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Weighted Fourier inequalities in Lebesgue and Lorentz spaces

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ABSTRACT. In this paper, we obtain sufficient conditions for the weighted Fourier-type transforms to be bounded in Lebesgue and Lorentz spaces. Two types of results are discussed. First, we review the method based on rearrangement inequalities and the corresponding Hardy's inequalities. Second, we present Hörmander-type conditions on weights so that Fourier-type integral operators are bounded in Lebesgue and Lorentz spaces. Both restricted weak- and strong-type results are obtained. In the case of regular weights necessary and sufficient conditions are given.

1. Introduction

We will let $\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{ix\xi} f(x) dx$, $\xi \in \mathbb{R}^n$, be the Fourier transform in $L_1(\mathbb{R}^n)$, and $\|\cdot\|_p$ be the norm in $L_p(\mathbb{R}^n)$. Let ν and ω be nonnegative weights. A weighted Fourier inequality states that $\|\nu\widehat{f}\|_q \leq C\|\omega f\|_p$, that is,

$$(1.1) \quad \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)\nu(\xi)|^q d\xi \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx \right)^{\frac{1}{p}}.$$

Throughout this paper, $F \lesssim G$ means that $F \leq CG$; by C we denote positive constants depending only on non-essential parameters that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G \lesssim F$.

The standard method to obtain sufficient conditions for (1.1) is to use the Hardy–Littlewood–Pólya inequality, which claims that $\int_{\mathbb{R}^n} fg \leq \int_0^\infty f^* g^*$. Here h^* is the decreasing rearrangements of h , i.e.,

$$h^*(t) = \inf\{\sigma : m(\sigma, h) \leq t\}, \quad \text{where} \quad m(\sigma, f) = |\{x \in \mathbb{R}^n : |f(x)| > \sigma\}|.$$

In more detail, denote

$$(1.2) \quad \mu(x) = \frac{1}{\omega(x)}.$$

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Then by the Hardy–Littlewood–Pólya inequality the estimate

$$(1.3) \quad \left(\int_0^\infty (\widehat{f})^*(t)^q \nu^*(t)^q dt \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^*(t)^p \frac{1}{\mu^*(t)^p} dt \right)^{\frac{1}{p}}$$

implies (1.1) for any $0 < p, q < \infty$, with the same constant C . In its turn, the usual way to obtain sufficient conditions for (1.3) to hold is to apply Hardy's inequalities and the following Calderón-type rearrangement inequality by Jodeit and Torchinsky [**JT**]:

$$(1.4) \quad \int_0^s (\widehat{f})^*(t)^2 dt \leq C \int_0^s \left(\int_0^{1/t} f^*(t) dt \right)^2 ds.$$

More explicitly, some sufficient conditions for (1.3) can be described as a collection of those couples ν^* and μ^* such that the inequality

$$(1.5) \quad \left(\int_0^\infty \left(\frac{\nu^*(1/t)}{t^{2/q}} \int_0^t \mu^*(s) f(s) ds \right)^q dt \right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty f(t)^p dt \right)^{\frac{1}{p}}$$

holds for any non-negative f in either of the following cases: (i) $1 < p \leq q < \infty$, (ii) $2 \leq q < p < \infty$, (iii) $1 < q < p \leq 2$; see [**BH**, **RS**].

In particular, if $1 < p \leq q < \infty$ the sufficient condition for inequality (1.1) to hold is given by

$$(1.6) \quad \sup_{s>0} \left(\int_0^s \nu^*(t)^q dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{1}{s}} \mu^*(t)^{p'} dt \right)^{\frac{1}{p'}} < \infty.$$

This condition was obtained already in the 1980s by Heinig [**He1**], Jurkat–Sampson [**JS**], and Muckenhoupt [**Mu1**, **Mu2**].

The case $q < 2 < p$ requires some additional assumptions. Recently, Rastegari and Sinnamon [**RS**] obtained that if $1 < p < \infty$, $0 < q < p$, condition (1.5) holds for any non-negative f and any one of the following conditions holds, then so does inequality (1.1):

$$(1.7) \quad x^{q/2} \int_x^\infty \nu^*(t)^q \frac{dt}{t^{q/2}} \leq C \left(\int_0^x \nu^*(t)^q dt + x^{\max(1,q)} \int_x^\infty \nu^*(t)^q \frac{dt}{t^{\max(1,q)}} \right), \quad x > 0,$$

$$(1.8) \quad \nu^*(t)^q \in B_{q/2},$$

$$(1.9) \quad q > 1 \quad \text{and} \quad \nu^*(t)^q t^{2-q} \quad \text{is a decreasing function of } t,$$

$$(1.10) \quad q > 1 \quad \text{and} \quad x^{p'/2} \int_x^\infty \mu^*(t)^{p'} \frac{dt}{t^{p'/2}} \leq C \left(\int_0^x \mu^*(t)^{p'} dt + x^{p'} \int_x^\infty \mu^*(t)^{p'} \frac{dt}{t^{p'}} \right), \quad x > 0,$$

$$(1.11) \quad q > 1 \quad \text{and} \quad \mu^*(t)^{p'} \in B_{p'/2},$$

$$(1.12) \quad q > 1 \quad \text{and} \quad \mu^*(t)^{p'} t^{2-p'} \quad \text{is a decreasing function of } t.$$

Recall that a non-negative function h , defined on \mathbb{R}_+ , satisfies the B_s condition (written $h \in B_s$) if, for some positive constant C ,

$$x^s \int_x^\infty h(t) \frac{dt}{t^s} \leq C \int_0^x h(t) dt, \quad x > 0.$$

Let $0 < p < \infty$. The Lorentz space $L_{p,q} = L_{p,q}(\mathbb{R}^n)$ is defined [BeSh, Ch. 4] by those measurable functions f such that

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty, \quad 0 < q < \infty,$$

with the standard modification for $q = \infty$. Denote $\|f\|_{L_p(\omega)} := \left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx \right)^{1/p}$.

We recall that a linear operator T is of strong type (p, q) (restricted weak type (p, q) , respectively) if $\|Tf\|_{L_q} \lesssim \|f\|_{L_p}$ (or $\|Tf\|_{L_{q,\infty}} \lesssim \|f\|_{L_{p,1}}$). Alternatively, the restricted weak type boundedness can be defined as in [Gr, Def. 1.4.22].

In this paper we address the following question. Let the kernel $K(x, y)$ be a complex-valued measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Let also the operator

$$(1.13) \quad Tf(y) := \int_{\mathbb{R}^n} K(x, y) f(x) dx$$

be of strong type $(1, \infty)$ and of restricted weak type $(2, 2)$. Under which conditions on weight functions μ and ν , the operator

$$(1.14) \quad Af(y) := \nu(y) \int_{\mathbb{R}^n} K(x, y) f(x) \mu(x) dx$$

is bounded in Lebesgue and Lorentz spaces?

Note that

$$A : L_p \rightarrow L_q \quad \text{if and only if} \quad T : L_p(\omega) \rightarrow L_q(\nu).$$

For the Fourier transform, i.e., $K(x, y) = e^{-ixy}$, the latter, taking into account (1.2), is equivalent to the weighted Fourier inequality (1.1). We also remark that if the operator A is bounded from $L_{p,\tau}$ to $L_{q,\tau}$ for $p \leq q$ and every positive τ , then it is of strong type (p, q) and, again, T is bounded from $L_p(\omega)$ to $L_q(\nu)$.

Our main goal is two-fold. Firstly, following similar ideas as in the previous study [BH, BHJ, GLT, He2, RS], we obtain (see Theorem 3.2 below) that sufficient conditions for boundedness of the operator A in Lebesgue spaces can be expressed in terms of Hardy's inequalities. In particular, we obtain that if, for any $1 < p < \infty$ and $0 < q \leq \infty$, (1.5) is valid for any non-negative function, then (1.1) holds provided that either

$$(1.15) \quad \nu^*(\lambda t) \lesssim \lambda^{-\alpha} \nu^*(t), \quad \lambda > 1, \quad \forall t > 0,$$

with some $\alpha > \frac{1}{q} - \frac{1}{2}$, or

$$(1.16) \quad \mu^*(\lambda t) \lesssim \lambda^{-\beta} \mu^*(t), \quad \lambda > 1, \quad \forall t > 0,$$

with some $\beta > \frac{1}{2} - \frac{1}{p}$. For example, if one of the following conditions is satisfied: (i) $1 < p \leq q \leq \infty$, (ii) $2 < q < p < \infty$, (iii) $1 < p < 2$, $0 < q < p$, then (1.15) and (1.16) always hold, cf. [BH]. If $q < 2 < p$, we will see that condition (1.15) is more restrictive than the condition $\nu^*(t)^q \in B_{q/2}$, but less restrictive than assuming that $\nu^*(t)^q t^{2-q}$ is a decreasing function (cf. (1.7)–(1.12)). A similar remark can be applied to condition (1.16).

We also obtain sufficient conditions on weights ν and w so that the Fourier transform is bounded in weighted Lorentz spaces, i.e., $\widehat{f} : \Lambda_q(\nu) \rightarrow \Lambda_p(w)$ for $0 < p, q \leq \infty$. See Theorem 3.4. These conditions are given in terms of Hardy's inequalities on the cone of monotone functions under the assumptions that $\nu(\lambda t) \lesssim \lambda^{-\alpha} \nu(t)$ with some $\alpha > 1/q - 1/2$ and $w(\lambda^{-1}t) \lesssim \lambda^{-\gamma} w(t)$ with some $\gamma > 1/2 - 1/p$.

Secondly, our objective is to give more tractable weight conditions for the operator A to be of restricted weak and strong (p, q) type. Namely, for $1 < p, q < \infty$ being either $p \leq 2$ or $q \geq 2$, we have

$$(1.17) \quad \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim \sup_{\xi \eta \leq 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^\eta \mu^*(t) dt.$$

Similar-looking necessary conditions are also given. See Theorem 4.1. Moreover, for radial monotone-type weights, inequality (1.17) becomes an equivalence (see Corollary 4.2 below).

