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Bernstein inequalities with nondoubling weights

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ABSTRACT. We answer Totik's question on weighted Bernstein's inequalities showing that

$$\|T'_n\|_{L_p(\omega)} \leq C(p, \omega) n \|T_n\|_{L_p(\omega)}, \quad 0 < p \leq \infty,$$

holds for all trigonometric polynomials T_n and certain nondoubling weights ω . Moreover, we find necessary conditions on ω for Bernstein's inequality to hold. We also prove weighted Bernstein-Markov, Remez, and Nikolskii inequalities for trigonometric and algebraic polynomials.

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1. Introduction

The famous Bernstein inequality for trigonometric polynomials T_n of degree at most n

$$(1.1) \quad \|T'_n\|_{L_p(\mathbb{T})} \leq Cn \|T_n\|_{L_p(\mathbb{T})}$$

plays an important role in the modern analysis. Here, $\|\cdot\|_{L_p(\mathbb{T})}$ is the L_p -(quasi)norm, i.e.,

$$\|f\|_{L_p(\mathbb{T})} = \left(\int_{\mathbb{T}} |f(t)|^p dt \right)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification for $p = \infty$. Bernstein proved (1.1) for $p = \infty$; the case $p < \infty$ was done by Zygmund [Zy]. The best constant C is equal to 1 for any $p \in (0, \infty]$, see [Ri, Zy, Ar].

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For algebraic polynomials P_n of degree at most n , the Bernstein inequality is given by

$$|P'_n(x)| \leq \frac{n}{\sqrt{1-x^2}} \|P_n\|_{C[-1,1]}, \quad x \in (-1, 1),$$

where $\|\cdot\|_{C[-1,1]}$ denotes the supremum norm on the $[-1, 1]$. Its L_p -version is written as follows:

$$(1.2) \quad \|\sqrt{1-x^2} P'_n(x)\|_{L_p[-1,1]} \leq C(p)n \|P_n\|_{L_p[-1,1]}, \quad 0 < p \leq \infty.$$

Another important inequality for the derivative of algebraic polynomials is the following Markov inequality:

$$(1.3) \quad \|P'_n\|_{L_p[-1,1]} \leq C(p)n^2 \|P_n\|_{L_p[-1,1]}, \quad 0 < p \leq \infty.$$

Both Bernstein and Bernstein–Markov inequalities for trigonometric and algebraic polynomials respectively were extended to the case of smaller intervals (Privalov, Jackson, and Bary; see, e.g., [Ba]) and several intervals (see the recent paper by Totik [To1]).

In this paper we study weighted analogues of Bernstein's inequality

$$(1.4) \quad \|T'_n\|_{L_p(\omega)} \leq C(p, \omega)n \|T_n\|_{L_p(\omega)},$$

where ω is a weight function, i.e., a nonnegative integrable function on \mathbb{T} . Here and in what follows, $\|T_n\|_{L_p(\omega)} = (\int_{\mathbb{T}} |T_n|^p \omega)^{1/p}$ if $p < \infty$ and $\|T_n\|_{L_\infty(\omega)} = \text{ess sup}_{t \in \mathbb{T}} |T_n(t)\omega(t)|$.

First, we note that Muckenhoupt's A_p condition on weights ensures that (1.4) holds for $1 < p < \infty$. This follows from the fact that the Marcinkiewicz multiplier theorem and Littlewood–Paley decomposition hold in $L_p(\omega)$ with $\omega \in A_p$. In [MT], Mastroianni and Totik proved a much stronger result that for any weight ω satisfying the *doubling* condition and for $1 \leq p < \infty$ inequality (1.4) holds. Later, a similar result was shown for $0 < p < 1$ (see [Er3]).

We recall that a periodic weight function ω satisfies the doubling condition if

$$(1.5) \quad W(2I) \leq L W(I)$$

for all intervals I , where L is a constant independent of I , $2I$ is the interval twice the length of I and with the midpoint coinciding with that of I , and

$$W(I) = \int_I \omega(t) dt.$$

Let us also recall that a weight ω satisfies the A_∞ condition if for every $\alpha > 0$ there is $\beta > 0$ such that

$$W(E) \geq \beta W(I)$$

for any interval I and any measurable set $E \subset I$ with $|E| \geq \alpha|I|$. It is known [St, Ch. V] that any A_∞ weight satisfies the doubling condition. Here and in what follows, $|E|$ denotes the Lebesgue measure of the set E .

For the supremum norm, in addition to the natural assumption that ω is bounded, one needs the A^* condition, i.e., there exists a constant L such that for all intervals $I \subset [-\pi, \pi]$ and $t \in I$ we have

$$\omega(t) \leq \frac{L}{|I|} W(I).$$

This condition is stronger than the A_∞ condition and it is sufficient for (1.4) to hold when $p = \infty$.

In [To2], Totik posed the following question: under which condition on a general (not necessary doubling) weight ω does the Bernstein inequality (1.4) hold for any trigonometric polynomial T_n of degree at most n ? In this paper we aim to answer this question. We deal with the weight functions from the class Ω .

DEFINITION. *Let*

$$\omega(t) = \exp(-F(g(t))), \quad t \in \mathbb{T},$$

where $g : \mathbb{T} \rightarrow [-A, A]$, $A > 0$, is an analytic function, e.g.,

$$(1.6) \quad |g^{(n)}(t)| \leq D^n n!, \quad t \in \mathbb{T}, \quad n = 1, 2, \dots,$$

such that each zero of g is of multiplicity one. Let also $F : [-A, A] \setminus \{0\} \rightarrow (0, \infty)$ be an even C^∞ function on $(0, A]$ such that

$$(F1) \quad F(x) \rightarrow \infty \quad \text{as } x \rightarrow 0+;$$

$$(F2) \quad F \quad \text{is decreasing on } (0, A];$$

$$(F3) \quad |F^{(n)}(x)| \leq B^n n^n \frac{F(x)}{x^n}, \quad x \in (0, A], \quad n = 1, 2, \dots;$$

$$(F4) \quad \text{there exist } A_1, A_2 > 0 \quad \text{such that}$$

$$A_2 \leq \frac{|F'(x)|x}{F(x)} \leq A_1, \quad x \in (0, A].$$

Then we write that $\omega \in \Omega$.

It is worth mentioning that all our results hold for weights $\omega(t) = \exp(-F(g(t)))$, where F satisfies $(F1) - (F4)$ only for $x \in (0, \varepsilon)$ for some $0 < \varepsilon < A$ and

$$|F^{(n)}(x)| \leq B^n n^n F(x), \quad x \in [\varepsilon, A], \quad n = 1, 2, \dots$$

The typical example of the function g is $\sin t$ or $\cos t$. Note that $\omega \in \Omega$ is nondoubling if and only if g has at least one zero on \mathbb{T} . In what follows this will be assumed to be the case. Below we give some examples of a function F satisfying properties $(F1) - (F4)$. Consider a positive even function F defined on $(0, A]$.

EXAMPLES.

1. Let

$$F(x) = x^{-\alpha}, \quad x^{-\alpha} |\log x|^{\xi_1}, \quad x^{-\alpha} |\log x|^{\xi_1} \dots |\log_k x|^{\xi_k}, \quad x^{-\alpha} \exp |\log x|^\xi,$$

where $\alpha > 0$, $\xi_j \in \mathbb{R}$, $\xi \in (0, 1)$, and $\log_j x = \log_{j-1} |\log x|$. Note that any such a function F is of regular variation of index $-\alpha$, i.e., for all $r > 0$,

$$(1.7) \quad \lim_{x \rightarrow 0+} \frac{F(rx)}{F(x)} = r^{-\alpha},$$

or, equivalently,

$$F(x) = \frac{1}{x^\alpha} \eta(x),$$

where η is a slowly varying function, i.e., $\lim_{x \rightarrow 0+} \frac{\eta(rx)}{\eta(x)} = 1$.

2. Note that there are functions satisfying $(F1) - (F4)$ which are not regularly varying. For example, the function

$$F(x) = \exp \left\{ -\log x (2 + \sin(\log_3 x)) \right\}$$

is such that

$$\limsup_{x \rightarrow 0+} F(x)x^3 = 1$$

and

$$\liminf_{x \rightarrow 0+} F(x)x = 1,$$

i.e., (1.7) does not hold. To show that F satisfies $(F3)$ one can use Faà di Bruno's formula.

The main results of the paper are the following Theorems 1.1–1.3.

THEOREM 1.1. *For $0 < p \leq \infty$ and $\omega = \omega_1 \dots \omega_s$ such that $\omega_i \in \Omega$, $i = 1, \dots, s$, the Bernstein inequality*

$$(1.8) \quad \|T'_n\|_{L_p(\omega u)} \leq C n \|T_n\|_{L_p(\omega u)}$$

holds for any trigonometric polynomial of degree at most n with $C = C(\omega, u, p)$, whenever u is doubling if $p < \infty$, and u satisfies the A^ condition if $p = \infty$.*

For example, inequality (1.8) holds for the following weight:

$$\omega(t) = \exp(-1/\sin^2 t - 1/\cos^4 t).$$

To prove Bernstein's inequality (1.8) in the case when $\omega = \omega_1 \in \Omega$, i.e., $s = 1$, we use approximation properties of ω . To verify (1.8) with the product of weights each of which is from the class Ω , we need a new technique based on introduction of weighted classes for which Bernstein and Remez inequalities hold. In particular, $\omega_i \in \Omega$ and u as in Theorem 1.1 belong to these classes. This technique is developed in Sections 5 and 6.

A necessary condition for Bernstein's inequality is given by the following result.

THEOREM 1.2. *Let $\omega \in C(\mathbb{T})$ be an arbitrary weight function satisfying the following conditions: $\omega \searrow$ on $(-\epsilon, 0)$, $\omega(0) = 0$, $\omega \nearrow$ on $(0, \epsilon)$, and moreover,*

$$(1.9) \quad \limsup_{t \rightarrow 0} \frac{\log \omega(rt)}{\log \omega(t)} = \infty \quad \text{for some } r \in (0, 1).$$

Then for each $0 < p \leq \infty$ there exist a sequence of positive integers $K_n \rightarrow \infty$ as $n \rightarrow \infty$, and a sequence of trigonometric polynomials Q_n of degree at most K_n such that

$$\lim_{n \rightarrow \infty} \frac{\|Q'_n\|_{L_p(\omega)}}{K_n \|Q_n\|_{L_p(\omega)}} = \infty.$$

Theorem 1.1 and Theorem 1.2 provide a sharp condition on the growth properties of a weight ω near the origin. Specifically, if a weight ω , satisfying $\omega \searrow$ on $(-\epsilon, 0)$, $\omega(0) = 0$, $\omega \nearrow$ on $(0, \epsilon)$, is such that Bernstein's inequality (1.4) holds, then ω necessarily satisfies the following condition: for all $r \in (0, 1)$,

$$(1.10) \quad \limsup_{t \rightarrow 0} \frac{\log \omega(rt)}{\log \omega(t)} = L < \infty.$$

On the other hand, any $\omega \in \Omega$ satisfies (1.10). Moreover, for each $r \in (0, 1)$ and $L > 1$, the weight $\omega(t) = \exp(-|\sin t|^{-\alpha})$ fulfills (1.10) with $\alpha = -\log_r L$. Then $\omega \in \Omega$ and by Theorem 1.1 Bernstein's inequality (1.4) holds for this weight.

