(\frac{1}{p}-1\right)_+; \\ \left(\frac{1}{\delta}\right)^{d(\frac{1}{p}-\frac{1}{q})-d(\frac{1}{p}-1)_+-\gamma}, & \gamma < d\left(\frac{1}{p}-\frac{1}{q}\right) - d\left(\frac{1}{p}-1\right)_+. \end{cases}$$

The proof of the corollary immediately follows from inequality (3.13) and Property 2.

Now we present a slightly stronger version of the sharp Ulyanov inequality than the one given in (1.21).

THEOREM 12. *Let $f \in L_p(\mathbb{R}^d)$, $0 < p < q \leq \infty$, $\alpha \in \mathbb{N} \cup ((1-1/q)_+, \infty)$, and $\gamma, m \geq 0$ be such that $\alpha + \gamma, \alpha + m, m - \gamma \in \mathbb{N} \cup ((1/p-1)_+, \infty)$. Then, for any $\delta \in (0, 1)$, we have*

$$(3.18) \quad \omega_\alpha(f, \delta)_q \lesssim \delta^\alpha \|f\|_p + \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^\gamma} \eta\left(\frac{1}{\delta}\right) + \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}},$$

where the function η is defined in (1.22) and (1.23).

PROOF. The case $\gamma = 0$ is given in (3.15). If $\gamma > 0$, we devide the proof into two cases: $0 < p \leq 1$ and $p > 1$. First, consider the case $0 < p \leq 1$ and $p < q \leq \infty$. Then we apply inequality (3.17) in the following form:

$$\omega_\alpha(f, \delta)_q \lesssim \delta^\alpha \|f\|_p + \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^{\gamma+d(\frac{1}{p}-1)}} \begin{cases} 1, & \gamma > d\left(1-\frac{1}{q}\right); \\ \ln^{1/\theta}\left(\frac{1}{\delta}+1\right), & \gamma = d\left(1-\frac{1}{q}\right); \\ \left(\frac{1}{\delta}\right)^{d(1-\frac{1}{q})-\gamma}, & 0 < \gamma < d\left(1-\frac{1}{q}\right). \end{cases}$$

Hence, (3.18) holds for the following parameters:

- (i) $\gamma > 0$ and $0 < q \leq 1$;
- (ii) $\gamma = d(1-1/q)$ and $1 < q \leq 2$ with $\alpha + \gamma \notin \mathbb{N}$;

- (iii) $\gamma = d(1 - 1/q)$ and $q = \infty$ with $\alpha + \gamma \notin \mathbb{N}$;
- (iv) $0 < \gamma < d(1 - 1/q)_+$.

Thus, in the case $0 < p \leq 1$ it remains to consider:

(v) $\gamma = d(1 - 1/q) \geq 1$, $\alpha + \gamma \notin \mathbb{N}$, and $2 < q < \infty$ (note that for such q and $d \geq 2$ we always have $d(1 - 1/q) \geq 1$);

(vi) $\gamma = d(1 - 1/q) \geq 1$, $1 < q \leq \infty$, and $\alpha + \gamma \in \mathbb{N}$;

(vii) $0 < \gamma = d\left(1 - \frac{1}{q}\right)_+ = 1$ and $d = 1$.

To obtain (3.18) in the cases (v)–(vii), we will use the Hardy-Littlewood-Nikolskii inequalities proved in Subsection 2.2 and the realizations of the K -functionals from Property 17.

Let $P_\sigma \in \mathcal{B}_{\sigma,p}$ be such that

$$(3.19) \quad \|f - P_\sigma\|_p \lesssim E_\sigma(f)_p.$$

From [17, Lemma 4.2] and Jackson's inequality (1.29), it follows that

$$(3.20) \quad \begin{aligned} \|f - P_{2^n}\|_q &\lesssim \left(\sum_{\nu=n}^{\infty} 2^{\nu q_1 d(\frac{1}{p} - \frac{1}{q})} \|f - P_{2^\nu}\|_p^{q_1} \right)^{\frac{1}{q_1}} \\ &\lesssim \left(\sum_{\nu=n}^{\infty} 2^{\nu q_1 d(\frac{1}{p} - \frac{1}{q})} \omega_{\alpha+m}(f, 2^{-\nu})_p^{q_1} \right)^{\frac{1}{q_1}}. \end{aligned}$$