If either $p \leq 2$ or $q \geq 2$, we derive in Theorem 6.1 that

$$(1.18) \quad \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s)$$

provided that either

$$(1.19) \quad \lambda^{-\alpha} \nu^*(t) \lesssim \nu^*(\lambda t), \quad \lambda > 1, \quad \forall t > 0,$$

for some $\alpha \leq \frac{1}{q'}$ or

$$(1.20) \quad \lambda^{-\beta} \mu^*(t) \lesssim \mu^*(\lambda t), \quad \lambda > 1, \quad \forall t > 0,$$

for some $\beta \leq \frac{1}{p'}$.

Some remarks are in order. (i) Besides the simple form of these sufficient conditions, another advantage of our result is that it demonstrates, in a transparent way, a strong relationship between the properties of weights and the range of parameters. In other words, the more regular the weights, the wider the range for p and q . (ii) It is worth mentioning that conditions (1.19) and (1.20) are reverse to (1.15) and (1.16). All these conditions can be given in term of the Matuszewska indices [BGT]. (iii) We also note that weak type inequalities for Fourier transforms have been recently studied in [BS].

Finally, we also obtain a similar result for the operator A to be bounded from $L_{p,\tau}$ to $L_{q,\tau}$. In more detail, assuming that $p < 2$ or $q > 2$, we claim that

$$(1.21) \quad \|A\|_{L_{p,\tau} \rightarrow L_{q,\tau}} \lesssim \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s), \quad \tau > 0,$$

provided that either (1.19) holds for some $\alpha < \frac{1}{q'}$ or (1.20) holds for some $\beta < \frac{1}{p'}$. We will see that conditions (1.19) and (1.20) cannot be improved. The condition on weights given by $\sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s) < \infty$ will be called the *Hörmander type* condition. This is because of the well-known result by Hörmander [Hö, Theorem 1.10 and Corollary 1.6] who proved that the one-weight inequality

$$(1.22) \quad \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi) \nu(\xi)|^q dx \right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 < p \leq q \leq p' < \infty,$$

holds provided

$$(1.23) \quad s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \lesssim 1.$$

See also the paper [AH]. The following result by Heinig and Sinnamon [HS] is closely related to Hörmander's theorem (for the one-dimensional version see [BHJ]).

THEOREM 1.1. [HS] *Suppose $w(x) = w_0(|x|)$ is a radial weight function on \mathbb{R}^n such that w_0 is nondecreasing on $(0, \infty)$. Let $1 < p \leq q \leq p' < \infty$, then*

$$(1.24) \quad \left(\int_{\mathbb{R}^n} \left(|\widehat{f}(\xi)| |\xi|^{-n(1/q-1/p')} w_0(1/|\xi|) \right)^q d\xi \right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}^n} |f(x) w(x)|^p dx \right)^{\frac{1}{p}}$$

holds, if and only if $w^p \in A_p$.

Recall that a nonnegative weight function w defined on \mathbb{R}^n belongs to the Muckenhoupt class A_p , $1 < p < \infty$, if there is a constant C such that for all n -balls $B \subset \mathbb{R}^n$ with volume $|B|$,

$$\left(\frac{1}{|B|} \int_B w(x) dx \right)^{\frac{1}{p}} \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{\frac{1}{p'}} \leq C < \infty.$$

A slightly more general result was obtained by Heinig [He2].

THEOREM 1.2. [He2] *Under the conditions of Theorem 1.1, one has*

$$(1.25) \quad \left(\int_{\mathbb{R}^n} \left(|\widehat{f}(\xi)| \varphi(\xi)^{1/q-1/p'} w_0(1/|\xi|) \right)^q d\xi \right)^{\frac{1}{q}} \lesssim \left(\int_{\mathbb{R}^n} |f(x)w(x)|^p dx \right)^{\frac{1}{p}}$$

provided that $w^p \in A_p$ and $\xi \varphi^*(\xi) \lesssim 1$.

For $w \equiv 1$ we arrive at Hörmander's result (1.23). We will see that if $w(x) = w_0(|x|)$ and w_0 is nondecreasing on $(0, \infty)$, the condition $w^p \in A_p$ is equivalent to condition (1.20) with $\beta < \frac{1}{p}$ (recall that $\mu = 1/\omega$ by (1.2)). Thus, our result (1.21) extends Theorem 1.2.

The paper is organized as follows. After useful auxiliary results in Section 2, we obtain in Section 3 sufficient conditions for the operator A to be bounded from L_p to L_q , $0 < p, q \leq \infty$, via Hardy's inequalities. We compare our results with those given in [RS]. Moreover, we study the boundedness of the operator T in the weighted Lorentz spaces, that is, $\|Tf\|_{\Lambda_q(\nu)} \lesssim \|f\|_{\Lambda_p(w)}$ for $0 < p, q \leq \infty$.

Section 4 provides sufficient and similar-looking necessary condition on weights (ν, μ) for the operator A to be of restricted weak type (p, q) . Section 5 contains interrelation between several known conditions on pairs of weights when at least one of them is so-called α -regular. In particular, we show that condition (1.6) can be equivalently written as the Hörmander-type condition $\sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s) < \infty$.

Applying the results of Sections 4 and 5, in Section 6 we obtain Hörmander-type conditions for the operator A to be bounded in the Lorentz spaces, namely (1.18) and (1.21). Moreover, for radial monotone-type weights, we obtain necessary and sufficient conditions, namely, $\|A\|_{L_p \rightarrow L_q} \asymp \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \asymp \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s)$, $1 < p \leq q \leq \infty$. Finally, in Appendix A we give the description of Hardy's inequalities on the cone of monotone functions.

Throughout, for a Lebesgue measurable set e , $|e|$ will denote its measure. If e and w are any subsets of \mathbb{R}^n , their Minkowski sum is defined by $e + w = \{x + y : x \in e \text{ and } y \in w\}$. The characteristic function of a set e is denoted by χ_e . Let ω_n be the volume of the unit ball in \mathbb{R}^n .

2. Lemmas

Define $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$. We will use the fact (see [BeSh, p. 53]) that f^{**} can be written as follows:

$$(2.1) \quad f^{**}(t) = \sup_{|e|=t} \frac{1}{|e|} \int_e |f(x)| dx.$$

LEMMA 2.1. *Let $\gamma \geq 0$ and $K : \mathbb{R}^n \rightarrow \mathbb{C}$ be integrable on any measurable set from \mathbb{R}^n of positive measure. Then, for any measurable set of positive measure $e \subset \mathbb{R}^n$, there exists $w \subset e$ such that $|w| \geq |e|/2$ and*

$$\frac{1}{|e|^\gamma} \int_e |K(x)| dx \leq \frac{8}{|w|^\gamma} \left| \int_w K(x) dx \right|.$$

PROOF. First, for the real-valued K we show that

$$(2.2) \quad \frac{1}{|e|^\gamma} \int_e |K(x)| dx \leq \frac{4}{|w|^\gamma} \left| \int_w K(x) dx \right|.$$

For any e such that $|e| = t > 0$, we define

$$e_+ := \left\{ x \in e : K(x) \geq 0 \right\} \quad \text{and} \quad e_- = \left\{ x \in e : K(x) < 0 \right\}.$$

Then

$$\int_e |K(x)| dx = \int_{e_+} K(x) dx - \int_{e_-} K(x) dx \leq 2 \max \left\{ \left| \int_{e_+} K(x) dx \right|, \left| \int_{e_-} K(x) dx \right| \right\}.$$

For definiteness, assume that

$$\left| \int_{e_+} K(x) dx \right| \geq \left| \int_{e_-} K(x) dx \right|.$$

Consider two cases: $|e_+| \geq \frac{t}{2}$ and $|e_+| < \frac{t}{2}$. In the first case, it suffices to take $w = e_+$. In the second case, there exist disjoint η_1 and η_2 such that $\eta_1 \cup \eta_2 = e_-$, $|\eta_1| = |\eta_2|$. Since K keeps its sign on e_- , we have

$$\begin{aligned} \left| \int_{e_+} K(x) dx \right| &\geq \left| \int_{e_-} K(x) dx \right| = \left| \int_{\eta_1} K(x) dx \right| + \left| \int_{\eta_2} K(x) dx \right| \\ &\geq 2 \min \left(\left| \int_{\eta_1} K(x) dx \right|, \left| \int_{\eta_2} K(x) dx \right| \right) = 2 \left| \int_{\eta} K(x) dx \right| \end{aligned}$$

with $\eta = \eta_j$ where $j = 1, 2$ being the set, where the minimum is attained.

Let now $w = \eta \cup e_+$. Then $|w| \geq \frac{t}{2}$ and

$$\begin{aligned} \left| \int_w K(x) dx \right| &= \left| \int_{e_+} K(x) dx + \int_{\eta} K(x) dx \right| \\ &\geq \left| \int_{e_+} K(x) dx \right| - \left| \int_{\eta} K(x) dx \right| \geq \frac{1}{2} \left| \int_{e_+} K(x) dx \right|. \end{aligned}$$

Therefore, in both cases, we obtain

$$\int_e |K(x)| dx \leq 2 \left| \int_{e_+} K(x) dx \right| \leq 4 \left| \int_w K(x) dx \right|,$$

and (2.2) follows because $w \subset e$.