If in (1.9) the limit (not only the limit superior) exists, then a stronger result is true:

THEOREM 1.3. *Let $\omega \in C(\mathbb{T})$ be an arbitrary weight function satisfying the following conditions: $w \searrow$ on $(-\epsilon, 0)$, $\omega(0) = 0$, $\omega \nearrow$ on $(0, \epsilon)$ and moreover,*

$$\lim_{t \rightarrow 0} \frac{\log \omega(rt)}{\log \omega(t)} = \infty \quad \text{for each } r \in (0, 1).$$

Then for each $0 < p \leq \infty$ there exists a sequence of trigonometric polynomials Q_n of degree at most n such that

$$\lim_{n \rightarrow \infty} \frac{\|Q'_n\|_{L_p(\omega)}}{n \|Q_n\|_{L_p(\omega)}} = \infty.$$

The paper is organized as follows. In Section 2 we discuss growth properties of weights from the class Ω which we will use further on. Section 3 presents the order of trigonometric approximation of functions from Ω as well as their derivatives. In Section 4 we give the proof of Bernstein's inequality with Ω weights in L_1 . We will use it as a model case to prove the general Bernstein inequality (1.8) in Section 6.

In Section 5 we establish weighted Remez inequalities for trigonometric and algebraic polynomials. Section 6 gives the proof of the general Bernstein inequality for $p \in (0, \infty]$. Theorem 1.1 is a corollary of this result. In Section 7 we study the weighted Bernstein and Markov inequalities for algebraic polynomials on $[-1, 1]$. Section 8 provides weighted Nikolskii's inequalities for trigonometric and algebraic polynomials.

Finally, in Section 9 we prove a necessary condition for Bernstein's inequality (1.4) to hold. Namely, we verify Theorems 1.2 and 1.3 as well as a result on the sharpness of Theorem 1.2.

Concerning algebraic polynomials on $[-1, 1]$, it is important to mention that for weights from the class \mathcal{W} the Markov-Bernstein inequalities were obtained by Lubinsky and Saff (cf. [LS] and the book [LL]), see discussion in Section 7. A typical example of weights from the class \mathcal{W} is $\omega_\alpha(x) = \exp(-(1-x^2)^\alpha)$, $\alpha > 0$. We note that using [LS] one can also derive weighted Bernstein's inequality

$$\|\sqrt{1-x^2} P'_n(x) \omega_\alpha(x)\|_{L_\infty[-1,1]} \leq C(\alpha) n \|P_n(x) \omega_\alpha(x)\|_{L_\infty[-1,1]},$$

see Remark 7.1. We also note that Bernstein's inequalities for algebraic polynomials were recently proved in [MN, No] for the weight $\omega = \omega_\alpha u$, where u is doubling. In Section 7 we deal with a more general class of weights. Our proof for the algebraic case is based on Bernstein's inequality for trigonometric polynomials from Section 6.

By C, C_i (c, c_i) we will denote positive large (small, respectively) constants that may be different on different occasions. Also, below we will write that $C(\omega)$ if $C(A, A_1, A_2, B, D)$, where A, A_1, A_2, B, D are from the definition of the class Ω . Moreover, for the positive sequences $\{a_n\}$ and $\{b_n\}$, $a_n \asymp b_n$ means that

$$c \leq \frac{a_n}{b_n} \leq C.$$

2. Growth properties of Ω -functions

Let $F : [-A, A] \setminus \{0\} \rightarrow (0, \infty)$ be an even C^∞ function on $(0, A]$ satisfying (F_1) – (F_4) .

DEFINITION. For each $n \geq F(A)$ we denote by $x_0(n)$ a unique positive solution of the equation

$$F(x) = n.$$

DEFINITION. For each $n \geq F(A)/A$ we denote by $x_1(n)$ a unique positive solution of the equation

$$F(x) = nx.$$

Note that both sequences $\{x_0(n)\}$ and $\{x_1(n)\}$ are decreasing.

LEMMA 2.1. There exist positive constants $C = C(A, A_1, A_2)$ and $c = c(A, A_1, A_2)$ such that

$$cx^{-A_2} < F(x) < Cx^{-A_1}, \quad x \in (0, A].$$

PROOF. By property (F_4) we have

$$|F'(x)| = -F'(x) \geq A_2 \frac{F(x)}{x}, \quad x \in (0, A].$$

Therefore,

$$\log F(x) - \log F(A) = \int_x^A -(\log F(t))' dt \geq \int_x^A \frac{A_2}{t} dt = A_2(\log A - \log x),$$

which yields

$$F(x) \geq F(A)A^{A_2}x^{-A_2}.$$

Similarly, the inequality

$$|F'(x)| \leq A_1 \frac{F(x)}{x}, \quad x \in (0, A],$$

implies

$$F(x) \leq F(A)A^{A_1}x^{-A_1}.$$

□

LEMMA 2.2. For each $R > 0$ there exist positive constants $C = C(R, A_1, A_2)$ and $c = c(R, A_1, A_2)$ such that

$$cx_0(Rn) < x_0(n) < Cx_0(Rn)$$

for all n large enough.

PROOF. Let us prove the lemma for $R \geq 1$. For $R < 1$ the proof is similar. Since F is decreasing on $(0, A]$ we take $c = 1$. So, it is enough to show that $x_0(n) < Cx_0(Rn)$. By definition of $x_0(n)$ and (F_4) we have

$$\begin{aligned} (R-1)n &= |F(x_0(n)) - F(x_0(Rn))| = \int_{x_0(Rn)}^{x_0(n)} -F'(t) dt \\ &\geq A_2 \int_{x_0(Rn)}^{x_0(n)} F(t) \frac{dt}{t} \geq A_2 n (\log x_0(n) - \log x_0(Rn)). \end{aligned}$$

Thus, one may choose $C = \exp((R-1)/A_2)$.

□

LEMMA 2.3. There exists a positive constant $\alpha = \alpha(A, A_1, A_2)$ such that

$$nx_1(n) \geq n^\alpha$$

for all n large enough.

PROOF. Since $F(x_1(n)) = nx_1(n)$ the proof of the lemma immediately follows from Lemma 2.1. □

By monotonicity of F we have $x_1(2n) < x_1(n)$ and hence, $2nx_1(2n) = F(x_1(2n)) > F(x_1(n)) = nx_1(n)$. In other words, $x_1(2n) < x_1(n) < 2x_1(2n)$. However, the following stronger statement holds.

LEMMA 2.4. There exists a positive constant $\epsilon = \epsilon(A, A_1, A_2)$ such that

$$(1 + \epsilon)x_1(2n) < x_1(n) < (2 - \epsilon)x_1(2n)$$

for all n large enough.

PROOF. Note that both $x_0(n)$ and $x_1(n)$ are monotonically decreasing to zero. First, we show that

$$(1 + \epsilon)x_1(2n) < x_1(n).$$

By definition of $x_1(n)$ and (F4) we have

$$\begin{aligned} 2nx_1(2n) - nx_1(n) &= F(x_1(2n)) - F(x_1(n)) = \int_{x_1(2n)}^{x_1(n)} -F'(t)dt \\ &\leq A_1 \int_{x_1(2n)}^{x_1(n)} \frac{F(t)}{t} dt \leq A_1(x_1(n) - x_1(2n)) \frac{F(x_1(2n))}{x_1(2n)} = 2nA_1(x_1(n) - x_1(2n)). \end{aligned}$$

Hence,

$$(2.1) \quad (2 + 2A_1)x_1(2n) \leq x_1(n)(2A_1 + 1).$$

Similarly,

$$\begin{aligned} 2nx_1(2n) - nx_1(n) &\geq A_2 \int_{x_1(2n)}^{x_1(n)} \frac{F(t)}{t} dt \\ &\geq A_2(x_1(n) - x_1(2n)) \frac{F(x_1(n))}{x_1(n)} = nA_2(x_1(n) - x_1(2n)), \end{aligned}$$

which gives

$$(2.2) \quad (1 + A_2)x_1(n) \leq (2 + A_2)x_1(2n).$$

Finally, by (2.1) and (2.2) we take

$$\epsilon = \frac{1}{2} \min \left\{ \frac{1}{1 + 2A_1}, \frac{A_2}{1 + A_2} \right\}.$$

□

COROLLARY 2.1. *For each $C > 0$ there exists $K = K(C, A, A_1, A_2)$ such that*

$$(2.3) \quad Cx_1(n) < Kx_1(Kn)$$

for all n large enough.

PROOF. By Lemma 2.4 (the right-hand side estimate) there exists a positive constant $\delta = \delta(A, A_1, A_2)$ such that $2x_1(2n) > (1 + \delta)x_1(n)$. Take an integer m such that $(1 + \delta)^m > C$. Then

$$2^m x_1(2^m n) > (1 + \delta)^m x_1(n) > Cx_1(n),$$

which is (2.3) with $K = 2^m$. □

Similarly, using Lemma 2.4 (the left-hand side estimate), we get

COROLLARY 2.2. *For each $L > 0$ there exists $Q = Q(L, A, A_1, A_2)$ such that*

$$(2.4) \quad x_1(Qn) < \frac{x_1(n)}{L}$$

for all n large enough.

COROLLARY 2.3. *For each $K > 0$ there exists $L = L(K, A, A_1, A_2)$ such that*

$$(2.5) \quad F\left(\frac{x_1(n)}{L}\right) > Kx_1(n)n$$

for all n large enough.

PROOF. First, by (2.3) there exists $L = L(K, A, A_1, A_2)$ such that $Lnx_1(Ln) > Knx_1(n)$. Second, on account of monotonicity of F ,

$$\frac{x_1(n)}{L} \leq x_1(Ln), \quad L \geq 1.$$

Therefore,

$$F\left(\frac{x_1(n)}{L}\right) \geq F(x_1(Ln)) = Lnx_1(Ln) > Knx_1(n).$$

□

3. Approximation of Ω -functions

The aim of this section is to obtain an order of approximation of functions from the class Ω by trigonometric polynomials.

3.1. Estimates for the Fourier coefficients of $\omega \in \Omega$. We use the classical estimate for the n -th Fourier coefficient of ω :

$$(3.1) \quad |\hat{\omega}_n| = \left| \frac{1}{\pi} \int_{\mathbb{T}} \omega(t) \cos nt \, dt \right| \leq 2 \inf_{k \geq 0} \frac{\|\omega^{(k)}\|_{C(\mathbb{T})}}{n^k}, \quad n \geq 1.$$

Below we obtain a uniform upper bound of the n -th derivative of the function $\omega \in \Omega$, where $\omega(t) = H(g(t))$, $H(x) = \exp(-F(x))$. To this end, we use Faà di Bruno's formula

$$(3.2) \quad (u(v(x)))^{(k)} = \sum \frac{k!}{m_1! \dots m_k!} u^{(m_1 + \dots + m_k)}(v(x)) \left(\frac{v'(x)}{1!} \right)^{m_1} \dots \left(\frac{v^{(k)}(x)}{k!} \right)^{m_k},$$

where summation is taken over all nonnegative integers such that $m_1 + \dots + km_k = k$. We start with the following technical lemma.