Thus, taking into account realization (1.38) as well as (3.19) and (3.20), we obtain

$$(3.21) \quad \begin{aligned} \omega_\alpha(f, 2^{-n})_q &\lesssim \|f - P_{2^n}\|_q + 2^{-\alpha n} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_{2^n}\|_q \\ &\lesssim \left(\sum_{\nu=n}^{\infty} \left(2^{\nu d(\frac{1}{p} - \frac{1}{q})} \omega_{\alpha+m}(f, 2^{-\nu})_p \right)^{q_1} \right)^{\frac{1}{q_1}} + 2^{-\alpha n} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_{2^n}\|_q. \end{aligned}$$

Now, considering the case (v), we estimate the first summand in the last inequality using Lemma 6 and (1.38). We derive

$$(3.22) \quad \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_{2^n}\|_q \lesssim 2^{\alpha n} 2^{d(\frac{1}{p}-1)n} n^{\frac{1}{q}} \omega_{\alpha+\gamma}(f, 2^{-n})_p + \|P_{2^n}\|_q.$$

To estimate $\|P_{2^n}\|_q$, we use the same arguments as in the proof of Lemma 10 and Corollary 11. We have

$$\begin{aligned}
 \|P_{2^n}\|_q &\lesssim \|f\|_q + E_{2^n}(f)_q \lesssim \|f\|_q \\
 &\lesssim \left(\int_0^1 \left(\frac{\omega_{\alpha+\gamma}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \|f\|_p \\
 (3.23) \quad &\lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \left(\int_\delta^1 \left(\frac{\omega_{\alpha+\gamma}(f, t)_p}{t^{\alpha+d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} + \|f\|_p^{q_1} \right)^{\frac{1}{q_1}} \\
 &\lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \|f\|_p + \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^{\gamma+d(\frac{1}{p}-1)}} \ln^{\frac{1}{q_1}} \left(\frac{1}{\delta} + 1 \right).
 \end{aligned}$$

Thus, combining (3.21)–(3.23), we arrive at (3.18) in the case (v).

To consider the case (vi), we use similar arguments applying Lemma 8. Remark that in this case we do not need to estimate $\|f\|_q$.

In the case (vii), the required estimate follows from the equality

$$P_\sigma(x) = \int_{-\infty}^x P'_\sigma(t) dt.$$

Note that by the Nikolskii inequality (2.6), $P_\sigma \in L_1(\mathbb{R}^d)$ whenever $P_\sigma \in L_p(\mathbb{R}^d)$ and using [46, Theorem 3.2.5] we have $P(x) \rightarrow 0$ as $x \rightarrow \infty$. See also [47, p. 118].

Let $1 < p < q < \infty$. In this case the proof is similar. First, we use inequality (3.17) as follows

$$\begin{aligned}
 \omega_\alpha(f, \delta)_q &\lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+m}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} + \delta^\alpha \|f\|_p \\
 &\quad + \frac{\omega_{\alpha+\gamma}(f, \delta)_p}{\delta^\gamma} \begin{cases} 1, & \gamma > d\left(\frac{1}{p} - \frac{1}{q}\right); \\ \ln^{1/\theta} \left(\frac{1}{\delta} + 1\right), & \gamma = d\left(\frac{1}{p} - \frac{1}{q}\right); \\ \left(\frac{1}{\delta}\right)^{d(\frac{1}{p}-\frac{1}{q})-\gamma}, & 0 < \gamma < d\left(\frac{1}{p} - \frac{1}{q}\right). \end{cases}
 \end{aligned}$$

Hence, (3.18) holds in all cases except the case $\gamma = d(\frac{1}{p} - \frac{1}{q})$. In the latter case, if $q < \infty$, the Hardy-Littlewood inequality $\|f\|_q \lesssim \|(-\Delta)^{\gamma/2} f\|_p$ and (3.21) yield (3.18). Indeed, by Corollary 5, we have that

$$\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_\sigma\|_q \asymp \|(-\Delta)^{\alpha/2} P_\sigma\|_q$$

and

$$\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^{\alpha+\gamma} P_\sigma\|_p \asymp \|(-\Delta)^{(\alpha+\gamma)/2} P_\sigma\|_p.$$

It remains to apply (1.38).