Assume now that K is a complex-valued function. Then there are w_1 and w_2 such that

$$\begin{aligned} \frac{1}{|e|^\gamma} \int_e |K(x)| dx &\leq 2 \max \left\{ \frac{1}{|e|^\gamma} \int_e |\operatorname{Re} K(x)| dx, \frac{1}{|e|^\gamma} \int_e |\operatorname{Im} K(x)| dx \right\} \\ &\leq 8 \max \left\{ \frac{1}{|w_1|^\gamma} \left| \operatorname{Re} \int_{w_1} K(x) dx \right|, \frac{1}{|w_2|^\gamma} \left| \operatorname{Im} \int_{w_2} K(x) dx \right| \right\} \leq \max_{j \in \{1, 2\}} \frac{8}{|w_j|^\gamma} \left| \int_{w_j} K(x) dx \right|. \end{aligned}$$

□

LEMMA 2.2. *We have*

$$(2.3) \quad \left(\int_{\mathbb{R}^n} \mu(x) K(x, \cdot) f(x) dx \right)^* (s) \lesssim \int_0^\infty \mu^*(t) \Phi(t, s) f^*(t) dt,$$

where

$$(2.4) \quad \Phi(s, t) = \sup_{|e| \geq s/2, |\omega| \geq t/2} \frac{1}{|e|} \frac{1}{|\omega|} \left| \int_e \int_\omega K(x, y) dx dy \right|.$$

PROOF. Following ideas from [NT3], by the Hardy–Littlewood–Pólya rearrangement inequality and (2.1), we have

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \mu(x) K(x, \cdot) f(x) dx \right)^* (s) &\leq \left(\int_{\mathbb{R}^n} \mu(x) K(x, \cdot) f(x) dx \right)^{**} (s) \\ &= \sup_{|\eta_1|=s} \frac{1}{|\eta_1|} \int_{\eta_1} \left| \int_{\mathbb{R}^n} \mu(x) K(x, y) f(x) dx \right| dy \lesssim \sup_{|\eta_1| \geq s/2} \frac{1}{|\eta_1|} \left| \int_{\mathbb{R}^n} \mu(x) f(x) \int_{\eta_1} K(x, y) dy dx \right|, \end{aligned}$$

where in the last estimate we used Lemma 2.1 with $\gamma = 1$. Further, we use similar estimates for the outer integral to derive

$$\begin{aligned} \left(\int_{\mathbb{R}^n} \mu(x) K(x, \cdot) f(x) dx \right)^* (s) &\lesssim \sup_{|\eta_1| \geq s/2} \int_0^\infty \mu^*(t) f^*(t) \sup_{|\eta_2| \geq t} \frac{1}{|\eta_2|} \frac{1}{|\eta_1|} \int_{\eta_2} \left| \int_{\eta_1} K(x, y) dx \right| dy dt. \end{aligned}$$

Applying again Lemma 2.1, we arrive at (2.3). \square

As a corollary we obtain the following version of Calderón's rearrangement inequality for the operator T given by (1.14); see [Ca].

COROLLARY 2.3. *Let $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$, $q_0 \neq q_1$, and*

$$B_i = \sup_{|\eta_1| > 0, |\eta_2| > 0} \frac{1}{|\eta_2|^{1/q_i}} \frac{1}{|\eta_1|^{1/p_i}} \left| \int_{\eta_2} \int_{\eta_1} K(x, y) dx dy \right| dt < \infty, \quad i = 0, 1.$$

Then for any $t > 0$ and $\xi = (1/q_0 - 1/q_1)/(1/p_0 - 1/p_1)$ we have

$$(2.5) \quad (Tf)^*(t) \lesssim \frac{B_0}{t^{1/q_0}} \int_0^{t^\xi} s^{1/p_0-1} f^*(s) \frac{ds}{s} + \frac{B_1}{t^{1/q_1}} \int_{t^\xi}^\infty s^{1/p_1-1} f^*(s) \frac{ds}{s}.$$

PROOF. Lemma 2.2 with $\mu \equiv 1$ implies that

$$(Tf)^*(t) \lesssim \int_0^\infty f^*(s) \sup_{|e| \geq s/2, |\omega| \geq t/2} \frac{1}{|e|} \frac{1}{|\omega|} \left| \int_e \int_\omega K(x, y) dx dy \right|.$$

Then

$$(Tf)^*(t) \lesssim B_0 \int_0^{t^\xi} f^*(s) (2/t)^{1/q_0} (2/s)^{1/p'_0} ds + B_1 \int_{t^\xi}^\infty f^*(s) (2/t)^{1/q_1} (2/s)^{1/p'_1} ds$$

and (2.5) follows. \square

The next result will play a crucial role in our proofs.

LEMMA 2.4. [NT2] *Let $1 < p < \infty$, $1 < q \leq \infty$, and $0 < r \leq 1$. We have*

$$(2.6) \quad \|T\|_{L_{p,r}(\mathbb{R}^n) \rightarrow L_{q,\infty}(\mathbb{R}^n)} \asymp \|T\|_{L_{p,1}(\mathbb{R}^n) \rightarrow L_{q,\infty}(\mathbb{R}^n)} \asymp \sup_{\substack{|e| > 0 \\ |\omega| > 0}} \frac{1}{|e|^{1/q'}} \frac{1}{|\omega|^{1/p}} \left| \int_e \int_\omega K(x, y) dx dy \right|.$$

This lemma immediately implies

LEMMA 2.5. *Let an operator T be of strong type $(1, \infty)$ and of restricted weak type $(2, 2)$. Then*

$$\sup_{|e|>0, |\omega|>0} \frac{1}{|e|} \frac{1}{|\omega|} \left| \int_e \int_\omega K(x, y) dx dy \right| + \sup_{|e|>0, |\omega|>0} \frac{1}{|e|^{\frac{1}{2}}} \frac{1}{|\omega|^{\frac{1}{2}}} \left| \int_e \int_\omega K(x, y) dx dy \right| \lesssim 1.$$

3. Fourier inequalities via Hardy inequalities

Let $0 < p, q \leq \infty$ and ν, μ, w be positive weights on $(0, \infty)$. Define

$$F_{p,q}^+(\nu, \mu, w) = \sup_{f \geq 0} \frac{\left(\int_0^\infty \left(\nu(t) \int_0^t \mu(s) f(s) ds \right)^q dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty (w(t) f(t))^p dt \right)^{\frac{1}{p}}}$$

and

$$(3.1) \quad F_{p,q}^\downarrow(\nu, \mu, w) = \sup_{f \downarrow} \frac{\left(\int_0^\infty \left(\nu(t) \int_0^t \mu(s) f(s) ds \right)^q dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty (w(t) f(t))^p dt \right)^{\frac{1}{p}}},$$

where the suprema are taken over all non-negative and non-increasing functions respectively. Note that the description of the class of weights ν, μ , and w for the condition $F_{p,q}^+(\nu, \mu, w) < \infty$ (or $F_{p,q}^\downarrow(\nu, \mu, w) < \infty$) to hold is an important problem in analysis, which has been studied extensively, see, e.g., [KP, GS, KO]. For the sake of completeness, necessary and sufficient conditions for $F_{p,q}^\downarrow(\nu, \mu, w) < \infty$ are given in Appendix A.

We remark that, in general, $F_{p,q}^\downarrow(\nu, \mu, w) \leq F_{p,q}^+(\nu, \mu, w)$. However, if μ is decreasing and w is increasing, then $F_{p,q}^+(\nu, \mu, w) = F_{p,q}^\downarrow(\nu, \mu, w)$.

LEMMA 3.1. *Let $0 < p, q \leq \infty$ and either of the following two conditions hold:*

(1) *the function ν satisfies the condition*

$$(3.2) \quad \nu(\lambda t) \lesssim \lambda^{-\alpha} \nu(t), \quad \lambda > 1, \quad \forall t > 0,$$

with some $\alpha > \frac{1}{q} - \frac{1}{2}$, or

(2) *the functions μ and w satisfy the conditions*

$$(3.3) \quad \begin{aligned} \mu(\lambda t) &\lesssim \lambda^{-\beta} \mu(t), & \lambda > 1, \quad \forall t > 0, \\ w(\lambda^{-1} t) &\lesssim \lambda^{-\gamma} w(t), & \lambda > 1, \quad \forall t > 0, \end{aligned}$$

with some β and γ such that $\beta + \gamma > \frac{1}{2} - \frac{1}{p}$.

Then, we have

$$\sup_{f \downarrow} \frac{\left(\int_0^\infty \left(\nu(t) \int_{\frac{1}{t}}^\infty \mu(s) f(s) (st)^{-\frac{1}{2}} ds \right)^q dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty (w(t) f(t))^p dt \right)^{\frac{1}{p}}} \lesssim F_{p,q}^\downarrow \left(\frac{\nu(t^{-1})}{t^{\frac{2}{q}}}, \mu, w \right).$$

PROOF. Let $\theta = 1$ if (3.2) holds and $\theta = 0$ if (3.3) holds. Assume first that $1 \leq q \leq \infty$. Then Minkowski's inequality yields

$$\begin{aligned} \left(\int_0^\infty \left(\nu(t) \int_{\frac{1}{t}}^\infty \mu(s) f(s) (st)^{-\frac{1}{2}} ds \right)^q dt \right)^{\frac{1}{q}} &= \left(\int_0^\infty \left(\nu(t) \sum_{k=1}^\infty \int_{\frac{2^{k-1}}{t}}^{\frac{2^k}{t}} \mu(s) f(s) (st)^{-\frac{1}{2}} ds \right)^q dt \right)^{\frac{1}{q}} \\ &\leq \sum_{k=1}^\infty \left(\int_0^\infty \left(\nu(t) \int_{\frac{2^{k-1}}{t}}^{\frac{2^k}{t}} \mu(s) f(s) (st)^{-\frac{1}{2}} ds \right)^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Changing variables twice and using our assumptions on the weights, we continue as follows

$$\begin{aligned} &\leq \sqrt{2} \sum_{k=1}^\infty 2^{(\frac{1}{2} - \frac{\theta}{q})k} \left(\int_0^\infty \left(\nu(2^{\theta k} t) \int_{\frac{1}{2t}}^{\frac{1}{t}} \mu(2^{(1-\theta)k} s) f(2^{(1-\theta)k} s) ds \right)^q dt \right)^{\frac{1}{q}} \\ &\lesssim \sum_{k=1}^\infty 2^{(\frac{1}{2} - \theta(\frac{1}{q} + \alpha) - (1-\theta)\beta)k} \left(\int_0^\infty \left(\nu(t) \int_{\frac{1}{2t}}^{\frac{1}{t}} \mu(s) f(2^{(1-\theta)k} s) ds \right)^q dt \right)^{\frac{1}{q}} \\ &\leq \sum_{k=1}^\infty 2^{(\frac{1}{2} - \theta(\frac{1}{q} + \alpha) - (1-\theta)\beta)k} \left(\int_0^\infty \left(\frac{\nu(t^{-1})}{t^{\frac{2}{q}}} \int_0^t \mu(s) f(2^{(1-\theta)k} s) ds \right)^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Further, taking into account definition (3.1), the latter can be estimated by

$$\begin{aligned} &\lesssim F_{p,q}^\downarrow \left(\frac{\nu(t^{-1})}{t^{\frac{2}{q}}}, \mu, w \right) \sum_{k=1}^\infty 2^{(\frac{1}{2} - \theta(\frac{1}{q} + \alpha) - (1-\theta)\beta)k} \left(\int_0^\infty (w(t) f(2^{(1-\theta)k} t))^p dt \right)^{\frac{1}{p}} \\ &\lesssim F_{p,q}^\downarrow \left(\frac{\nu(t^{-1})}{t^{\frac{2}{q}}}, \mu, w \right) \left(\int_0^\infty (w(t) f(t))^p dt \right)^{\frac{1}{p}} \sum_{k=1}^\infty 2^{(\frac{1}{2} - \theta(\frac{1}{q} + \alpha) - (1-\theta)(\frac{1}{p} + \beta + \gamma))k}. \end{aligned}$$

Conditions (3.2) and (3.3) and the choice of θ imply the convergence of the sum in the last expression. This completes the proof in the case $q \geq 1$.