LEMMA 3.1. *For each $k \in \mathbb{N}$ the following identity holds:*

$$(3.3) \quad \sum_{m_1 + 2m_2 + \dots + km_k = k} \frac{k!}{m_1! m_2! \dots m_k! (k - m_1 - \dots - m_k)!} = \frac{1}{2} \binom{2k}{k}.$$

PROOF. Denote the left-hand side of (3.3) by S_k . One can see that S_k is the coefficient of x^k of the polynomial

$$(1 + x + x^2 + \dots + x^k)^k,$$

and hence, it is equal to the coefficient of x^k in the Taylor series expansion of the function

$$f(x) = \frac{1}{(1-x)^k}.$$

Therefore,

$$S_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2} \binom{2k}{k}.$$

□

Now we are ready to estimate the maximum norm of the k -th derivative of the function H .

LEMMA 3.2. *Let $H(x) = \exp(-F(x))$, where F satisfies (F1) – (F4). Then $H \in C^\infty[-A, A]$, and there exists $C = C(A, B, A_1, A_2) > 0$ such that for all $k > F(A)$*

$$H^{(k)}(x) \leq \left(\frac{Ck}{x_0(k)} \right)^k, \quad x \in [-A, A].$$

PROOF. Consider $x \in (0, A]$. By (3.2),

$$H^{(k)}(x) = \sum_{m_1 + \dots + km_k = k} \frac{k!}{m_1! \dots m_k!} (-1)^{m_1 + \dots + m_k} \exp(-F(x)) \left(\frac{F'(x)}{1!} \right)^{m_1} \dots \left(\frac{F^{(k)}(x)}{k!} \right)^{m_k}.$$

By (F3) we have

$$\frac{|F^{(s)}(x)|}{s!} \leq C^s \frac{F(x)}{x^s}, \quad 1 \leq s \leq k.$$

Hence,

$$(3.4) \quad \begin{aligned} |H^{(k)}(x)| &\leq C^k \sum_{m_1 + \dots + km_k = k} \frac{k!}{m_1! \dots m_k!} \frac{H(x)(F(x))^{m_1 + \dots + m_k}}{x^k} \\ &= C^k \sum_{m_1 + \dots + km_k = k} \frac{k!}{m_1! \dots m_k!} G_{m,k}(x), \end{aligned}$$

where $m = m_1 + \dots + m_k$, and

$$G_{m,k}(x) := \frac{H(x)(F(x))^m}{x^k}, \quad x \in (0, A].$$

To estimate the maximum of $G_{m,k}(x)$ for $x \in (0, A)$, we write

$$G'_{m,k}(x) = \frac{H(x)(F(x))^{m-1}}{x^k} \left(-F'(x)F(x) + mF'(x) - \frac{k}{x}F(x) \right).$$

Therefore, if $F(x) < k/A_1$, then $G'_{m,k} < 0$. Indeed, by (F4), we get

$$-F'(x)F(x) + mF'(x) - \frac{k}{x}F(x) < F(x) \left(-F'(x) - \frac{k}{x} \right) < F(x) \left(\frac{A_1 F(x)}{x} - \frac{k}{x} \right) < 0.$$

Similarly, if $F(x) > \max\{2, 2/A_2\}k$, then $G'_{m,k} > 0$. In this case $F(x) > 2k \geq 2m$ and therefore,

$$\begin{aligned} -F'(x)F(x) + mF'(x) - \frac{k}{x}F(x) &\geq -\frac{F'(x)F(x)}{2} - \frac{k}{x}F(x) \\ &= \frac{F(x)}{2} \left(-F'(x) - \frac{2k}{x} \right) \geq \frac{F(x)}{2} \left(\frac{A_2 F(x)}{x} - \frac{2k}{x} \right) > 0. \end{aligned}$$

Using the fact that each $G_{m,k}$ is a continuously differentiable function on $(0, A]$, we get that $\max_{0 < x \leq A} G_{m,k}(x)$ exists for all $1 \leq m \leq k$ and is attained at a point x^* such that

$$(3.5) \quad \frac{k}{A_1} \leq F(x^*) \leq \max\{2, 2/A_2\}k.$$

Now, Lemma 2.1 implies that

$$G_{m,k}(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0+.$$

Then, it follows from (3.4) that $H^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, and hence $H \in C^\infty[-A, A]$. Set $R = \max\{2, 2/A_2\}$. Since F is decreasing, then we get by (3.5) that

$$G_{m,k}(x) \leq \exp(-F(A)) \frac{(Rk)^m}{x_0^k(Rk)} \leq \frac{C^k k^m}{x_0^k(k)}, \quad k > F(A).$$

Here the last inequality follows from Lemma 2.2. Combining the latter with (3.4) we obtain that

$$|H^{(k)}(x)| \leq \frac{C^k k!}{x_0^k(k)} \sum_{m_1 + \dots + m_k = k} \frac{k^{m_1 + \dots + m_k}}{m_1! \dots m_k!}.$$

Finally, taking into account that $k^{k-m} \geq (k-m)!$ for $1 \leq m \leq k$, we get by (3.3)

$$|H^{(k)}(x)| \leq \frac{C^k k!}{x_0^k(k)} \sum_{m_1 + \dots + m_k = k} \frac{k!}{m_1! \dots m_k! (k - m_1 - \dots - m_k)!} \leq \frac{C^k k!}{x_0^k(k)}, \quad x \in (0, A].$$

For $x \in [-A, 0)$ the same inequality holds because F and hence H are even. \square

We are now in a position to give the uniform estimate of $\omega^{(k)}$, where $\omega \in \Omega$.

LEMMA 3.3. *Let $\omega \in \Omega$, then there exists $C = C(\omega) > 0$ such that for all k large enough*

$$\omega^{(k)}(t) \leq \frac{C^k k^k}{x_0^k(k)}, \quad t \in \mathbb{T}.$$

PROOF. Take $k \geq F(A)$ so that $x_0(k)$ is well defined. Since g is an analytic function on \mathbb{T} , and $H \in C^\infty[-A, A]$, then $\omega \in C^\infty(\mathbb{T})$. By Faà di Bruno's formula, it follows that, for each $k \in \mathbb{N}$,

$$|\omega^{(k)}(t)| = |(H(g(t)))^{(k)}| = \sum_{m_1 + \dots + m_k = k} \frac{k!}{m_1! \dots m_k!} H^{(m)}(g(t)) \left(\frac{g'(t)}{1!} \right)^{m_1} \dots \left(\frac{g^{(k)}(t)}{k!} \right)^{m_k},$$

where $m = m_1 + \dots + m_k$. We rewrite the last sum as $\sum_{m < F(A)} + \sum_{m \geq F(A)}$. Since $H^{(m)}(x) \leq C(\omega)$ for any $m < F(A)$, we have

$$(3.6) \quad \sum_{m < F(A)} \leq C(\omega) D^k \sum_{m < F(A)} \frac{k!}{m_1! \dots m_k!} \leq C^k k! \sum_{m < F(A)} \frac{1}{m_1! \dots m_k!}.$$

To estimate $\sum_{m \geq F(A)}$, we can use

$$(3.7) \quad H^{(m)}(x) \leq \left(\frac{Cm}{x_0(m)} \right)^m, \quad m \geq F(A),$$

provided by Lemma 3.2 and (1.6) to get

$$\sum_{m \geq F(A)} \leq D^k \sum_{m_1 + \dots + km_k = k} \frac{k!}{m_1! \dots m_k!} \left(\frac{Cm}{x_0(m)} \right)^m.$$

Combining this with (3.6), we get

$$|\omega^{(k)}(t)| \leq \frac{C^k k!}{x_0^k(k)} \sum_{m_1 + \dots + km_k = k} \frac{m^m}{m_1! \dots m_k!}, \quad t \in \mathbb{T}.$$

Noting that for each integers $1 \leq m \leq k$

$$m^m \leq \frac{k^k}{(k-m)!} \leq \frac{C^k k!}{(k-m)!},$$

we have by (3.3)

$$|\omega^{(k)}(t)| \leq \frac{C^k k!}{x_0^k(k)} \sum_{m_1 + \dots + km_k = k} \frac{k!}{m_1! \dots m_k! (k - m_1 - \dots - m_k)!} \leq \frac{C^k k!}{x_0^k(k)}, \quad t \in \mathbb{T}.$$

□

The next result provides a near optimal k in estimate (3.1) for the n -th Fourier coefficient of $\omega \in \Omega$.

LEMMA 3.4. *Let F be a function satisfying (F1) – (F4). Then for each $C > e$ and n large enough there exists an integer $k = k(C, n, F)$ such that*

$$\frac{C^k k^k}{n^k x_0^k(k)} \leq \exp \left(-\frac{1}{C^2} n x_1(n) + 1 \right).$$

PROOF. Let k be the minimal integer such that

$$(3.8) \quad \frac{Ck}{n x_0(k)} > \frac{1}{e}.$$

If $k < n x_1(n)/C^2$, then

$$\frac{Ck}{n x_0(k)} < \frac{1}{C} \frac{n x_1(n)}{n x_0(\frac{1}{C^2} n x_1(n))} < \frac{1}{C} \frac{x_1(n)}{x_0(n x_1(n))} = \frac{1}{C},$$

where in the last equation we used the definitions of $x_0(n)$ and $x_1(n)$. This contradicts (3.8). Thus, $k \geq n x_1(n)/C^2$. Finally, applying again (3.8) we get

$$\left(\frac{C(k-1)}{n x_0(k-1)} \right)^{k-1} \leq \left(\frac{1}{e} \right)^{\frac{1}{C^2} n x_1(n) - 1},$$

and the claim easily follows. □

We will also need the following technical result.

LEMMA 3.5. *For each $\omega \in \Omega$ and $c > 0$ we have*

$$\sum_{v=n}^{\infty} \exp(-c v x_1(v)) \leq \exp \left(-\frac{c}{2} n x_1(n) \right)$$

for all n large enough, i.e., for $n \geq n_0(\omega, c)$.

PROOF. Indeed, since the sequence $n x_1(n)$ is increasing to infinity,

$$\sum_{v=n}^{\infty} \exp(-c v x_1(v)) = \sum_{s=0}^{\infty} \sum_{k=2^s n}^{2^{s+1} n - 1} \exp(-c k x_1(k)) \leq \sum_{s=0}^{\infty} n 2^s \exp(-c 2^s n x_1(2^s n)).$$

By Lemma 2.4 there exists $\epsilon = \epsilon(A, A_1, A_2)$ such that $2 x_1(2n) \geq (1 + \epsilon) x_1(n)$ for all n large enough. Then

$$\sum_{v=n}^{\infty} \exp(-c v x_1(v)) \leq \sum_{s=0}^{\infty} n 2^s \exp(-c(1 + \epsilon)^s n x_1(n)) := \sum_{s=0}^{\infty} h_s.$$

It is easy to check that, for $s \geq 0$ and $n \geq n_0(\omega, c)$,

$$\frac{h_{s+1}}{h_s} \leq 2 \exp(-c\epsilon n x_1(n)) \leq \frac{1}{2}.$$

Thus, Lemma 2.3 gives

$$\sum_{v=n}^{\infty} \exp(-cvx_1(v)) \leq 2h_0 = 2n \exp(-cnx_1(n)) \leq \exp\left(-\frac{c}{2}nx_1(n)\right), \quad n \geq n_0(\omega, c).$$

□

3.2. Remez inequality for trigonometric polynomials. We will need the following Remez inequality answering how large can be $\|T_n\|_{L_\infty(\mathbb{T})}$ if

$$\left| \left\{ t \in \mathbb{T} : |T_n(t)| > 1 \right\} \right| \leq \varepsilon$$

for some $0 < \varepsilon \leq 1$ holds.