Finally, to show (3.18) in the case $1 < p < q = \infty$ and $\gamma = d/p$, we note that by Lemma 9 and (1.38), we derive

$$\sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_{2^n}\|_\infty \lesssim 2^{(\alpha+\gamma)n} n^{\frac{1}{p'}} \omega_{\alpha+\gamma}(f, 2^{-n})_p + \|f\|_p.$$

This together with (3.21) implies (3.18). $\square$

COROLLARY 13. *Let $0 < p \leq 1 < q \leq \infty$ and $d \geq 2$. We have*

$$\omega_\alpha(f, \delta)_q \lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+d(1-\frac{1}{q})}(f, t)_p}{t^{d(\frac{1}{p}-\frac{1}{q})}} \right)^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}}$$

provided that $\alpha + d(1 - 1/q) \in \mathbb{N}$. In particular, for any $d, \alpha \in \mathbb{N}$, we have

$$\omega_\alpha(f, \delta)_\infty \lesssim \int_0^\delta \frac{\omega_{\alpha+d}(f, t)_1}{t^d} \frac{dt}{t}.$$

Finally, we note that unlike the case of Lebesgue spaces, the sharp Ulyanov inequality in Hardy spaces has the same form both in quasi-Banach spaces ($0 < p < 1$) and Banach spaces ($p \geq 1$). Recall that the real Hardy spaces $H_p(\mathbb{R}^d)$, $0 < p < \infty$, is the class of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H_p} = \|f\|_{H_p(\mathbb{R}^d)} = \left\| \sup_{t>0} |\varphi_t * f(x)| \right\|_{L_p(\mathbb{R}^d)} < \infty,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\widehat{\varphi}(0) \neq 0$, and $\varphi_t(x) = t^{-d}\varphi(x/t)$ (see [58, Ch. III]). The moduli of smoothness in $H_p(\mathbb{R}^d)$ are defined by $\omega_\alpha(f, \delta)_{H_p} := \sup_{|h|<\delta} \|\Delta_h^\alpha f\|_{H_p}$, where the fractional difference $\Delta_h^\alpha f$ is given by (1.3). Using boundedness properties of the fractional integrals in the Hardy spaces, the following sharp Ulyanov inequality in $H_p(\mathbb{R}^d)$ was derived in [36].

THEOREM 14. *Let $f \in H_p(\mathbb{R}^d)$, $0 < p < q < \infty$, $\alpha \in \mathbb{N} \cup ((1/q - 1)_+, \infty)$ and $\alpha + \theta \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, and $\theta = d(1/p - 1/q)$. Then, for any $\delta \in (0, 1)$, we have*

$$\omega_\alpha(f, \delta)_{H_q} \lesssim \left(\int_0^\delta \left(\frac{\omega_{\alpha+\theta}(f, t)_{H_p}}{t^\theta} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Proof of Property 10. See [67, Theorem 2.6].

The following generalization of (1.25) for Hardy spaces has been recently obtained in [36, Theorem 15].

THEOREM 15. *Let $f \in H_p(\mathbb{R}^d)$, $0 < p < q < \infty$, $\theta = d(1/p - 1/q)$, and $\alpha \in \mathbb{N} \cup ((1/p - 1)_+, \infty)$, $\alpha > \theta$. Then*

$$\delta^{\alpha-\theta} \left(\int_\delta^\infty \left(\frac{\omega_\alpha(f, t)_{H_q}}{t^{\alpha-\theta}} \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \lesssim \left(\int_0^\delta \left(\frac{\omega_\alpha(f, t)_{H_p}}{t^\theta} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Proof of Property 11. See [19, Theorem 12.2] for the proof of inequality (1.28) and [67, Theorem 2.3] for inequality (1.27).

Proof of Property 12. The proof of the Jackson inequality (1.29) in the multi-dimensional case is based on the following lemma.

LEMMA 16. (SEE [5].) *Let $f \in L_p(\mathbb{R})$, $0 < p < 1$, $r \in \mathbb{N}$, and $\sigma > 0$. Then*

$$\frac{1}{2\sigma} \int_{[-\sigma, \sigma]} \|f - V_{\sigma, \lambda}(f)\|_{L_p(\mathbb{R})}^p d\lambda \lesssim \omega_{2r+1}(f, \sigma^{-1})_{L_p(\mathbb{R})}^p$$

and

$$\frac{1}{2\sigma} \int_{[-\sigma, \sigma]} \|V_{\sigma, \lambda}(f)\|_{L_p(\mathbb{R})}^p d\lambda \lesssim \|f\|_{L_p(\mathbb{R})}^p,$$

where

$$V_{\sigma, \lambda}(f)(x) = 2\pi \sum_{k \in \mathbb{Z}} f\left(\frac{k}{\sigma} + \lambda\right) (\mathcal{F}\varphi)(\sigma(x - \lambda) - k), \quad \varphi(\xi) = (1 + i\xi^{2r+1})\eta(\xi),$$

and $\eta \in C^\infty(\mathbb{R})$, $\eta(\xi) = 1$ for $|\xi| \leq 1/2$, and $\eta(\xi) = 0$ for $|\xi| \geq 1$.