Now assume that $q < 1$.

$$\begin{aligned} \left(\int_0^\infty \left(\nu(t) \int_{\frac{1}{t}}^\infty \mu(s) f(s) (st)^{-\frac{1}{2}} ds \right)^q dt \right)^{\frac{1}{q}} &\leq \left(\int_0^\infty \left(\sum_{k=1}^\infty \nu(t) \int_{\frac{2^{k-1}}{t}}^{\frac{2^k}{t}} \mu(s) f(s) (st)^{-\frac{1}{2}} ds \right)^q dt \right)^{\frac{1}{q}} \\ &\leq \left(\sum_{k=1}^\infty \int_0^\infty \left(\nu(t) \int_{\frac{2^{k-1}}{t}}^{\frac{2^k}{t}} \mu(s) f(s) (st)^{-\frac{1}{2}} ds \right)^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

The rest of the proof is similar. \square

3.1. Fourier inequalities in weighted Lebesgue spaces. The main result in this section is the following

THEOREM 3.2. *Let an operator T be of strong type $(1, \infty)$ and of restricted weak type $(2, 2)$. Let $0 < p, q \leq \infty$ and either of the following two conditions hold:*

(1) *the function ν^* satisfies the condition*

$$(3.4) \quad \nu^*(\lambda t) \leq c \lambda^{-\alpha} \nu^*(t), \quad \lambda > 1, \quad \forall t > 0,$$

- with some $\alpha > \frac{1}{q} - \frac{1}{2}$, or
 (2) the function μ^* satisfies the condition

$$(3.5) \quad \mu^*(\lambda t) \leq c\lambda^{-\beta}\mu^*(t), \quad \lambda > 1, \quad \forall t > 0,$$

with some $\beta > \frac{1}{2} - \frac{1}{p}$.

Then

$$(3.6) \quad \|Af\|_{L_q} \lesssim F_{p,q}^+ \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}}, \mu^*(t), 1 \right) \|f\|_{L_p}.$$

Note that conditions (3.4) and (3.5) always hold for $\alpha = 0$ and $\beta = 0$. In particular, if either $p < 2$ or $2 < q$, then (3.6) is valid. We also observe that conditions (3.4) and (3.5) provide a clear link between the regularity of weights and the range of parameters p and q .

For the Fourier transform we arrive at the following result.

COROLLARY 3.3. (i) Under the conditions of Theorem 3.2, if $1 < p \leq \infty$, (1.1) holds true provided that $F_{p,q}^+ \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}}, \mu^*(t), 1 \right) < \infty$.

(ii) If $p = q = 2$, then (1.1) holds true provided that $F_{2,2}^+ \left(\frac{\nu^*(t^{-1})}{t}, 1, \frac{1}{\mu^*(t)} \right) < \infty$.

Part (i) follows directly from Theorem 3.2. For the second part see Theorem 1(b) in [BH].

PROOF OF THEOREM 3.2. First, we recall that the function $\Phi(s, t)$ is defined by (2.4) and note that $F_{p,q}^+ \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}}, \mu^*(t), 1 \right) = F_{p,q}^\downarrow \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}}, \mu^*(t), 1 \right)$. By the Hardy–Littlewood–Pólya inequality and Lemmas 2.2 and 2.5, we have

$$\begin{aligned} \|Af\|_{L_q(\mathbb{R}^n)} &\leq \left(\int_0^\infty \left(\nu^*(t) \left(\int_{\mathbb{R}^n} \mu(x) K(x, \cdot) f(x) dx \right)^* (t) \right)^q dt \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty \left(\nu^*(t) \int_0^\infty \mu^*(s) f^*(s) \Phi(s, t) ds \right)^q dt \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty \left(\nu^*(t) \int_0^{\frac{1}{t}} \mu^*(s) f^*(s) ds \right)^q dt \right)^{\frac{1}{q}} \\ &\quad + \left(\int_0^\infty \left(\nu^*(t) \int_{\frac{1}{t}}^\infty \frac{\mu^*(s) f^*(s)}{(st)^{\frac{1}{2}}} ds \right)^q dt \right)^{\frac{1}{q}} =: I_1 + I_2. \end{aligned}$$

Changing variables, we estimate the first integral as follows:

$$\begin{aligned} I_1 &\leq \left(\int_0^\infty \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}} \int_0^t \mu^*(s) f^*(s) ds \right)^q dt \right)^{\frac{1}{q}} \\ &\lesssim F_{p,q}^\downarrow \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}}, \mu^*(t), 1 \right) \|f^*\|_{L_p(0,\infty)} = F_{p,q}^+ \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}}, \mu^*(t), 1 \right) \|f\|_{L_p(\mathbb{R}^n)}. \end{aligned}$$

To estimate I_2 , we apply Lemma 3.1 with $w(x) \equiv 1$ and $\gamma = 0$. □

3.2. Comparison of Theorem 3.2 and Theorem 2.1 in [RS]. The main results of Theorem 2.1 in [RS] were explained in the introduction (see (1.7)–(1.12)).

Let us now show that, for $q < 2$, (3.4) is more restrictive than both (1.7) and (1.8) but less restrictive than (1.9). Similarly, (3.5) is more restrictive than (1.10) and (1.11) but less restrictive than (1.12).

We observe that condition (1.8), that is, $\nu^*(t)^q \in B_{q/2}$ is equivalent to the following condition: $\int_t^\infty V(s)s^{-q/2-1}ds \lesssim V(t)t^{-q/2}$, where $V(x) = \int_0^x \nu^*(t)^q dt$ (see [So]). The latter can be equivalently written as

$$(3.7) \quad V(x)x^{-q/2+\varepsilon} \quad \text{is almost decreasing}$$

(see [BaSt]). Recall that f is almost decreasing if $f(x) \leq Cf(y)$ for all $y \leq x$ (see, e.g., [Be]). On the other hand, it is easy to see that condition (3.4) is equivalent to

$$(3.8) \quad \nu^*(x)^q x^{1-q/2+\varepsilon} \quad \text{is almost decreasing.}$$

Let us prove that (3.8) always implies (3.7) and the reverse does not hold in general. Let (3.8) be valid. Assume that ε is sufficiently small. Setting $y \leq x$ and changing variables $z = \frac{y}{x}t$, we get

$$\frac{\int_0^x \nu^*(t)^q dt}{x^{q/2-\varepsilon}} = \frac{\int_0^y \nu^*(xz/y)^q \frac{x}{y} dz}{x^{q/2-\varepsilon}} \lesssim \frac{\int_0^y \nu^*(z)^q (\frac{x}{y})^{q/2-\varepsilon} dz}{x^{q/2-\varepsilon}} = \frac{\int_0^y \nu^*(t)^q dt}{y^{q/2-\varepsilon}},$$

i.e., (3.7) holds.

Define now

$$\nu(x)^q = \sum_{k=1}^{\infty} c_k \chi_{(1/(k+1)!, 1/k!]}(x), \quad x \in (0, 1],$$

where $c_k = (k-1)!/k$. It is clear that $\nu(x) = \nu^*(x)$ since c_k is increasing. Moreover, taking $x = 1/(k!)$ and $y = 1/(kk!)$, we see that

$$k^{1-q/2+\varepsilon} = \frac{\nu^*(x)^q x^{1-q/2+\varepsilon}}{\nu^*(y)^q y^{1-q/2+\varepsilon}} \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Thus (3.8) does not hold.

On the other hand, note that $\frac{1}{k+1} \leq \int_0^x \nu(t)^q dt \leq \frac{1}{k}$ for $x \in (1/(k+1)!, 1/k!]$ and $\int_0^x \nu(t)^q dt = \frac{1}{2}$ for $x \geq \frac{1}{2}$. Take now $x \in (1/(k+1)!, 1/k!]$ and $y \in (1/(m+1)!, 1/m!]$, $2 \leq k \leq m$, and set $0 \leq \xi = m - k$. Note that $\frac{x}{y} > \frac{m!}{(k+1)!} = m(m-1) \cdots (k+2) \geq 4^{\xi-1}$ for $\xi \geq 2$ and $\frac{x}{y} > 1$ for $\xi = 0, 1$. Hence,

$$\frac{x^{-q/2+\varepsilon} \int_0^x \nu^*(t)^q dt}{y^{-q/2+\varepsilon} \int_0^y \nu^*(t)^q dt} \leq \left(\frac{y}{x}\right)^{q/2-\varepsilon} \frac{m+1}{k} \leq \begin{cases} (1 + \frac{\xi+1}{m})(4^{q/2-\varepsilon})^{1-\xi}, & \xi \geq 2 \\ 2, & \xi = 0, 1 \end{cases},$$

which is bounded for all ξ . Thus, (3.7) holds for the function ν .

Finally, condition (1.9) clearly implies (3.7) but the reverse is not true (take $\nu(x) = x^{1/2-1/q-\alpha}$ for sufficiently small α).

We note that the complete description of the condition $F_{p,q}^+ \left(\frac{\nu^*(t^{-1})}{t^{\frac{2}{q}}}, \mu^*(t), 1 \right) < \infty$ can be found in [KP].