LEMMA 3.6. [Er1], [Er2] *For any Lebesgue measurable set $B \subset \mathbb{T}$ such that $|B| < \pi/2$ we have*

$$(3.9) \quad \|T_n\|_{L_\infty(\mathbb{T})} \leq \exp(4n|B|) \|T_n\|_{L_\infty(\mathbb{T} \setminus B)}.$$

If $0 < p < \infty$ and $|B| < 1/4$ we have

$$(3.10) \quad \|T_n\|_{L_p(\mathbb{T})} \leq \left(1 + \exp(4n|B|p)\right) \|T_n\|_{L_p(\mathbb{T} \setminus B)}.$$

3.3. Two approximation theorems for the Ω -weights. We are now ready to prove the following result on simultaneous trigonometric approximation of functions from the class Ω and their derivatives.

THEOREM 3.1. *For each $\omega \in \Omega$ there exists a positive constant $c = c(\omega)$ such that*

$$(3.11) \quad \|\omega - \omega_n\|_{C(\mathbb{T})} \leq \exp(-cnx_1(n))$$

and

$$(3.12) \quad \|\omega' - \omega'_n\|_{C(\mathbb{T})} \leq \exp(-cnx_1(n))$$

hold for n large enough, where ω_n is the n -th partial sum of the Fourier series of ω .

PROOF. Integrating by parts and Lemma 3.3 imply that, for some $C > 0$,

$$|\hat{\omega}_n| \leq 2 \frac{\|\omega^{(k)}\|_{C(\mathbb{T})}}{n^k} \leq \frac{C^k k!}{x_0^k(k)n^k}.$$

Hence, by Lemma 3.4, there exists $c = c(\omega)$ such that, for $n \geq n_0(\omega)$,

$$|\hat{\omega}_n| \leq \exp(-cnx_1(n)).$$

Let ω_n be the n -th partial sum of the Fourier series of ω , i.e.,

$$\omega_n(t) = \frac{\hat{\omega}_0}{2} + \sum_{k=1}^n \hat{\omega}_k \cos kt.$$

Since $\omega \in C^\infty(\mathbb{T})$ then for each $t \in \mathbb{T}$

$$\omega_n(t) \rightarrow \omega(t) \quad \text{and} \quad \omega'_n(t) \rightarrow \omega'(t) \quad \text{as } n \rightarrow \infty.$$

Therefore, taking into account Lemma 3.5, we have for each $t \in \mathbb{T}$

$$|\omega(t) - \omega_n(t)| \leq \sum_{v=n+1}^{\infty} |\hat{\omega}_v| \leq \sum_{v=n+1}^{\infty} \exp(-cvx_1(v)) \leq \exp\left(-\frac{c}{2}nx_1(n)\right)$$

and

$$|\omega'(t) - \omega'_n(t)| \leq \sum_{v=n+1}^{\infty} v |\hat{\omega}_v| \leq \sum_{v=n+1}^{\infty} \exp\left(-\frac{c}{2}vx_1(v)\right) \leq \exp\left(-\frac{c}{4}nx_1(n)\right).$$

□

Let g be an analytic function such as in the definition of the class Ω , i.e., satisfying (1.6) and such that each zero of g is of multiplicity one. Let $\{a_1, \dots, a_m\}$ be the set of all zeros of g on \mathbb{T} . For each $\epsilon > 0$ denote

$$B_\epsilon := \left\{ t \in \mathbb{T} : |g(t)| < \epsilon \right\}.$$

Let us show that the measure of B_ϵ is at most linear in ϵ .

LEMMA 3.7. *For an arbitrary $\epsilon > 0$ we have*

$$|B_\epsilon| \leq C(g) \epsilon.$$

PROOF. Since all zeros of g have multiplicity one, then

$$|g(t)| = |(t - a_1) \dots (t - a_m) h(t)|,$$

where $\min_{t \in \mathbb{T}} |h(t)| = b(g) =: b > 0$. Set

$$S := \left(\frac{3}{\min_{1 \leq i < j \leq m} |a_i - a_j|} \right)^{m-1}.$$

For given $\epsilon > 0$, let $t_0 \in \mathbb{T}$ be such that

$$|t_0 - a_i| > \frac{S\epsilon}{b} \quad \text{for all } i = \overline{1, m}.$$

Since the inequality

$$|t_0 - a_j| \leq \frac{\min_{1 \leq i < j \leq m} |a_i - a_j|}{3}$$

may hold at most for one $j \in \overline{1, m}$ we have

$$|g(t_0)| \geq \frac{S\epsilon}{b} \left(\frac{\min_{1 \leq i < j \leq m} |a_i - a_j|}{3} \right)^{m-1} b = \epsilon.$$

Hence, $t_0 \notin B_\epsilon$. Therefore, for each $t \in B_\epsilon$, there exists $j \in \overline{1, m}$ such that

$$|t - a_j| \leq \frac{S\epsilon}{b}.$$

Thus,

$$|B_\epsilon| \leq \frac{2mS}{b} \epsilon.$$

□

Now we are in a position to prove the following approximation theorem.

THEOREM 3.2. *For each $\omega \in \Omega$ there exists an integer constant $K = K(\omega)$ such that for each trigonometric polynomial T_n we have*

$$(3.13) \quad \frac{1}{2} \int_{\mathbb{T}} |T_n(t) \omega_{Kn}(t)| dt \leq \int_{\mathbb{T}} |T_n(t) \omega(t)| dt \leq 2 \int_{\mathbb{T}} |T_n(t) \omega_{Kn}(t)| dt,$$

where ω_n is the n -th partial Fourier sum of ω .

PROOF. It is enough to verify (3.13) for sufficiently large n . Using Theorem 3.1 we get

$$\int_{\mathbb{T}} |T_n(t)| |\omega(t) - \omega_{Kn}(t)| dt \leq \exp(-cKn x_1(Kn)) \int_{\mathbb{T}} |T_n(t)| dt.$$

We define

$$B_{x_1(n)} = \left\{ t \in \mathbb{T} : |g(t)| < x_1(n) \right\}.$$

Then, Lemma 3.7 implies that $|B_{x_1(n)}| \leq C x_1(n)$, where C depends only on ω . Then, by the Remez inequality we get

$$\begin{aligned} \int_{\mathbb{T}} |T_n(t)| |\omega(t) - \omega_{Kn}(t)| dt &\leq \exp(-cKn x_1(Kn)) \exp(4n |B_{x_1(n)}|) \int_{\mathbb{T} \setminus B_{x_1(n)}} |T_n(t)| dt \\ &\leq \exp(-cKn x_1(Kn) + Cn x_1(n)) \int_{\mathbb{T} \setminus B_{x_1(n)}} |T_n(t)| dt. \end{aligned}$$

Note that for each $t \in \mathbb{T} \setminus B_{x_1(n)}$,

$$(3.14) \quad \omega(t) = \exp(-F(g(t))) \geq \exp(-F(x_1(n))) = \exp(-nx_1(n)).$$

Therefore,

$$\int_{\mathbb{T}} |T_n(t)| |\omega(t) - \omega_{Kn}(t)| dt \leq \exp\left(-cKn x_1(Kn) + Cn x_1(n) + nx_1(n)\right) \int_{\mathbb{T} \setminus B_{x_1(n)}} |T_n(t)| \omega(t) dt.$$

Now, by Corollary 2.1 we can choose integer K large enough such that

$$\int_{\mathbb{T}} |T_n(t)| |\omega(t) - \omega_{Kn}(t)| dt \leq \frac{1}{2} \int_{\mathbb{T} \setminus B_{x_1(n)}} |T_n(t)| \omega(t) dt \leq \frac{1}{2} \int_{\mathbb{T}} |T_n(t)| \omega(t) dt.$$

This immediately implies the statement of the theorem. \square

4. Weighted Bernstein inequality in L_1

In this section we prove the Bernstein inequality in $L_1(\omega)$, where $\omega \in \Omega$.

THEOREM 4.1. *Let $\omega \in \Omega$. Then for each trigonometric polynomial T_n of degree at most n*

$$(4.1) \quad \int_{\mathbb{T}} |T'_n(t)| \omega(t) dt \leq C(\omega) n \int_{\mathbb{T}} |T_n(t)| \omega(t) dt.$$

PROOF. Since the inequality

$$(4.2) \quad \int_{\mathbb{T}} |T'_n(t)| \omega(t) dt \leq C(\omega, n) \int_{\mathbb{T}} |T_n(t)| \omega(t) dt$$

holds for any continuous weight ω , it is enough to prove (4.1) for n large enough. The proof is in three steps.

Step 1. By Theorem 3.2 there exists an integer $K = K(\omega)$ large enough such that the Kn -partial Fourier sum ω_{Kn} satisfies the following:

$$(4.3) \quad \begin{aligned} \int_{\mathbb{T}} |T'_n(t)| \omega(t) dt &\leq 2 \int_{\mathbb{T}} |T'_n(t)| \omega_{Kn}(t) dt \\ &\leq 2 \int_{\mathbb{T}} |(T_n(t) \omega_{Kn}(t))'| dt + 2 \int_{\mathbb{T}} |T_n(t) \omega'_{Kn}(t)| dt =: I_1 + I_2. \end{aligned}$$

Then by the classical Bernstein inequality and Theorem 3.2 we have

$$I_1 \leq CKn \int_{\mathbb{T}} |T_n(t) \omega_{Kn}(t)| dt \leq C(\omega) n \int_{\mathbb{T}} |T_n(t)| \omega(t) dt.$$

Further,

$$I_2 \leq 2 \int_{\mathbb{T}} |T_n(t)| |\omega'(t)| dt + 2 \int_{\mathbb{T}} |T_n(t)| |\omega'(t) - \omega'_{Kn}(t)| dt =: I_{21} + I_{22}.$$

Step 2. To estimate I_{21} , let us define the set

$$B_{n,M} := \left\{ t \in \mathbb{T} : g(t) \neq 0, \text{ and } |F'(g(t))g'(t)| \geq Mn \right\}.$$

Note that, for any $t \in B_{n,M}$, it follows from (F4) that

$$A_1 \frac{F(g(t))}{|g(t)|} \|g'\|_{C(\mathbb{T})} \geq Mn,$$

and therefore,

$$(4.4) \quad \frac{F(g(t))}{|g(t)|} \geq M_2 n, \quad \text{where } M_2 = \frac{M}{A_1 D}.$$