PROOF OF INEQUALITY (1.29). We consider only the case $0 < p < 1$. The proof of the inequality in the case $p \geq 1$ can be found, e.g., in [64, 5.3.2].

First, we consider the case $\alpha = r \in \mathbb{N}$. Denote

$$V_{\sigma, \lambda, j}(f)(x) = 2\pi \sum_{k \in \mathbb{Z}} f\left(x_1, \dots, x_{j-1}, \frac{k}{\sigma} + \lambda, x_{j+1}, \dots, x_d\right) (\mathcal{F}\varphi)(\sigma(x_j - \lambda) - k).$$

Then, we have

$$E_{d^{-\frac{1}{2}\sigma}}(f)_p^p \leq \frac{1}{(2\sigma)^d} \int_{[-\sigma, \sigma]^d} \|f - V_{\sigma, \lambda_1, 1} \circ \dots \circ V_{\sigma, \lambda_d, d}(f)\|_p^p d\lambda_1 \dots d\lambda_d.$$

Using the equality

$$\begin{aligned} f - V_{\sigma, \lambda_1, 1} \circ \dots \circ V_{\sigma, \lambda_d, d}(f) &= f - V_{\sigma, \lambda_1, 1}(f) \\ &\quad + V_{\sigma, \lambda_1, 1}(f - V_{\sigma, \lambda_2, 2}(f)) + \dots + V_{\sigma, \lambda_1, 1} \circ \dots \circ V_{\sigma, \lambda_{d-1}, d-1}(f - V_{\sigma, \lambda_d, d}(f)), \end{aligned}$$

Lemma 16, and Fubini's theorem, we derive

$$\begin{aligned} &\frac{1}{(2\sigma)^d} \int_{[-\sigma, \sigma]^d} \|f - V_{\sigma, \lambda_1, 1} \circ \dots \circ V_{\sigma, \lambda_d, d}(f)\|_p^p d\lambda_1 \dots d\lambda_d \\ &\leq \frac{1}{2\sigma} \int_{[-\sigma, \sigma]} \|f - V_{\sigma, \lambda_1, 1}(f)\|_p^p d\lambda_1 + \frac{1}{(2\sigma)^2} \int_{[-\sigma, \sigma]^2} \|V_{\sigma, \lambda_1, 1}(f - V_{\sigma, \lambda_2, 2}(f))\|_p^p d\lambda_1 d\lambda_2 \\ &\quad + \dots + \frac{1}{(2\sigma)^d} \int_{[-\sigma, \sigma]^d} \|V_{\sigma, \lambda_1, 1} \circ \dots \circ V_{\sigma, \lambda_{d-1}, d-1}(f - V_{\sigma, \lambda_d, d}(f))\|_p^p d\lambda_1 \dots d\lambda_d \\ &\lesssim \sum_{j=1}^d \frac{1}{2\sigma} \int_{[-\sigma, \sigma]} \|f - V_{\sigma, \lambda_j, j}(f)\|_p^p d\lambda_j \lesssim \sum_{j=1}^d \omega_{2r+1}^{(j)}(f, \sigma^{-1})_p^p, \end{aligned}$$

where $\omega_{2r+1}^{(j)}(f, \sigma^{-1})_p$ is the partial modulus of smoothness given in (1.9).