3.3. Fourier inequalities in weighted Lorentz spaces. Let $0 < q \leq \infty$ and ν be a non-negative weight on $(0, \infty)$. Define the weighted Lorentz space

$$\Lambda_q(\nu) = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ such that } \|f\|_{\Lambda_q(\nu)} = \left(\int_0^\infty (f^*(t)\nu(t))^q dt \right)^{\frac{1}{q}} < \infty \right\}.$$

Replacing f^* by $f^{**}(t) = \frac{1}{t} \int_0^t f^*$ we analogously define the space $\Gamma_q(\nu)$.

Sinnamon [Si] gave some sufficient (and some necessary) conditions to ensure that the Fourier transform is bounded from $\Lambda_p(w)$ to $\Gamma_q(\nu)$. In particular, in the case $0 < p \leq q = 2$, he fully characterized the weights w and ν such that $\widehat{f} : \Lambda_p(w) \rightarrow \Gamma_q(\nu)$. Recently, a sufficient condition for $\|\widehat{f}\|_{\Lambda_q(\nu)} \lesssim \|f\|_{\Lambda_p(w)}$ has been obtained [RS] in the case $1 < p \leq q < \infty$, $2 \leq q$. It is given in terms of the level function of the weights ν . See also the previous results in [BH, Theorems 2 and 3] and the recent paper [BS].

We continue this line of research. The next result provides sufficient conditions for $T : \Lambda_p(w) \rightarrow \Lambda_q(\nu)$, $0 < p, q \leq \infty$ under suitable conditions on the weights.

THEOREM 3.4. *Let an operator T be of strong type $(1, \infty)$ and of restricted weak type $(2, 2)$. Let $0 < p, q \leq \infty$, and either of the following two conditions hold:*

(1) *the function ν satisfies the condition*

$$\nu(\lambda t) \lesssim \lambda^{-\alpha} \nu(t), \quad \lambda > 1, \quad \forall t > 0,$$

with some $\alpha > \frac{1}{q} - \frac{1}{2}$, or

(2) *the function μ satisfies the condition*

$$w(\lambda^{-1}t) \lesssim \lambda^{-\gamma} w(t), \quad \lambda > 1, \quad \forall t > 0,$$

with some $\gamma > \frac{1}{2} - \frac{1}{p}$.

Then, we have

$$\|Tf\|_{\Lambda_q(\nu)} \lesssim F_{p,q}^\downarrow \left(\frac{\nu(t^{-1})}{t^{\frac{2}{q}}}, 1, w(t) \right) \|f\|_{\Lambda_p(w)}.$$

PROOF. As in the proof of Theorem 3.2 we use Lemmas 2.2 and 2.5 to obtain

$$\begin{aligned} \|Tf\|_{\Lambda_q(\nu)} &= \left(\int_0^\infty \left[\nu(t) \left(\int_{\mathbb{R}^n} K(x, \cdot) f(x) dx \right)^*(t) \right]^q dt \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty \left(\nu(t) \int_0^\infty f^*(s) \Phi(s, t) ds \right)^q dt \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty \left(\nu(t) \int_0^{\frac{1}{t}} f^*(s) ds \right)^q dt \right)^{\frac{1}{q}} + \left(\int_0^\infty \left(\nu(t) \int_{\frac{1}{t}}^\infty \frac{f^*(s)}{(st)^{\frac{1}{2}}} ds \right)^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

We have

$$\begin{aligned} \left(\int_0^\infty \left(\nu(t) \int_0^{\frac{1}{t}} f^*(s) ds \right)^q dt \right)^{\frac{1}{q}} &\leq \left(\int_0^\infty \left(\frac{\nu(t^{-1})}{t^{\frac{2}{q}}} \int_0^t f^*(s) ds \right)^q dt \right)^{\frac{1}{q}} \\ &\leq F_{p,q}^\downarrow \left(\frac{\nu(t^{-1})}{t^{\frac{2}{q}}}, 1, w(t) \right) \|f\|_{\Lambda_p(w)}. \end{aligned}$$

Lemma 3.1 with $\mu(x) \equiv 1$ and $\beta = 0$ completes the proof. \square

4. Weak type inequalities for Fourier transforms

Our main objective in the section is to present sufficient and similar-looking necessary condition on the weights (ν, μ) for the operator A to be of restricted weak type (p, q) .

THEOREM 4.1. *Let an operator T be of strong type $(1, \infty)$ and of restricted weak type $(2, 2)$. Let $1 < p, q < \infty$ be such that either $p \leq 2$ or $q \geq 2$. If*

$$I = \sup_{\xi \eta \leq 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^\eta \mu^*(t) dt < \infty,$$

then

$$(4.1) \quad \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim I.$$

Moreover, if A is bounded from $L_{p,1}$ to $L_{q,\infty}$, then

$$(4.2) \quad \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \gtrsim \sup_{x_0, y_0 \in \mathbb{R}^n} \sup_{\xi \eta \leq 1} \frac{1}{|B_\xi(0)|^{1/q'} |B_\eta(0)|^{1/p}} \int_{B_\xi(y_0)} \nu(y) dy \int_{B_\eta(x_0)} \mu(x) dx,$$

where $B_r(x) = \{y \in \mathbb{R}^n : |y - x| \leq r\}$.

In light of (2.1) it is easy to compare the right-hand side of (4.2) and I , namely,

$$I \asymp \sup_{|e||w| \leq 1} \frac{1}{|e|^{\frac{1}{q'}}} \frac{1}{|w|^{\frac{1}{p}}} \int_e \nu(t) dt \int_w \mu(s) ds.$$

Moreover, assuming certain monotonicity-type conditions on the weights, we derive necessary and sufficient condition for A to be of restricted weak type (p, q) .

COROLLARY 4.2. *Under conditions of Theorem 4.1, if for some $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n$ one has*

$$(4.3) \quad \int_0^{|B_r(x_0)|} \nu^*(t) dt \lesssim \int_{B_r(x_0)} \nu(x) dx \quad \text{and} \quad \int_0^{|B_r(y_0)|} \mu^*(t) dt \lesssim \int_{B_r(y_0)} \mu(x) dx,$$

then $\|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \asymp I$.

The proof immediately follows from Theorem 4.1 and the fact that, for any fixed $\lambda > 1$, one has $\int_0^T \nu^*(t) dt \asymp \int_0^{\lambda T} \nu^*(t) dt$, $T > 0$. Observe that condition (4.3) holds for any radial weights $\nu(x) = \nu_0(|x|)$ and $\mu(x) = \mu_0(|x|)$ such that ν_0 and μ_0 are non-increasing on \mathbb{R}_+ .

Note also that necessary conditions for $A : L_{p,1} \rightarrow L_{q,\infty}$ can be given with the help of polar sets **[DGT]**.

PROOF OF THEOREM 4.1. Let first $1 < p \leq 2$. By Lemmas 2.2, 2.4, and 2.5, we establish

$$\begin{aligned} \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} &\asymp \sup_{\substack{|e| > 0 \\ |w| > 0}} \frac{1}{|e|^{\frac{1}{q'}}} \frac{1}{|w|^{\frac{1}{p}}} \left| \int_e \nu(y) \int_w \mu(x) K(x, y) dx dy \right| \\ &\lesssim \sup_{\substack{|e| > 0 \\ |w| > 0}} \frac{1}{|e|^{\frac{1}{q'}}} \frac{1}{|w|^{\frac{1}{p}}} \int_0^{|e|} \nu^*(t) \int_0^{|w|} \mu^*(s) \Phi(s, t) ds dt \\ &\lesssim \sup_{\xi \eta \leq 1} \frac{1}{\xi^{\frac{1}{q'}}} \frac{1}{\eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^\eta \mu^*(t) dt + \sup_{\xi \eta > 1} \frac{1}{\xi^{\frac{1}{q'}}} \frac{1}{\eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) \int_0^\eta \mu^*(s) \Phi(s, t) ds dt. \end{aligned}$$

Therefore, using again Lemma 2.5, we derive

$$\begin{aligned}
& \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \\
& \lesssim I + \sup_{\xi \cdot \eta > 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \left(\int_0^{1/\eta} \nu^*(t) \int_0^\eta \mu^*(s) ds dt + \int_{1/\eta}^\xi \nu^*(t) \int_0^\eta \mu^*(s) \Phi(s, t) ds dt \right) \\
& \lesssim I + \sup_{\eta > 0} \eta^{\frac{1}{q'} - \frac{1}{p}} \int_0^{1/\eta} \nu^*(t) \int_0^\eta \mu^*(s) ds dt \\
& + \sup_{\xi \cdot \eta > 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \left(\int_{1/\eta}^\xi \nu^*(t) \int_0^{1/t} \mu^*(s) ds dt + \int_{1/\eta}^\xi \nu^*(t) \int_{1/t}^\eta \mu^*(s) \frac{ds dt}{(st)^{\frac{1}{2}}} \right) \\
& =: I + J_1 + J_2 + J_3.
\end{aligned}$$

It is clear that $J_1 \leq I$. To estimate J_2 , noting that $\xi \eta > 1$, we write

$$\begin{aligned}
\frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_{1/\eta}^\xi \nu^*(t) \int_0^{1/t} \mu^*(s) ds dt & \leq \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_{\frac{1}{\eta}}^\xi \nu^{**}(t) \int_0^{\frac{1}{t}} \mu^*(s) ds dt \\
& \leq \begin{cases} \frac{1}{\xi^{\frac{1}{q'} - \frac{1}{p}}} \int_{\frac{1}{\eta}}^\xi t^{\frac{1}{q'} - \frac{1}{p} - 1} dt \cdot I \lesssim I, & \frac{1}{q'} - \frac{1}{p} > 0, \\ \frac{1}{\eta^{\frac{1}{p} - \frac{1}{q'}}} \int_{\frac{1}{\eta}}^\xi t^{\frac{1}{q'} - \frac{1}{p} - 1} dt \cdot I \lesssim I, & \frac{1}{q'} - \frac{1}{p} < 0, \\ \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_{\frac{1}{\eta}}^\xi dt \cdot I = \frac{\ln(\xi \eta)}{(\xi \eta)^{\frac{1}{p}}} I \leq \sup_{x \geq 1} \frac{\ln x}{x^{\frac{1}{p}}} I \lesssim I, & \frac{1}{q'} - \frac{1}{p} = 0. \end{cases}
\end{aligned}$$

Thus, $J_2 \lesssim I$ for any $1 < p, q < \infty$.