Using Corollary 2.2 we have that for each $L > 0$ there exists $Q = Q(L, \omega) > 1$ such that

$$F\left(\frac{x_1(n)}{L}\right) \leq F(x_1(Qn)) = Qn x_1(Qn) < Qn x_1(n)$$

for n large enough. Then, for all $x \in [x_1(n)/L, A]$, we get

$$(4.5) \quad F(x) \leq F\left(\frac{x_1(n)}{L}\right) < Qn x_1(n) \leq xQLn.$$

Therefore, if

$$(4.6) \quad M_2 = \frac{M}{A_1 D} > QL,$$

then (4.4) and (4.5) imply

$$|g(t)| < \frac{x_1(n)}{L}, \quad t \in B_{n,M}.$$

Now, for each $K \in \mathbb{N}$, taking $L = L(K, \omega)$ as in Corollary 2.3 we have

$$(4.7) \quad F(g(t)) \geq F\left(\frac{x_1(n)}{L}\right) \geq Kx_1(n)n.$$

Moreover, by Lemma 2.1 we have

$$\frac{F(g(t))}{|g(t)|} \leq C(\omega) (F(g(t)))^{1+\frac{1}{A_2}}, \quad t \in B_{n,M}.$$

Let us estimate $|\omega'(t)|$ from above for $t \in B_{n,M}$. In view of (F_1) and (F_4) , we get

$$(4.8) \quad \begin{aligned} |\omega'(t)| &= \omega(t) \left| F'(g(t))g'(t) \right| \leq A_1 D \omega(t) \frac{F(g(t))}{|g(t)|} \\ &\leq C(\omega) \exp(-F(g(t))) (F(g(t)))^{1+\frac{1}{A_2}} \\ &\leq C(\omega) \exp(-F(g(t))/2), \quad t \in B_{n,M}, \end{aligned}$$

where in the last estimate we have used (4.7) and the fact that $n x_1(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Hence, (4.7) and (4.8) imply

$$(4.9) \quad |\omega'(t)| \leq C(\omega) \exp(-Kx_1(n)n/2), \quad t \in B_{n,M}.$$

Step 3. Now we are ready to estimate I_{21} . We have

$$I_{21} = 2 \int_{B_{n,M}} |T_n(t)| |\omega'(t)| dt + 2 \int_{\mathbb{T} \setminus B_{n,M}} |T_n(t)| |\omega'(t)| dt =: I_{211} + I_{212}.$$

Let us estimate I_{211} . Thanks to (4.9), we obtain

$$\begin{aligned} I_{211} &= 2 \int_{B_{n,M}} |T_n(t)| |\omega'(t)| dt \leq C(\omega) \exp(-Kx_1(n)n/2) \int_{B_{n,M}} |T_n(t)| dt \\ &\leq C(\omega) \exp(-Kx_1(n)n/2) \int_{\mathbb{T}} |T_n(t)| dt. \end{aligned}$$

Now, as in the proof of Theorem 3.2, we consider

$$B_{x_1(n)} = \left\{ t \in \mathbb{T} : |g(t)| < x_1(n) \right\}.$$

By the Remez inequality and Lemma 3.7 we get

$$(4.10) \quad \begin{aligned} I_{211} &\leq C(\omega) \exp(-Kx_1(n)n/2) \exp(4n|B_{x_1(n)}|) \int_{\mathbb{T} \setminus B_{x_1(n)}} |T_n(t)| dt \\ &\leq C(\omega) \exp(-Kx_1(n)n/2 + C(\omega)n x_1(n)) \int_{\mathbb{T} \setminus B_{x_1(n)}} |T_n(t)| \omega(t) dt \\ &\leq C(\omega) \int_{\mathbb{T}} |T_n(t)| \omega(t) dt \end{aligned}$$

for $K \in \mathbb{N}$ large enough. On the other hand, it follows from the definition of $B_{n,M}$ that

$$I_{212} = 2 \int_{\mathbb{T} \setminus B_{n,M}} |T_n(t)| |\omega'(t)| dt \leq 2Mn \int_{\mathbb{T}} |T_n(t)| \omega(t) dt.$$

Thus,

$$I_{21} \leq C(\omega)n \int_{\mathbb{T}} |T_n(t)|\omega(t)dt.$$

Regarding I_{22} , we first note that Theorem 3.1 yields

$$I_{22} \leq \exp(-c(\omega)Kn x_1(Kn)) \int_{\mathbb{T}} |T_n(t)|dt.$$

Similarly as we proceed in the estimates of I_{211} , we use Remez's inequality for the set $B_{x_1(n)}$ and Lemma 3.7 to get

$$(4.11) \quad I_{22} \leq C(\omega) \int_{\mathbb{T}} |T_n(t)|\omega(t)dt,$$

for $K \in \mathbb{N}$ large enough.

Let us explain how we choose the constants K, L, Q , and M . First, $K \in \mathbb{N}$ is taking large enough such that (4.3), (4.10), and (4.11) hold. Further we choose $L = L(K, \omega)$ as in Corollary 2.3, $Q = Q(L, \omega)$ as in Corollary 2.2, and finally $M > QLA_1D$ so that (4.6) holds. \square

5. Weighted Remez inequalities

For an arbitrary measurable set E , denote $\|T_n\|_{L_p(\omega, E)} = (\int_E |T_n|^p \omega)^{1/p}$ if $p < \infty$ and $\|T_n\|_{L_\infty(\omega, E)} = \text{ess sup}_{t \in E} |T_n(t)\omega(t)|$. We use the notation $\|T_n\|_{L_p(\omega)}$ rather than $\|T_n\|_{L_p(\omega, \mathbb{T})}$.

The following classes play an important role in our further study.

DEFINITION. We say that a weight u satisfies the $\mathcal{R}(p)$ condition, $0 < p \leq \infty$, and write $u \in \mathcal{R}(p)$, if for any trigonometric polynomial T_n the weighted Remez inequality holds, that is, there exists $C = C(p, u) > 0$ such that

$$(5.1) \quad \|T_n\|_{L_p(u, \mathbb{T})} \leq \exp(Cn|E|) \|T_n\|_{L_p(u, \mathbb{T} \setminus E)}$$

for all measurable sets E with $|E| \leq 1$.

DEFINITION. We say that a weight u satisfies the $\mathcal{R}_{int}(p)$ condition, $0 < p \leq \infty$, and write $u \in \mathcal{R}_{int}(p)$, if for any trigonometric polynomial T_n the restricted weighted Remez inequality holds, that is, there exists $C = C(p, u) > 0$ such that

$$(5.2) \quad \|T_n\|_{L_p(u, \mathbb{T})} \leq \exp(Cn|E|) \|T_n\|_{L_p(u, \mathbb{T} \setminus E)}$$

for all sets E which are a finite union of intervals of length $\geq 1/n$ and such that $|E| \leq 1$.

REMARK 5.1. One can define the class $\mathcal{R}_{int}(p, d)$ such that for any T_n and the set E being a finite union of intervals of length $\geq d/n$ we have $\|T_n\|_{L_p(u, \mathbb{T})} \leq \exp(Cn|E|) \|T_n\|_{L_p(u, \mathbb{T} \setminus E)}$ for some constant $C = C(p, u, d)$. Then $\mathcal{R}_{int}(p, d) = \mathcal{R}_{int}(p)$.

We will need the following approximation inequalities for the weight $\omega^{1/p}$ that are similar to Theorems 3.1 and 3.2.

LEMMA 5.1. Let $\omega = \exp(-F(g(t))) \in \Omega$ and $v = \omega^{1/p}$ for $p \in (0, \infty)$. Let v_n is the n -th partial Fourier sum of v .

(A). We have

$$(5.3) \quad \|v^p - |v_n|^p\|_{C(\mathbb{T})} \leq \exp(-c(p, \omega)nx_1(n))$$

and

$$(5.4) \quad \|v' - v'_n\|_{C(\mathbb{T})} \leq \exp(-c(p, \omega)nx_1(n))$$

for n large enough, where $x_1(n)$ is the unique positive solution of the equation $F(x_1(n)) = nx_1(n)$.

(B). For any $u \in \mathcal{R}_{int}(p)$, there exists $K = K(\omega, u, p)$ such that

$$(5.5) \quad \frac{1}{2} \int_{\mathbb{T}} |T_n(t)|^p |v_{Kn}(t)|^p u(t) dt \leq \int_{\mathbb{T}} |T_n(t)|^p \omega(t) u(t) dt \leq 2 \int_{\mathbb{T}} |T_n(t)|^p |v_{Kn}(t)|^p u(t) dt.$$

(C). For any $u \in \mathcal{R}_{int}(\infty)$, there exists $K = K(\omega, u)$ such that

$$(5.6) \quad \frac{1}{2} \|T_n \omega_{K_n} u\|_{L_\infty(\mathbb{T})} \leq \|T_n \omega u\|_{L_\infty(\mathbb{T})} \leq 2 \|T_n \omega_{K_n} u\|_{L_\infty(\mathbb{T})},$$

where ω_n is the n -th partial Fourier sum of ω .

PROOF. We may assume that n is large enough. For any $\omega = \exp(-F(g(t))) \in \Omega$ and for any $p \in (0, \infty)$ we have, by definition of the class Ω ,

$$v(t) = \omega^{1/p}(t) = \exp(-H(g(t))) \in \Omega, \quad t \in \mathbb{T},$$

where $H(x) = F(x)/p$ satisfies $(F_1) - (F_4)$. Moreover, by Corollary 2.3

$$x_1^\omega(n) \asymp x_1^v(n),$$

where $x_1^\omega(n)$ is a unique positive solution of the equation $F(x_1^\omega(n)) = nx_1^\omega(n)$ and $x_1^v(n)$ is a unique positive solution of the equation $H(x_1^v(n)) = nx_1^v(n)$.

To verify (5.3) and (5.4), we use Theorem 3.1 and the following inequality:

$$(5.7) \quad \left| v^p(t) - |v_{K_n}(t)|^p \right| \leq C(p, \omega) \left| v(t) - v_{K_n}(t) \right|^{\min\{1, p\}}, \quad 0 < p < \infty, \quad t \in \mathbb{T}.$$

For $0 < p < 1$, the latter follows from the inequality $|a^p - b^p| \leq C(p)|a - b|^p$, where $a, b \geq 0$. For $p > 1$, we get (5.7) using the fact that if $a > b > 0$ then $a^p - b^p = p\xi^{p-1}(a - b)$ for some $\xi \in (b, a)$. Thus, the proof of part (A) is complete.

To show (B) and (C), we follow the proof of Theorem 3.2 using (5.3) and the following remark.

REMARK 5.2. In the proofs of Theorems 3.1 and 3.2, we use the Remez inequalities only for the set

$$B_{x_1(n)} = \left\{ t \in \mathbb{T} : |g(t)| < x_1(n) \right\}.$$

Analyzing the proof of Lemma 3.7, we note that there exists $\widehat{B}_{x_1(n)} \subset \mathbb{T}$ such that $B_{x_1(n)} \subseteq \widehat{B}_{x_1(n)}$, $|\widehat{B}_{x_1(n)}| \leq Cx_1(n)$, and $\widehat{B}_{x_1(n)}$ is a union of m intervals of length $> 1/n$, where m is a number of zeros of g on \mathbb{T} . Therefore, in the proof of Theorems 3.1 and 3.2 we can apply the Remez inequality for the set $\widehat{B}_{x_1(n)}$. \square

In this section we prove the following general Remez inequality in L_p .