Thus, using the fact that $\omega_{2r+1}^{(j)}(f, d^{1/2}\delta)_p \lesssim \omega_{2r+1}^{(j)}(f, \delta)_p$, see, e.g., [11, p. 370], and relations (1.19) and (1.7), we have

$$E_\sigma(f)_p \lesssim \sum_{j=1}^d \omega_{2r+1}^{(j)}(f, d^{\frac{1}{2}}\sigma^{-1})_p \lesssim \sum_{j=1}^d \omega_r^{(j)}(f, \sigma^{-1})_p \lesssim \omega_r(f, \sigma^{-1})_p.$$

Finally, using the above estimate for $r = \alpha + k \in \mathbb{N}$ with $k > (1/p - 1)_+$ and Property 8, we get

$$E_\sigma(f)_p \lesssim \omega_{\alpha+k}(f, \sigma^{-1})_p \lesssim \omega_\alpha(f, \sigma^{-1})_p.$$

□

Proof of Property 13. The proof of (1.31) follows the standard argument using the Littlewood-Paley decomposition in the case $1 < p < \infty$, see [10], and telescoping sums in the cases $0 < p \leq 1$ and $p = \infty$, see, e.g., [36]. Since we deal with fractional moduli of smoothness, we also need the inequality $\omega_\alpha(P_\sigma, \delta)_p \lesssim \sigma^\alpha \|P_\sigma\|_p$, $0 < \delta \leq \pi/\sigma$, which follows from (2.3) and (2.4). □

Proof of Property 14. In the case $0 < p < 1$, the property can be proved repeating step-by-step the proof of Theorem 1 from [31] using also inequalities (1.5), (1.29), and (1.31). For the case $p \geq 1$ see [24, Theorem 8.2]. □

Proof of Property 15. In the case $0 < p < 1$, the proof of the first inequality in (1.32) easily follows from (1.29) and (2.1):

$$2^{-n\alpha} \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha P_{2^n}\|_p \lesssim \omega_\alpha(P_{2^n}, 2^{-n})_p \lesssim \|f - P_{2^n}\|_p + \omega_\alpha(f, 2^{-n})_p \lesssim \omega_\alpha(f, 2^{-n})_p.$$

To obtain the second inequality in (1.32), one needs to apply [37, Theorem 2.1], taking into account inequalities (1.29) and (2.1).

For the case $1 < p < \infty$ see [37, Theorem 6.1]. □

Proof of Property 16. The estimate $\gtrsim$ follows from the realization result given in (1.37). To obtain the estimate $\lesssim$, we use the fact that the function

$$m(\xi) = \left(\frac{1 - e^{i(\xi_1 + \dots + \xi_d)}}{\xi_1 + \dots + \xi_d} \right)^\alpha$$

is a Fourier multiplier in $L_p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$, see, e.g., [71, Proposition 1]. Then by standard arguments, we derive for any function $g \in L_p(\mathbb{R}^d)$

$$\omega_\alpha(f, \delta)_p \lesssim \omega_\alpha(f - g, \delta)_p + \omega_\alpha(g, \delta)_p \lesssim \|f - g\|_p + \delta^\alpha \sup_{|\zeta|=1, \zeta \in \mathbb{R}^d} \|D_\zeta^\alpha g\|_p.$$

Taking the infimum over all g , we obtain the estimate $\lesssim$ in (1.36). □

Proof of Property 17. First, let us prove the estimate from above in (1.37). Let $P_\sigma \in \mathcal{B}_{\sigma,p}$, $\delta \in (0, 1/\sigma)$. By Corollary 2, we estimate

$$\begin{aligned}\omega_\alpha(f, \delta)_p &\lesssim \|f - P_\sigma\|_p + \omega_\alpha(P_\sigma, \delta)_p \\ &\lesssim \|f - P_\sigma\|_p + \delta^\alpha \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha P_\sigma\|_p.\end{aligned}$$

It remains to take infimum over all $P_\sigma \in \mathcal{B}_{\sigma,p}$.

Now let $P_\sigma \in \mathcal{B}_{\sigma,p}$ be such that $\|f - P_\sigma\|_p \lesssim E_\sigma(f)_p$. Then by the Jackson inequality (1.29) and the Nikolskii-Stechkin inequality (2.3), we obtain

$$\begin{aligned}\omega_\alpha(f, \sigma^{-1})_p &\lesssim \mathcal{R}_\alpha(f, \sigma^{-1})_p \lesssim \|f - P_\sigma\|_p + \sigma^{-\alpha} \sup_{\zeta \in \mathbb{R}^d, |\zeta|=1} \|D_\zeta^\alpha P_\sigma\|_p \\ &\lesssim \omega_\alpha(f, \sigma^{-1})_p + \omega_\alpha(P_\sigma, \sigma^{-1})_p \lesssim \omega_\alpha(f, \sigma^{-1})_p,\end{aligned}$$

that is, (1.37) and (1.38) follow.

Similarly, taking into account Corollary 4, we can prove (1.39). $\square$

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