Now we proceed with the estimate of J_3 as follows:

$$\begin{aligned}
J_3 & \leq \sup_{\xi \cdot \eta > 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_{1/\eta}^\xi \nu^*(t) \mu^*(1/t) \int_{\frac{1}{t}}^\eta \frac{1}{(st)^{\frac{1}{2}}} ds dt \\
& \lesssim \sup_{\xi \cdot \eta > 1} \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_{1/\eta}^\xi \nu^*(t) \mu^*(1/t) \frac{1}{t^{\frac{1}{2}}} dt \\
& \leq \sup_{\xi \cdot \eta > 1} \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_{1/\eta}^\xi t^{-1/2} \int_0^t \nu^*(s) ds \int_0^{\frac{1}{t}} \mu^*(s) ds dt.
\end{aligned}$$

Taking into account that $p \leq 2$ or $q \geq 2$ gives

$$\begin{aligned}
J_3 & \lesssim I \sup_{\xi \cdot \eta > 1} \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_{\frac{1}{\eta}}^\xi t^{\frac{1}{q'} - \frac{1}{p} - \frac{1}{2}} dt \\
& \lesssim I \sup_{\xi \cdot \eta > 1} \begin{cases} \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \xi^{\frac{1}{q'} - \frac{1}{p} + \frac{1}{2}} = (\xi \eta)^{\frac{1}{2} - \frac{1}{p}} \leq 1, & \frac{1}{q'} - \frac{1}{p} + \frac{1}{2} > 0, \\ \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \eta^{-\frac{1}{q'} + \frac{1}{p} - \frac{1}{2}} = (\xi \eta)^{-\frac{1}{q'}} \leq 1, & \frac{1}{q'} - \frac{1}{p} + \frac{1}{2} < 0, \\ \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \ln \xi \eta = (\xi \eta)^{-\frac{1}{q'}} \ln \xi \eta \lesssim 1, & \frac{1}{q'} - \frac{1}{p} + \frac{1}{2} = 0. \end{cases}
\end{aligned}$$

Thus,

$$\|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim \sup_{\xi \eta \leq 1} \frac{1}{\xi^{\frac{1}{q'}}} \frac{1}{\eta^{\frac{1}{p}}} \int_0^\xi \nu^*(s) ds \int_0^\eta \mu^*(s) ds,$$

that is, (4.1) is proved when $p \leq 2$ or $q \geq 2$.

Let us prove the second part of the theorem. Let the operator (1.14) be bounded from $L_{p,1}(\mathbb{R}^n)$ to $L_{q,\infty}(\mathbb{R}^n)$. Defining the operator

$$A_{x_0,y_0}f := \nu(y-y_0) \int_{\mathbb{R}^n} f(x)\mu(x-x_0)e^{-iyx}dx$$

for fixed $x_0, y_0 \in \mathbb{R}^n$, we claim (for example, with the help of Lemma 2.1) that $\|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \asymp \|A_{x_0,y_0}\|_{L_{p,1} \rightarrow L_{q,\infty}}$. Then

$$\begin{aligned} \|A_{x_0,y_0}\|_{L_{p,1} \rightarrow L_{q,\infty}} &\asymp \sup_{\substack{|E|>0 \\ |W|>0}} \frac{1}{|E|^{1/q'}} \frac{1}{|W|^{1/p}} \left| \int_E \nu(y-y_0) \int_W \mu(x-x_0)e^{-ixy}dxdy \right| \\ &\gtrsim \sup_{\xi\eta \leq 1} \frac{1}{|B_\xi(0)|^{1/q'}|B_\eta(0)|^{1/p}} \left| \int_{B_\xi(0)} \nu(y-y_0) \int_{B_\eta(0)} \mu(x-x_0)e^{-ixy}dxdy \right|. \end{aligned}$$

Since $|xy| \leq 1$ for any $x \in B_\xi(0)$, $y \in B_\eta(0)$, the latter expression can be estimated from below by

$$\sup_{\xi\eta \leq 1} \frac{\cos 1}{|B_\xi(0)|^{1/q'}|B_\eta(0)|^{1/p}} \left| \int_{B_\xi(0)} \nu(y-y_0) \int_{B_\eta(0)} \mu(x-x_0)dxdy \right|.$$

Thus, (4.2) follows. \square

As a consequence of Theorem 4.1 we obtain the following strong type result for the operator A .

COROLLARY 4.3. *Let $1 < p, q, p_0, q_0 < \infty$ and $0 < \tau \leq \infty$ be such that*

$$(4.4) \quad \frac{1}{p_0} - \frac{1}{q'_0} = \frac{1}{p} - \frac{1}{q'}$$

and

$$(4.5) \quad I(p_0, q_0) := \sup_{\substack{\xi, \eta \leq 1 \\ \xi > 0, \eta > 0}} \frac{1}{\xi^{1/q'_0}\eta^{1/p_0}} \int_0^\xi \nu^*(t)dt \int_0^\eta \mu^*(t)dt < \infty.$$

If either $p_0 < p < 2$ or $2 < q < q_0$, the operator A is bounded from $L_{p,\tau}$ to $L_{q,\tau}$ and

$$\|A\|_{L_{p,\tau} \rightarrow L_{q,\tau}} \lesssim I(p_0, q_0).$$

PROOF. First, we note that

$$I(p, q) \leq I(p_0, q_0),$$

where

$$I(p, q) = \sup_{\substack{\xi, \eta \leq 1 \\ \xi > 0, \eta > 0}} \frac{1}{\xi^{1/q'}\eta^{1/p}} \int_0^\xi \nu^*(t)dt \int_0^\eta \mu^*(t)dt.$$

Indeed, for $\xi\eta \leq 1$ one has

$$\frac{1}{\xi^{1/q'}\eta^{1/p}} = \frac{\xi^{\frac{1}{p_0} - \frac{1}{p}}\eta^{\frac{1}{q'_0} - \frac{1}{q'}}}{\xi^{1/q'_0}\eta^{1/p_0}} = \frac{(\xi\eta)^{\frac{1}{p_0} - \frac{1}{p}}}{\xi^{1/q'_0}\eta^{1/p_0}} \leq \frac{1}{\xi^{1/q'_0}\eta^{1/p_0}}.$$

In view of Theorem 4.1, we have

$$\|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim I(p_0, q_0)$$

for any couple (p, q) such that $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q}$ and $p_0 \leq p \leq 2$.

Then the Marcinkiewicz interpolation theorem [BL, Ch. 5.3] yields that A is bounded from $L_{p,\tau}$ to $L_{q,\tau}$ for any couple (p, q) satisfying $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q}$ and $p_0 < p < 2$. We treat the case $2 < q < q_0$ similarly. \square

5. Some useful conditions on weights

Before presenting applications of Theorem 4.1, we introduce the concept of α -regularity.

DEFINITION 5.1. Let $\alpha > 0$. We will say that a measurable function μ is α -regular, if the function $t^\alpha \mu^*(t)$ is almost increasing, that is, there exists $C > 0$ such that

$$(5.1) \quad t^\alpha \mu^*(t) \leq C s^\alpha \mu^*(s) \quad \text{for all } 0 < t < s.$$

This condition is equivalent to the condition

$$(5.2) \quad \frac{\mu^*(t)}{\mu^*(\lambda t)} \leq C \lambda^\alpha \quad t > 0, \quad \forall \lambda > 1,$$

It is well known that α -regularity can be rewritten in terms of the Matuszewska indices or Bary–Stechkin’s conditions (see [BGT, BaSt]).

PROPOSITION 5.2. Let $1 < p, q < \infty$ and either of the following two conditions hold:

- (1) the function $\mu(x)$ is α -regular with some $\alpha < \frac{1}{p'}$, or
- (2) the function $\nu(x)$ is α -regular with some $\alpha < \frac{1}{q}$.

Then the following conditions are equivalent:

$$(5.3) \quad \mathbb{D}_1 := \sup_{s>0} \left(\int_0^s \nu^*(t)^q dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{1}{s}} \mu^*(t)^{p'} dt \right)^{\frac{1}{p'}} < \infty,$$

$$(5.4) \quad \mathbb{D}_2 := \sup_{s>0} s^{\frac{1}{p} - \frac{1}{q'}} \int_0^s \nu^*(t) dt \int_0^{1/s} \mu^*(t) dt < \infty,$$

$$(5.5) \quad \mathbb{D}_3 := \sup_{s>0} s^{\frac{1}{p} - \frac{1}{q'}} \nu^*(s) \mu^*(1/s) < \infty.$$

We remark that conditions (5.3)–(5.5) play a crucial role in studying weighted Fourier inequalities. Namely, the condition $\mathbb{D}_1 < \infty$ coincides with condition (1.6), $\mathbb{D}_2 < \infty$ is related to the condition $I < \infty$ in Theorem 4.1, and $\mathbb{D}_3 < \infty$ will be discussed in detail in Section 6.

PROOF. It is easy to see using Hölder’s inequality and monotonicity of decreasing rearrangements that $(5.3) \implies (5.4) \implies (5.5)$.

Let us show that $(5.5) \implies (5.3)$. First we assume that ν is α -regular and $\alpha < \frac{1}{q}$. Then

$$\left(\int_0^s \nu^*(t)^q dt \right)^{\frac{1}{q}} \leq C s^\alpha \nu^*(s) \left(\int_0^s t^{-\alpha q} dt \right)^{\frac{1}{q}} = C_1 \nu^*(s) s^{\frac{1}{q}}$$

and

$$\mathbb{D}_1 \lesssim \sup_{s>0} \nu^*(s) s^{\frac{1}{q}} \left(\int_s^\infty \mu^*(1/t)^{p'} \frac{dt}{t^2} \right)^{\frac{1}{p'}}.$$

We take into account that (5.5) yields

$$\mu^*(1/t) \leq \mathbb{D}_3 t^{\frac{1}{q'} - \frac{1}{p}} (\nu^*(t))^{-1}.$$

Using this and the α -regularity of ν with $\alpha < \frac{1}{q}$, we derive that

$$\mathbb{D}_1 \lesssim \mathbb{D}_3 \sup_{s>0} \nu^*(s) s^{\frac{1}{q}} \left(\int_s^\infty \left(\frac{1}{t^{\frac{1}{q}} \nu^*(t)} \right)^{p'} \frac{dt}{t} \right)^{\frac{1}{p'}} \lesssim \mathbb{D}_3.$$

Similarly, one can verify that (5.5) implies (5.3) in the case when μ is α -regular with $\alpha < \frac{1}{p'}$. \square

Note that conditions (5.4) and (5.5) are equivalent under less restrictive assumptions.