THEOREM 5.1. Let $0 < p \leq \infty$, $\omega \in \Omega$, and $u \in \mathcal{R}(p)$. Then for each trigonometric polynomial T_n we have

$$(5.8) \quad \|T_n\|_{L_p(\omega u)} \leq \exp(Cn|E|) \|T_n\|_{L_p(\omega u, \mathbb{T} \setminus E)},$$

where $C = C(\omega, u, p)$ and E is a measurable set of positive measure $|E| \leq 1$.

Since any A_∞ weight u satisfies the $\mathcal{R}(p)$ condition, $0 < p < \infty$ (see [MT, Th. 5.2] and [Er1, Th. 7.2]) and any A^* -weight u satisfies the $\mathcal{R}(\infty)$ condition (see [MT, (6.10)]), Theorem 5.1 immediately implies the following result.

COROLLARY 5.1. For $0 < p < \infty$, the Remez inequality (5.8) holds for any measurable set E , $|E| \leq 1$ whenever $\omega \in \Omega$ and $u \in A_\infty$ and for $p = \infty$, whenever $\omega \in \Omega$ and $u \in A^*$. Moreover, applying Theorem 5.1 several times we obtain inequality (5.8) for the weight $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$.

Conditions on the weight u in Corollary 5.1 can be relaxed in the case when the set E is a finite union of intervals. First, we give an analogue of Theorem 5.1 for this case.

THEOREM 5.2. Let $0 < p \leq \infty$, $\omega \in \Omega$, and $u \in \mathcal{R}_{int}(p)$. Then for each trigonometric polynomial T_n we have

$$(5.9) \quad \|T_n\|_{L_p(\omega u)} \leq \exp(Cn|E|) \|T_n\|_{L_p(\omega u, \mathbb{T} \setminus E)},$$

where $C = C(\omega, u, p)$ and E is a finite union of intervals of length $\geq 1/n$.

In particular, this and [MT, Th. 5.3] give a refinement of Corollary 5.1 for such sets E .

COROLLARY 5.2. *For $0 < p < \infty$ the Remez inequality (5.9) holds whenever $\omega \in \Omega$, u is doubling, and E is a union of intervals of length $\geq 1/n$. Moreover, applying Theorem 5.2 several times we obtain inequality (5.9) for the weight $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$.*

PROOF OF THEOREM 5.1. It is sufficient to show (5.8) for n large enough. Let first $p \in (0, \infty)$. It follows from Lemma 5.1 that for $v = \omega^{1/p} \in \Omega$ we have

$$(5.10) \quad \|v^p - |v_n|^p\|_{C(\mathbb{T})} \leq \exp(-c(p, \omega)nx_1(n)),$$

where v_n is the n -th partial Fourier sum of the function v . Moreover, by (5.5)

$$(5.11) \quad \int_{\mathbb{T}} |T_n|^p v^p u \asymp \int_{\mathbb{T}} |T_n|^p |v_{Kn}|^p u$$

for $K = K(\omega, u, p)$ large enough. Let us also remind that

$$B = B_{x_1(n)} = \left\{ t \in \mathbb{T} : |g(t)| \leq x_1(n) \right\}.$$

Case 1. Let $|B| \leq |E|$. Using (5.11) and (5.1) for $u \in \mathcal{R}(p)$, we have

$$\begin{aligned} \int_{\mathbb{T}} |T_n|^p \omega u &\leq \exp(C(p, u)Kn(|E| + |B|)) \int_{\mathbb{T} \setminus (E \cup B)} |T_n|^p |v_{Kn}|^p u \\ &\leq \exp(C(p, u)Kn|E|) \int_{\mathbb{T} \setminus (E \cup B)} |T_n|^p |v_{Kn}|^p u. \end{aligned}$$

The latter can be estimated by $I_1 + I_2$, where

$$I_1 := \exp(C(p, u)Kn|E|) \int_{\mathbb{T} \setminus (E \cup B)} |T_n|^p v^p u,$$

and

$$I_2 := \exp(C(p, u)Kn|E|) \int_{\mathbb{T} \setminus (E \cup B)} |T_n|^p |v^p - |v_{Kn}|^p| u.$$

Corollary 2.1 implies that, for any $c > 0$, there exists $K = K(c, \omega)$ such that $x_1(n) < cKx_1(Kn)$ and therefore $\exp(-cKn x_1(Kn)) \leq \exp(-nx_1(n))$. Then, by (5.10) for $c = c(p, \omega)$,

$$|v^p - |v_{Kn}|^p| \leq \exp(-cKn x_1(Kn)) \leq \exp(-nx_1(n)) \leq \omega(t), \quad t \in \mathbb{T} \setminus B,$$

where the last inequality follows from (3.14). Thus,

$$I_1 + I_2 \leq 2I_1 \leq 2 \exp(C(p, \omega, u)n|E|) \int_{\mathbb{T} \setminus E} |T_n|^p \omega u.$$

Case 2. Let $|B| > |E|$. Similarly to Case 1, using (5.1), we get

$$\int_{\mathbb{T}} |T_n|^p \omega u \leq I_1 + I_2,$$

where

$$I_1 := \exp(C(p, u)Kn|E|) \int_{\mathbb{T} \setminus E} |T_n|^p v^p u \quad \text{and} \quad I_2 := \exp(C(p, u)Kn|E|) \int_{\mathbb{T} \setminus E} |T_n|^p |v^p - |v_{Kn}|^p| u.$$

By (5.10),

$$I_2 \leq \exp(C(p, u)Kn|E|) \exp(-c(p, \omega)Kn x_1(Kn)) \int_{\mathbb{T}} |T_n|^p u.$$

Applying again the Remez inequality (5.1), we have

$$I_2 \leq \exp(C(p, u)Kn|E|) \exp(-c(p, \omega)Kn x_1(Kn)) \exp(C(p, u)n(|B| + |E|)) \int_{\mathbb{T} \setminus (E \cup B)} |T_n|^p u.$$

Since $\omega(t) \geq \exp(-nx_1(n))$, $t \in \mathbb{T} \setminus B$, we get

$$I_2 \leq \exp(C(p, u)Kn|E|) \exp(-c(p, \omega)Kn x_1(Kn)) \exp(C(p, u)n|B|) \exp(nx_1(n)) \int_{\mathbb{T} \setminus (E \cup B)} |T_n|^p \omega u.$$

Taking into account that $|B| \leq C(\omega)x_1(n)$, we obtain that

$$\exp\left(C(p, u)Kn|E| - c(p, \omega)Kn x_1(Kn) + C(p, u)n|B| + nx_1(n)\right) \leq \exp\left(C(p, u)Kn|E|\right)$$

for $K = K(\omega, u, p)$ large enough. Thus,

$$I_2 \leq \exp(C(p, u)Kn|E|) \int_{\mathbb{T} \setminus E} |T_n|^p \omega u.$$

Collecting estimates for I_1 and I_2 , we arrive at

$$\int_{\mathbb{T}} |T_n|^p \omega u \leq \exp(C(p, \omega, u)n|E|) \int_{\mathbb{T} \setminus E} |T_n|^p \omega u, \quad p \in (0, \infty),$$

which is the required inequality.

The proof in the case $p = \infty$ follows along the same lines as above and left for the reader. \square

Proof of Theorem 5.2 is similar to the proof of Theorem 5.1 thanks to Remark 5.2.

We now give the following important corollary of the Remez inequalities for the product of weights.

COROLLARY 5.3. *Let $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$. Let also $0 < p \leq \infty$ and $u \in \mathcal{R}_{int}(p)$. Then*

$$\int_{\mathbb{T}} |T_n|^p \omega u \asymp \int_{\mathbb{T}} |T_n|^p |v_{Kn}^{(1)}|^p \dots |v_{Kn}^{(s)}|^p u, \quad 0 < p < \infty,$$

where $v_n^{(i)}$ is the n -th partial Fourier sum of $\omega_i^{1/p}$, $i = 1, \dots, s$, and $K = K(\omega, u, p)$ is large enough. Moreover,

$$\|T_n \omega u\|_{L_\infty(\mathbb{T})} \asymp \|T_n v_{Kn}^{(1)} \dots v_{Kn}^{(s)} u\|_{L_\infty(\mathbb{T})}$$

where $v_n^{(i)}$ is the n -th partial Fourier sum of ω_i , $i = 1, \dots, s$, and $K = K(\omega, u)$ is large enough.

To prove this, we use induction, Lemma 5.1 **(B)**, and the following result provided by Corollary 5.2 for $p < \infty$ and Theorem 5.2 for $p = \infty$: if $\omega_i \in \Omega$ and $u \in \mathcal{R}_{int}(p)$, we have $\omega_1 \dots \omega_l u \in \mathcal{R}_{int}(p)$ for any integer $1 \leq l \leq s - 1$.

We finish this section by proving the following Remez inequality for algebraic polynomials P_n .

COROLLARY 5.4. *Let $0 < p < \infty$, $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$. Then the following inequality*

$$(5.12) \quad \|P_n\|_{L_p(\omega u, [-1, 1])} \leq \exp(C(p, \omega, u)n\sqrt{|E|}) \|P_n\|_{L_p(\omega u, [-1, 1] \setminus E)}$$

holds for all measurable sets E with $|E| \leq 1/4$ and a weight $u \in A_\infty$. For $p = \infty$, (5.12) holds for a weight $u \in A^*$.

PROOF. To prove (5.12), we use change of variables $x = \cos t$, Corollary 5.1, and the following two facts:

$$(5.13) \quad u \in A_\infty \quad \text{on } [-1, 1] \quad \text{if and only if} \quad u(\cos t)|\sin t| \in A_\infty \quad \text{on } \mathbb{T};$$

see [MT, p. 63] and

$$(5.14) \quad u \in A^* \quad \text{on } [-1, 1] \quad \text{if and only if} \quad u(\cos t) \in A^* \quad \text{on } \mathbb{T};$$

see [MT, p. 68].

To conclude the proof, we remark that for the map $\Phi(t) = \cos t$ and any measurable set $E \subset [-1, 1]$ with $|E| \leq 1/4$, we get $|\Phi^{-1}(E)| \leq 2\sqrt{|E|} \leq 1$. \square

An analogue of Theorem 5.2 for algebraic polynomials can be written similarly.

6. Weighted Bernstein inequality in L_p

The goal of this section is to establish the weighted Bernstein inequality in L_p for the case of product of weights generalizing Theorem 4.1. The proof combines the approximation technique that we used in Theorem 4.1 and the Remez inequalities from Section 5.

DEFINITION. We say that a weight u satisfies the $\mathcal{B}(p)$ condition, $0 < p \leq \infty$, and write $u \in \mathcal{B}(p)$, if for any trigonometric polynomial T_n of degree at most n the weighted Bernstein inequality holds, that is,

$$(6.1) \quad \|T'_n\|_{L_p(u)} \leq C(p, u) n \|T_n\|_{L_p(u)}.$$

THEOREM 6.1. Let $0 < p \leq \infty$, $\omega \in \Omega$, $u \in \mathcal{B}(p) \cap \mathcal{R}_{int}(p)$, then for any trigonometric polynomial T_n of degree at most n we have

$$(6.2) \quad \|T'_n\|_{L_p(\omega u)} \leq C n \|T_n\|_{L_p(\omega u)},$$

where $C = C(\omega, u, p)$.

PROOF. It is enough to prove (6.2) for n large enough. We start with the case $0 < p < \infty$.

First, by (5.5), we have that, for some $K = K(\omega, u, p)$,

$$\begin{aligned} \int_{\mathbb{T}} |T'_n|^p \omega u &\leq 2 \int_{\mathbb{T}} |T'_n|^p |v_{Kn}|^p u \\ &\leq 2^{1+p} \left(\int_{\mathbb{T}} |(T_n v_{Kn})'|^p u + \int_{\mathbb{T}} |T_n v'_{Kn}|^p u \right). \end{aligned}$$

Since $u \in \mathcal{B}(p)$, we get

$$\int_{\mathbb{T}} |(T_n v_{Kn})'|^p u \leq C(\omega, u, p) n^p \int_{\mathbb{T}} |T_n v_{Kn}|^p u \leq C(\omega, u, p) n^p \int_{\mathbb{T}} |T_n|^p \omega u.$$

Also,

$$\int_{\mathbb{T}} |T_n v'_{Kn}|^p u \leq 2^p \left(\int_{\mathbb{T}} |T_n v'|^p u + \int_{\mathbb{T}} |T_n|^p |v' - v'_{Kn}|^p u \right).$$

To conclude the proof, we follow estimates of I_{21} and I_{22} in the proof of Theorem 4.1 taking into account (5.4). Note that in view of Remark 5.2 it suffices to assume $u \in \mathcal{R}_{int}(p)$.

Finally, we arrive at

$$\int_{\mathbb{T}} |T'_n|^p \omega u \leq C(\omega, u, p) n^p \int_{\mathbb{T}} |T_n|^p \omega u.$$

The proof for the case $p = \infty$ repeats the same lines as the proof in the case $0 < p < \infty$ using Theorem 3.1 and the inequality

$$(6.3) \quad \frac{1}{2} \|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})} \leq \|T_n \omega u\|_{L_\infty(\mathbb{T})} \leq 2 \|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})},$$

for K large enough provided by Lemma 5.1 (C). First,

$$\|T'_n \omega u\|_{L_\infty(\mathbb{T})} \leq C \left(\|(T_n \omega_{Kn})' u\|_{L_\infty(\mathbb{T})} + \|T_n \omega'_{Kn} u\|_{L_\infty(\mathbb{T})} \right) \leq C \left(n \|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})} + \|T_n \omega'_{Kn} u\|_{L_\infty(\mathbb{T})} \right),$$

where $C = C(\omega, u, p)$. In view of (6.3), $n \|T_n \omega_{Kn} u\|_{L_\infty(\mathbb{T})} \leq 2n \|T_n \omega u\|_{L_\infty(\mathbb{T})}$. To estimate the second term, we write

$$\|T_n \omega'_{Kn} u\|_{L_\infty(\mathbb{T})} \leq \left(\text{ess sup}_{t \in B_{n,M}} + \text{ess sup}_{t \in \mathbb{T} \setminus B_{n,M}} \right) |T_n(t) \omega'_{Kn}(t) u(t)|$$

and use Remez's inequality with $u \in \mathcal{R}_{int}(p)$ and Theorem 3.1 to get $\|T_n \omega'_{Kn} u\|_{L_\infty(\mathbb{T})} \leq Cn \|T_n \omega u\|_{L_\infty(\mathbb{T})}$. \square

Now we are in position to prove Theorem 1.1 stated in Introduction.

PROOF OF THEOREM 1.1. First, any doubling weight u satisfies Bernstein's inequality (6.1) for $0 < p < \infty$ (see [MT, Th. 4.1] and [Er1, Th. 3.1]). Concerning the restricted Remez inequality, (5.2) holds for any doubling weight u (see [Er1, Th. 7.2]) and therefore, $u \in \mathcal{B}(p) \cap \mathcal{R}_{int}(p)$, $0 < p < \infty$. Then, by Corollary 5.2 $\omega_1 \dots \omega_{s-1} u \in \mathcal{R}_{int}(p)$. Thus, if $0 < p < \infty$, the statement of Theorem 1.1 follows from Theorem 6.1 by induction.

Let now $p = \infty$ and $u \in A^*$. Bernstein's inequality (6.1) is proved in [MT, (6.7)] and Remez's inequality in [MT, (6.10)]. Therefore, $u \in A^*$ implies $u \in \mathcal{B}(\infty) \cap \mathcal{R}_{int}(\infty)$. Similarly to the case $p < \infty$, Theorem 1.1 immediately follows from Corollary 5.1 and Theorem 6.1. \square

7. Weighted Bernstein and Markov inequalities for algebraic polynomials

In this section, we deal with weights ω and $u : [-1, 1] \rightarrow [0, \infty)$. The weight u is either doubling or satisfies the A^* condition on $[-1, 1]$ which are defined similar to those on \mathbb{T} (see, e.g., [MT, p. 62]). First, we obtain the weighted Bernstein inequality for algebraic polynomials on $[-1, 1]$.

THEOREM 7.1. *Let $0 < p < \infty$, $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and a weight u is doubling. Then*

$$(7.1) \quad \int_{-1}^1 \varphi^p(x) |P'_n(x)|^p \omega(x) u(x) dx \leq C(p, \omega, u) n^p \int_{-1}^1 |P_n(x)|^p \omega(x) u(x) dx, \quad \varphi(x) = \sqrt{1-x^2}.$$

PROOF. This result immediately follows from Theorem 1.1, change of variables $x = \cos t$, and the fact that

$$u \text{ is doubling on } [-1, 1] \text{ if and only if } u(\cos t) |\sin t| \text{ is doubling on } \mathbb{T},$$

see [MT, p. 63]. \square

A counterpart for $p = \infty$ reads as follows.

THEOREM 7.2. *Let $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and $u \in A^*$. Then*

$$(7.2) \quad \|\varphi P'_n \omega u\|_{L_\infty[-1,1]} \leq C(\omega, u) n \|P_n \omega u\|_{L_\infty[-1,1]}.$$

The proof is similar to the proof of Theorem 7.1 using the fact (5.14).

Let us now discuss Markov's inequality for algebraic polynomials.

THEOREM 7.3. *Let $0 < p < \infty$, $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and a weight u is doubling. Then*

$$(7.3) \quad \int_{-1}^1 |P'_n(x)|^p \omega(x) u(x) dx \leq C(p, \omega, u) n^{2p} \int_{-1}^1 |P_n(x)|^p \omega(x) u(x) dx.$$

PROOF. First, applying the Bernstein inequality (7.1),

$$C n^{2p} \int_{-1}^1 |P_n(x)|^p \omega(x) u(x) dx \geq n^p \int_{-1}^1 \varphi^p(x) |P'_n(x)|^p \omega(x) u(x) dx.$$

Therefore, it is enough to show that

$$C n^p \int_{-1}^1 \varphi^p(x) |P'_n(x)|^p \omega(x) u(x) dx \geq \int_{-1}^1 |P'_n(x)|^p \omega(x) u(x) dx,$$

or, taking an even trigonometric polynomial $T_n(t) = P'_n(\cos t)$,

$$C n^p \int_{\mathbb{T}} |T_n(t) \sin t|^p \omega(\cos t) u(\cos t) |\sin t| dt \geq \int_{\mathbb{T}} |T_n(t)|^p \omega(\cos t) u(\cos t) |\sin t| dt,$$

or, equivalently,

$$C n^p \int_{\mathbb{T}} |T_n(t) \sin t|^p \bar{\omega}(t) \bar{u}(t) dt \geq \int_{\mathbb{T}} |T_n(t)|^p \bar{\omega}(t) \bar{u}(t) dt,$$

where $\bar{\omega}(t) = \bar{\omega}_1(t) \dots \bar{\omega}_s(t)$, $\bar{\omega}_i = \omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and $\bar{u}(t) = u(\cos t) |\sin t|$ is doubling on \mathbb{T} .

Since any doubling weight \bar{u} satisfies $\bar{u} \in \mathcal{R}_{int}(p)$, $0 < p < \infty$, using Corollary 5.3, it remains to obtain

$$(7.4) \quad C n^p \int_{\mathbb{T}} |T_n(t) \sin t|^p |v_{Kn}(t)|^p \bar{u}(t) dt \geq \int_{\mathbb{T}} |T_n(t)|^p |v_{Kn}(t)|^p \bar{u}(t) dt,$$

where $v_{Kn} = v_{Kn}^{(1)} \dots v_{Kn}^{(s)}$ and $v_n^{(i)}$ is the n -th partial Fourier sum of $\bar{\omega}_i^{1/p}$.

Moreover, by Theorem 3.1 and Lemma 3.2 from the paper [MT] for $1 \leq p < \infty$ and Theorem 2.1 from the paper [Er1] for $0 < p < 1$, it follows that for any doubling weight \bar{u} there exists a nonnegative trigonometric polynomial \bar{u}_n of degree at most n such that

$$\int_{\mathbb{T}} |T_n(t)|^p \bar{u}(t) dt \asymp \int_{\mathbb{T}} |T_n(t)|^p \bar{u}_n^p(t) dt, \quad 0 < p < \infty.$$

Then (7.4) follows from

$$C(p)n^p \int_{\mathbb{T}} |T_n(t) \sin t|^p dt \geq \int_{\mathbb{T}} |T_n(t)|^p dt$$

for any trigonometric polynomial T_n . The latter is known for $1 \leq p < \infty$ (see [Ba, Theorem 1]) and the proof in the case of $0 < p < 1$ is similar. \square

Markov's inequality for the case $p = \infty$ is written as follows.

THEOREM 7.4. *Let $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$, and u is an A^* weight on $[-1, 1]$. Then*

$$\|P'_n \omega u\|_{L_\infty[-1,1]} \leq C(\omega, u) n^2 \|P_n \omega u\|_{L_\infty[-1,1]}.$$

The proof repeats the argument of the proof of Theorem 7.3 using the following inequality:

$$\|T_n(t)\|_{C(\mathbb{T})} \leq (n+1) \|T_n(t) \sin t\|_{C(\mathbb{T})};$$

see [Be2, Ba].