PROPOSITION 5.3. *Let $1 < p, q < \infty$ and either of the following two conditions hold:*

- (1) *the function μ is α -regular with some $\alpha < \min \left\{ \frac{1}{q'} + \frac{1}{p'}, 1 \right\}$, or*
- (2) *the function ν is α -regular with $\alpha < \min \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}$.*

Then $\mathbb{D}_2 \asymp \mathbb{D}_3$.

PROOF. Clearly, (5.4) \implies (5.5). Let first ν be α -regular with $\alpha < \min \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}$. Then $\alpha < 1$ and therefore

$$\int_0^s \nu^*(t) dt \asymp \nu^*(s)s.$$

In view of $\mu^*(1/t) \leq \mathbb{D}_3 t^{\frac{1}{q'} - \frac{1}{p}} (\nu^*(t))^{-1}$, we have

$$\begin{aligned} \mathbb{D}_2 &\asymp \sup_{s>0} s^{1+\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \int_s^\infty \mu^*(1/t) \frac{dt}{t^2} \\ &\leq \mathbb{D}_3 \sup_{s>0} s^{\frac{1}{p}+\frac{1}{q}} \nu^*(s) \int_s^\infty \frac{1}{t^{\frac{1}{p}+\frac{1}{q}} \nu^*(t)} \frac{dt}{t} \lesssim \mathbb{D}_3. \end{aligned}$$

The case when μ is α -regular with $\alpha < \min \left\{ \frac{1}{p'} + \frac{1}{q'}, 1 \right\}$ can be established similarly. \square

Finally, we remark that if $t^\alpha \mu^*(t) \leq C s^\alpha \mu^*(s)$ holds for all $t \leq s$, then the same property holds for $\mu^{**}(t) = \frac{1}{t} \int_0^t \mu^*(x) dx$ as well. More precisely, we have the following

PROPOSITION 5.4. *If μ is α -regular for some positive α , then $t^\alpha \mu^{**}(t) \leq C s^\alpha \mu^{**}(s)$, $0 < t < s$, or equivalently, for all $\lambda > 1$,*

$$(5.6) \quad \int_0^\xi \mu^*(t) dt \leq \frac{C}{\lambda^{1-\alpha}} \int_0^{\lambda\xi} \mu^*(t) dt.$$

PROOF. Indeed, for $\lambda > 1$ we have

$$\int_0^\xi \mu^*(t) dt = \frac{1}{\lambda} \int_0^{\lambda\xi} \mu^*(t/\lambda) dt \leq C \frac{1}{\lambda^{1-\alpha}} \int_0^{\lambda\xi} \mu^*(t) dt.$$

\square

Note that if μ is 1-regular, then the reverse statement also holds, namely, condition (5.6) with any fixed positive α implies that μ is α -regular. Indeed, in this case one has

$$\mu^*(\xi)\xi \asymp \int_0^\xi \mu^*(t) dt \lesssim \frac{1}{\lambda^{1-\alpha}} \int_0^{\lambda\xi} \mu^*(t) dt \asymp \mu^*(\lambda\xi) \frac{\lambda\xi}{\lambda^{1-\alpha}}.$$

6. Hörmander-type conditions for weak and strong type results

We start with two applications of Theorem 4.1, namely restricted weak and strong (p, q) boundedness of the operator A . We would like to stress that in both results (6.1) and (6.3) below the condition on weights depends only on $1/p + 1/q$ and therefore for every pairs (p_1, q_1) and (p_2, q_2) such that $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ we arrive at the same results.

THEOREM 6.1. *Let an operator T be of strong type $(1, \infty)$ and of restricted weak type $(2, 2)$. Let $1 < p, q < \infty$ be such that either $p \leq 2$, or $q \geq 2$. Suppose that either of the following two conditions hold:*

- (1) *the function μ is α -regular with $\alpha \leq \frac{1}{p'}$, or*
- (2) *the function ν is α -regular with $\alpha \leq \frac{1}{q}$.*

Then the operator A is bounded from $L_{p,1}$ to $L_{q,\infty}$ and moreover,

$$(6.1) \quad \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \lesssim \sup_{s>0} s^{\frac{1}{p} - \frac{1}{q'}} \nu^*(s) \mu^*(1/s).$$

PROOF. Recall that \mathbb{D}_2 and \mathbb{D}_3 are given by (5.4) and (5.5) respectively. Let μ be α -regular with $0 < \alpha \leq \frac{1}{p'}$. Then $\mu(x)x^{\frac{1}{p'}}$ is almost increasing. Hence, in light of Proposition 5.4, μ satisfies the following condition

$$(6.2) \quad \int_0^\xi \mu^*(t) dt \lesssim \frac{1}{\lambda^{1/p}} \int_0^{\lambda\xi} \mu^*(t) dt$$

for all $\lambda > 1$.

Theorem 4.1 and condition (6.2) imply

$$\begin{aligned} \|A\|_{L_{p,1} \rightarrow L_{q,\infty}} &\lesssim \sup_{\xi \cdot \eta \leq 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^\eta \mu^*(t) dt \\ &\lesssim \sup_{\xi \cdot \eta \leq 1} \frac{\eta^{1/p-1/p}}{\xi^{\frac{1}{q'} - \frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^{1/\xi} \mu^*(t) dt = \mathbb{D}_2. \end{aligned}$$

Moreover, it is clear that if $\alpha \leq \frac{1}{p'}$, then the condition $\alpha < \min\left\{\frac{1}{q'} + \frac{1}{p'}, 1\right\}$, is valid and thus Proposition 5.3 yields that $\mathbb{D}_2 \asymp \mathbb{D}_3$.

If ν is α -regular with $0 < \alpha \leq \frac{1}{q}$, then we use similar reasonings taking into account that

$$\int_0^\xi \nu^*(t) dt \lesssim \frac{1}{\lambda^{1/q'}} \int_0^{\lambda\xi} \nu^*(t) dt,$$

and therefore,

$$\sup_{\xi \cdot \eta \leq 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^\eta \mu^*(t) dt \lesssim \sup_{\xi \cdot \eta \leq 1} \frac{\xi^{1/q'-1/q'}}{\eta^{\frac{1}{p}-\frac{1}{q'}}} \int_0^{1/\eta} \nu^*(t) dt \int_0^\eta \mu^*(t) dt.$$

Theorem 4.1 completes the proof. \square

Similar strong type results for Lorentz spaces require slightly more restrictive conditions.

THEOREM 6.2. *Let an operator T be of strong type $(1, \infty)$ and of restricted weak type $(2, 2)$. Let $1 < p, q < \infty$ be such that either $p < 2$, or $q > 2$. Suppose that either of the following two conditions hold:*

- (1) *the function μ is α -regular with $\alpha < \frac{1}{p'}$, or*
- (2) *the function ν is α -regular with $\alpha < \frac{1}{q}$.*

Then the operator A is bounded from $L_{p,\tau}$ to $L_{q,\tau}$, $0 < \tau \leq \infty$, and moreover,

$$(6.3) \quad \|A\|_{L_{p,\tau} \rightarrow L_{q,\tau}} \lesssim \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s).$$

We will show in Subsection 6.1 below that conditions (1) and (2) of Theorem 6.2 are sharp.

PROOF. For any admissible (p, q) , one can find two couples (p_0, q_0) and (p_1, q_1) satisfying all conditions of the corollary (in particular, either (1) or (2) holds for these parameters) and such that $p_0 < p < p_1$, $q_0 < q < q_1$, and

$$\frac{1}{q'_0} - \frac{1}{p_0} = \frac{1}{q'_1} - \frac{1}{p_1} = \frac{1}{q'} - \frac{1}{p}.$$

Then Theorem 6.1 implies that $A : L_{p_i,1}(\mathbb{R}^n) \rightarrow L_{q_i,\infty}(\mathbb{R}^n)$, $i = 0, 1$ and $\|A\|_{L_{p_i,1} \rightarrow L_{q_i,\infty}} \lesssim \mathbb{D}_3$. Finally, the Marcinkiewicz multiplier theorem [BL, Th. 5.3.4] implies (6.3). \square

Similarly to Corollary 4.2, assuming certain monotonicity-type conditions on the weights, we obtain necessary and sufficient conditions for A to be bounded from L_{p,τ_1} to L_{q,τ_2} .

COROLLARY 6.3. *Under conditions of Theorem 6.2, if for some $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n$ one has*

$$(6.4) \quad \int_0^{|B_r(x_0)|} \nu^*(t) dt \lesssim \int_{B_r(x_0)} \nu(x) dx \quad \text{and} \quad \int_0^{|B_r(y_0)|} \mu^*(t) dt \lesssim \int_{B_r(y_0)} \mu(x) dx,$$

then, for any $1 \leq \tau_1 \leq \tau_2 \leq \infty$,

$$\|A\|_{L_{p,\tau_1} \rightarrow L_{q,\tau_2}} \asymp \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s)$$

and, in particular, $\|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \asymp \|A\|_{L_{p,\tau_1} \rightarrow L_{q,\tau_2}}$. If, additionally, $p \leq q$, then

$$\|A\|_{L_{p,1} \rightarrow L_{q,\infty}} \asymp \|A\|_{L_p \rightarrow L_q} \asymp \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s).$$

PROOF. The second part of theorem readily follows from the first one. To show that

$$\|A\|_{L_{p,\tau} \rightarrow L_{q,\tau}} \asymp \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s),$$

in light of Theorems 4.1 and 6.2, it is enough to verify that

$$\mathbb{D}_3 \lesssim \sup_{x_0, y_0 \in \mathbb{R}^n} \sup_{\xi \eta \leq 1} \frac{1}{|B_\xi(0)|^{1/q'} |B_\eta(0)|^{1/p}} \int_{B_\xi(y_0)} \nu(y) dy \int_{B_\eta(x_0)} \mu(x) dx,$$

where we take into account that $\mathbb{D}_2 \asymp \mathbb{D}_3$ by Proposition 5.2.

Let x_0 and y_0 be such that (6.4) hold. Then

$$\begin{aligned} & \sup_{x_0, y_0 \in \mathbb{R}^n} \sup_{\xi \eta \leq 1} \frac{1}{|B_\xi(0)|^{1/q'} |B_\eta(0)|^{1/p}} \int_{B_\xi(y_0)} \nu(y) dy \int_{B_\eta(x_0)} \mu(x) dx \\ & \gtrsim \sup_{\xi \eta \leq w_n^2} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^\eta \mu^*(t) dt \gtrsim \sup_{\xi \eta \leq 1} \frac{1}{\xi^{\frac{1}{q'}} \eta^{\frac{1}{p}}} \int_0^\xi \nu^*(t) dt \int_0^\eta \mu^*(t) dt \\ & \gtrsim \sup_{k \in \mathbb{Z}} 2^{k(1/p-1/q')} \int_{2^{k-1}}^{2^k} \nu^*(t) dt \int_{2^{-k-1}}^{2^{-k}} \mu^*(t) dt \asymp \mathbb{D}_3, \end{aligned}$$

where in the last equivalence we have used that either μ or ν is α -regular. \square

REMARK 6.4. Since any radial weights $\nu(x) = \nu_0(|x|)$ and $\mu(x) = \mu_0(|x|)$ such that ν_0 and μ_0 are non-increasing on \mathbb{R}_+ satisfy (6.4), then $\|A\|_{L_p \rightarrow L_q} \asymp \sup_{s>0} s^{\frac{1}{p} - \frac{1}{q'}} \nu^*(s) \mu^*(1/s)$ provided that conditions of Theorem 6.2 hold. For example, let $1 < p, q < \infty$ be such that either $p < 2$, or $q > 2$ and take $\mu(x) = |x|^{-\sigma}$, $\nu(y) = |y|^{-\gamma}$ with $0 \leq \gamma < \frac{n}{q}$, $0 \leq \sigma < \frac{n}{p'}$, $\sigma - \gamma = n - n(\frac{1}{p} + \frac{1}{q})$. Then the operator $Af(y) = |y|^{-\gamma} \int_{\mathbb{R}^n} |x|^{-\sigma} f(x) e^{-ixy} dx$ is bounded from L_{p,τ_1} to L_{q,τ_2} , $1 \leq \tau_1 \leq \tau_2 \leq \infty$. For $1 < p \leq q < \infty$ we arrive at the classical Pitt inequality [St1] given by $\left(\int_{\mathbb{R}^n} (|y|^{-\gamma} |\widehat{f}(y)|)^q dy \right)^{1/q} \lesssim \left(\int_{\mathbb{R}^n} (|x|^\sigma |f(x)|)^p dx \right)^{1/p}$.

6.1. Comparison of Theorems 1.1, 1.2, 6.2 and Corollary 6.3. We first compare Corollary 6.3 and Theorem 1.1.

As in Theorem 1.1, let $w(x) = w_0(|x|)$ be such that w_0 is nondecreasing on $(0, \infty)$. Considering a particular choice of weights (μ, ν) (see Remark 6.4):

$$\mu(x) = \mu_0(|x|) = 1/w(x) = 1/w_0(|x|)$$

and

$$\nu(x) = \nu_0(|x|) = |x|^{-n(1/q-1/p')} w_0(1/|x|),$$

we observe that (1.24) can be written as $\|Af\|_q \lesssim \|f\|_p$, where $Af(y) := \nu(y) \int_{\mathbb{R}^n} f(x) e^{-ixy} \mu(x) dx$, cf. (1.14).

We are going to show that the condition $w^p \in A_p$ is in fact equivalent to either of the conditions (1) and (2) of Theorem 6.2. First, in view of relationship between ν , μ , and w , we note that, for any $\varepsilon > 0$,

$$\nu_0(s^{1/n}) s^{1/q-\varepsilon} = \frac{s^{1/p'-\varepsilon}}{\mu_0(s^{-1/n})}$$

and hence,

$$\nu^*(s) s^{1/q-\varepsilon} = \frac{\omega_n^{1/q-1/p'}}{\mu^*(\omega_n^2/s) (1/s)^{1/p'-\varepsilon}}.$$

Therefore, conditions (1) and (2) are equivalent for our choice of μ and ν .

Let now $w^p \in A_p$. Since for any function $w \in A_p$, there is $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$ (see [St2, p.202]) and using the fact that $w \in A_p$ implies

$$\int_{|x| \geq s} |x|^{-np} w(x) dx \lesssim s^{-np} \int_{s \geq |x|} w(x) dx$$

(see [HS, Lemma 2.3]), we derive the following result: if $w^p \in A_p$, then

$$\int_s^\infty t^{n-n(p-\varepsilon)-1} w_0^p(t) dt \lesssim s^{-n(p-\varepsilon)} \int_0^s t^{n-1} w_0^p(t) dt.$$

Using the monotonicity of w_0 , this gives for any $z > s$

$$z^{n-n(p-\varepsilon)} w_0^p(z) \lesssim \int_z^{2z} t^{n-n(p-\varepsilon)-1} w_0^p(t) dt \lesssim s^{n-n(p-\varepsilon)} w_0^p(s).$$

In other words, we arrive at the following condition:

$$(6.5) \quad \text{there exists } \varepsilon > 0 \text{ such that } s^{n-n(p-\varepsilon)} w_0^p(s) \text{ is almost decreasing.}$$

In particular, this implies that $w_0(2s) \asymp w_0(s)$. Taking into account that $w_0(|x|) = 1/\mu_0(|x|)$, (6.5) can be equivalently written as

$$\text{there exists } \varepsilon > 0 \text{ such that } \mu_0(s^{1/n}) s^{1/p'-\varepsilon} \text{ is almost increasing.}$$

The latter is in fact the condition (1) of Theorem 6.2.

On the other hand, it is a routine to check that condition (6.5) implies that $w^p \in A_p$ provided that w_0 is nondecreasing. Moreover, we clearly have $\sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s) \lesssim 1$.

In particular, this comparison and Theorem 1.1 provide the sharpness of conditions (1) and (2) of Theorem 6.2.

Let us now show that Theorem 6.3 implies Theorem 1.2. Setting

$$\mu(x) = \mu_0(|x|) = 1/w(x) = 1/w_0(|x|)$$

and

$$\nu(x) = \varphi(\xi)^{1/q-1/p'} w_0(1/|\xi|),$$

we can similarly obtain that $w^p \in A_p$ and $\xi \varphi^*(\xi) \lesssim 1$ imply that condition (1) of Theorem 6.2 holds and

$$\begin{aligned} \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \nu^*(s) \mu^*(1/s) &\lesssim \sup_{s>0} s^{\frac{1}{p}-\frac{1}{q'}} \varphi^*(s/2)^{1/q-1/p'} w_0((2\omega_n)^{1/n}/s^{1/n}) \mu^*(1/s) \\ &\lesssim \sup_{s>0} \frac{w_0((2\omega_n)^{1/n}/s^{1/n})}{w_0((\omega_n)^{1/n}/s^{1/n})} \lesssim 1. \end{aligned}$$

Here we have used the known estimate $(f_1 f_2)^*(t) \leq f_1^*(t/2) f_2^*(t/2)$. Thus, $\|Af\|_q \lesssim \|f\|_p$, i.e., (1.25) holds.

Appendix A. Hardy's inequalities on monotone functions

In this section we address the characterization of the three-weight Hardy's inequality

$$(6.6) \quad \left(\int_0^\infty \left(\nu(x) \int_0^x f(y) \mu(y) dy \right)^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty [f(x)w(x)]^p dx \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^\downarrow,$$

(see, e.g., [GS]). Here \mathfrak{M}^\downarrow is the cone of all non-increasing Lebesgue-measurable functions on \mathbb{R}_+ . Let

$$M(t) := \int_0^t \mu, \quad W(t) := \int_0^t w^p.$$

THEOREM 6.5. [GS] *Let C be the smallest possible constant in inequality (6.6).*

(a) *If $1 < p \leq q < \infty$, then $C \asymp A_0 + A_1$, where*

$$A_0 = \sup_{t>0} A_0(t) := \sup_{t>0} \left(\int_0^t M^q \nu^q \right)^{\frac{1}{q}} W^{-\frac{1}{p}}(t)$$

and

$$A_1 := \sup_{t>0} \left(\int_t^\infty \nu^q \right)^{\frac{1}{q}} \left(\int_0^t \left(\frac{M}{W} \right)^{p'} w^p \right)^{\frac{1}{p'}}.$$

(b) *If $0 < q < p < \infty$ and $1 < p < \infty$, then $C \asymp B_0 + B_1$, where*

$$B_0 = B_0(p, q) := \left(\int_0^\infty W^{-\frac{r}{p}}(t) \left(\int_0^t M^q \nu^q \right)^{\frac{r}{p}} M^q(t) \nu^q(t) dt \right)^{\frac{1}{r}}$$

and

$$B_1 = B_1(p, q) := \left(\int_0^\infty \left(\int_t^\infty \nu^q \right)^{\frac{r}{p}} \left(\int_0^t \left(\frac{M}{W} \right)^{p'} w^p \right)^{\frac{r}{p'}} \nu^q(t) dt \right)^{\frac{1}{r}}.$$

(c) If $0 < q < p \leq 1$, then $C \asymp B_0 + B'_1$, where

$$B'_1 = B'_1(p, q) := \left(\int_0^\infty \left(\operatorname{esssup}_{s \in [0, t]} \frac{M^p(s)}{W(s)} \right)^{\frac{r}{p}} \left(\int_t^\infty \nu^q \right)^{\frac{r}{p}} \nu^q(t) dt \right)^{\frac{1}{r}}.$$

(d) If $0 < p \leq q < \infty$ and $0 < p \leq 1$, then

$$C = \sup_{t>0} W^{-\frac{1}{p}}(t) \left(\int_0^\infty M^q(\min\{s, t\}) \nu^q(s) ds \right)^{\frac{1}{q}}.$$

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