REMARK 7.1. Note that for some weights the Bernstein inequality (7.2) for algebraic polynomials can be derived from known results. First, let us recall the definition of the Mhaskar-Rakhmanov-Saff number, which is a crucial concept to analyze weighted inequalities. Let us suppose that $\omega(x) = \exp(Q(x))$, where $Q : (-1, 1) \rightarrow \mathbb{R}$ is even, and differentiable on $(0, 1)$. Also suppose that $xQ'(x)$ is positive and increasing in $(0, 1)$ with limits zero and infinity at zero and 1, respectively, and

$$\int_0^1 \frac{xQ'(x)}{\sqrt{1-x^2}} dx = \infty.$$

Then the n -th Mhaskar-Rakhmanov-Saff number, $a_n = a_n(Q)$, is defined to be the root of

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n x Q'(a_n x)}{\sqrt{1-x^2}} dx, \quad n \geq 1.$$

The importance of this number lies in the Mhaskar-Saff identity

$$\|P_n \omega\|_{C[-1,1]} = \|P_n \omega\|_{C[-a_n, a_n]}, \quad n \geq 1,$$

and asymptotically as $n \rightarrow \infty$, a_n is the smallest such number; see [MS1, MS2]. In particular, for the weight

$$(7.5) \quad \omega_\alpha(x) = \exp(-(1-x^2)^\alpha), \quad \alpha > 0$$

we have

$$(7.6) \quad 1 - a_n \asymp n^{-1/(\alpha+\frac{1}{2})}, \quad n \rightarrow \infty.$$

For this weight, Lubinsky and Saff proved the following inequalities [LS, p. 531]:

$$(7.7) \quad \left| P'_n(x) \omega_\alpha(x) \sqrt{1 - \frac{|x|}{a_n}} \right| \leq C(\alpha) n \|P_n \omega_\alpha\|_{C[-1,1]}, \quad |x| < a_n,$$

and

$$(7.8) \quad \|P'_n \omega_\alpha\|_{C[-1,1]} \leq C(\alpha) n^{\frac{2\alpha+2}{2\alpha+1}} \|P_n \omega_\alpha\|_{C[-1,1]}.$$

In fact, similar results hold for wide class of functions denoted by \mathcal{W} . By definition, $\omega = \exp(-Q) \in \mathcal{W}$, if

- (i) Q is even and continuously differentiable in $(-1, 1)$, while Q'' is continuous in $(0, 1)$;
- (ii) $Q' \geq 0$ and $Q'' \geq 0$ in $(0, 1)$;
- (iii) $\int_0^1 xQ'(x)/(\sqrt{1-x^2}) dx = \infty$;

- (iv) for $T(x) = 1 + \frac{xQ''(x)}{Q'(x)}$, $x \in (0, 1)$ one has: T is increasing in $(0, 1)$, $T(0+) > 1$, and $T(x) = O(Q'(x))$, $x \in 1 -$.

Let us show that both (7.7) and (7.8) imply (7.2) for ω_α given by (7.5) and $u(x) \equiv 1$. Indeed, let $x \in (0, 1)$. If $1 - C^2 n^{-1/(\alpha+1/2)} \leq x$ for some positive $C = C(\alpha)$, we have $n^{\frac{2\alpha+2}{2\alpha+1}} \leq 2C \frac{n}{\sqrt{1-x^2}}$ and (7.8) implies

$$(7.9) \quad |P'_n(x) \varphi(x) \omega_\alpha(x)| \leq Cn \|P_n \omega_\alpha\|_{C[-1,1]}$$

for such x .

If $x \leq (C^2 - 1)/(C^2/a_n - 1)$, then $\sqrt{1-x^2} \leq 2C \sqrt{1 - \frac{|x|}{a_n}}$ and (7.7) implies (7.9) for such x . Further, (7.6) yields that $a_n > 1 - Bn^{-1/(\alpha+1/2)}$ for some $B = B(\alpha) > 0$. Then, taking $C^2 = 2B + 2$, we have

$$1 - C^2 n^{-1/(\alpha+1/2)} < (C^2 - 1)/(C^2/a_n - 1)$$

for sufficiently large n . Finally, we have

$$\|P'_n \varphi \omega_\alpha\|_{C[-1,1]} \leq Cn \|P_n \omega_\alpha\|_{C[-1,1]}.$$

We also mention that in the recent papers [MN, No] the authors obtained the weighted Bernstein, Nikolskii, and Remez inequalities for algebraic polynomials for the weights $\omega(x) = \exp(-(1-x^2)^\alpha)u(x)$, $\alpha > 0$, where u is doubling on $[-1, 1]$.

8. Weighted Nikolskii inequalities

Nikolskii's inequality for trigonometric polynomials, that is,

$$\|T_n\|_{L_q(\mathbb{T})} \leq C n^{1/p-1/q} \|T_n\|_{L_p(\mathbb{T})}, \quad p < q,$$

plays an important role in approximation theory and functional analysis, in particular, to prove embedding theorems for function spaces (see, e.g., [DW]). It is known that if u is an A_∞ weight, then for any $0 < p \leq q < \infty$ there is a constant $C = C(u, p, q)$ such that

$$(8.1) \quad \left(\int_{\mathbb{T}} |T_n|^q u \right)^{1/q} \leq C n^{1/p-1/q} \left(\int_{\mathbb{T}} |T_n|^p u^{p/q} \right)^{1/p};$$

see [MT, Th. 5.5] and [Er3, Th. 8.1]. Moreover, if $u \in A^*$, then for any $1 \leq p < \infty$ there is a constant $C = C(u, p)$ such that

$$(8.2) \quad \|T_n u\|_{L_\infty(\mathbb{T})} \leq C n^{1/p} \left(\int_{\mathbb{T}} |T_n|^p u^p \right)^{1/p};$$

see [MT, (6.9)]. Note that (8.2) holds for $0 < p < 1$ as well, provided $u \in A^*$. Indeed, we first apply (8.2) with $p = 1$ to get

$$(8.3) \quad \|T_n\|_{L_\infty(u)} \leq Cn \|T_n\|_{L_1(u)}.$$

Second, since $u \in A^*$ yields $u \in A_\infty$, we use (8.1) with $0 < p < 1$ and $q = 1$:

$$(8.4) \quad \|T_n\|_{L_1(u)} \leq C n^{\frac{1}{p}-1} \|T_n\|_{L_p(u^p)}.$$

We prove the following weighted Nikolskii inequalities for trigonometric polynomials.

THEOREM 8.1. *Let $0 < p \leq q \leq \infty$, $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$, and $u \in \mathcal{R}_{int}(q)$.*

(A). *Let $q < \infty$ and $u^{p/q} \in \mathcal{R}_{int}(p)$. Suppose u is such that inequality (8.1) holds for each trigonometric polynomial T_n . Then*

$$(8.5) \quad \|T_n\|_{L_q(\omega u)} \leq C n^{1/p-1/q} \|T_n\|_{L_p((\omega u)^{p/q})},$$

where $C = C(\omega, u, p, q)$.

(B). *Let $p < q = \infty$ and $u^p \in \mathcal{R}_{int}(p)$. Suppose u is such that inequality (8.2) holds for each trigonometric polynomial T_n . Then*

$$(8.6) \quad \|T_n\|_{L_\infty(\omega u)} \leq C n^{1/p} \|T_n\|_{L_p((\omega u)^p)},$$

where $C = C(\omega, u, p)$.

In particular, this implies

COROLLARY 8.1. *Let $\omega = \omega_1 \dots \omega_s$, where $\omega_i \in \Omega$, $i = 1, \dots, s$. Then inequality (8.5) holds provided $u \in A_\infty$ and $0 < p \leq q < \infty$ and (8.6) holds provided $u \in A^*$ and $0 < p < \infty$.*

PROOF OF THEOREM 8.1. First, by definition of the class Ω , any weight $\omega_i \in \Omega$, $1 \leq i \leq s-1$, is such that $\omega_i^{p/q} \in \Omega$ for any $0 < p, q < \infty$. Then, by Corollary 5.2 we get that $(\omega_1 \dots \omega_{s-1} u) \in \mathcal{R}_{int}(q)$ and $(\omega_1 \dots \omega_{s-1} u)^{p/q} \in \mathcal{R}_{int}(p)$. Thus, it is enough to prove (8.5) and (8.6) for $\omega = \omega_s \in \Omega$.

(A). By (5.5) we have

$$(8.7) \quad \int_{\mathbb{T}} |T_n|^q \omega u \asymp \int_{\mathbb{T}} |T_n|^q |v_{Kn}|^q u, \quad u \in \mathcal{R}_{int}(q),$$

where v_n is the n -th partial Fourier sum of $\omega^{1/q}$ and $K = K(\omega, u)$ is large enough. Moreover, applying again (5.5) for the weight $\omega^{p/q}$, where $0 < p \leq q < \infty$, we have

$$(8.8) \quad \int_{\mathbb{T}} |T_n|^p \omega^{p/q} u^{p/q} \asymp \int_{\mathbb{T}} |T_n|^p |v_{Kn}|^p u^{p/q}$$

for $K = K(\omega, u, p, q)$ large enough, provided that $u^{p/q} \in \mathcal{R}_{int}(p)$. Now we apply (8.1) to get (8.5).

(B). The case $q = \infty$ is similar since $(\omega_1 \dots \omega_{s-1} u) \in \mathcal{R}_{int}(\infty)$ and $(\omega_1 \dots \omega_{s-1} u)^p \in \mathcal{R}_{int}(p)$. \square

PROOF OF COROLLARY 8.1. To show (8.5) for $0 < p < q < \infty$ and (8.6) for $1 \leq p < \infty$, we use results from [MT], [Er3], and the following two facts:

- (i) $u^{p/q} \in A_\infty$ whenever $u \in A_\infty$ and $0 < p < q < \infty$ (see [St, Ch. V]) and
- (ii) $u^p \in A^* \subset A_\infty$ whenever $u \in A^*$ and $p > 1$. The latter follows from Jensen's inequality.

To prove (8.6) for $0 < p < 1$, we first apply (8.6) with $p = 1$ and then (8.5) with $0 < p < 1$ and $q = 1$ as in (8.3) and (8.4). \square

We finish this section by proving Nikolskii's inequalities for algebraic polynomials.

COROLLARY 8.2. *Let $0 < p \leq q \leq \infty$ and $\omega = \omega_1 \dots \omega_s$, where $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$. Then for each algebraic polynomial P_n we have*

$$(8.9) \quad \|P_n\|_{L_q([-1,1], \omega u)} \leq C(p, q, \omega, u) n^{2/p-2/q} \|P_n\|_{L_p([-1,1], (\omega u)^{p/q})}, \quad 0 < p \leq q < \infty,$$

provided $u \in A_\infty$ and

$$(8.10) \quad \|P_n\|_{L_\infty([-1,1], \omega u)} \leq C(p, q, \omega, u) n^{2/p} \|P_n\|_{L_p([-1,1], (\omega u)^p)}, \quad 0 < p < \infty,$$

provided $u \in A^*$.

PROOF. First, let $0 < p \leq q < \infty$. We give a straightforward proof applying the Remez inequalities for algebraic polynomials given by Corollary 5.4. Define

$$E := \left\{ x \in [-1, 1] : n^2 \int_{-1}^1 |P_n|^q \omega u \leq |P_n(x)|^q \omega(x) u(x) \right\}.$$

Then, since $|E| \leq n^{-2}$ inequality (5.12) yields

$$\begin{aligned} \|P_n\|_{L_q([-1,1], \omega u)}^q &\leq C(q, \omega, u) \int_{[-1,1] \setminus E} |P_n|^q \omega u \\ &\leq C(q, \omega, u) \left\| |P_n|^q \omega u \right\|_{L_\infty([-1,1] \setminus E)}^{(q-p)/q} \int_{[-1,1] \setminus E} |P_n|^p (\omega u)^{p/q} \\ &\leq C(q, \omega, u) n^{2(q-p)/q} \left(\int_{-1}^1 |P_n|^q \omega u \right)^{(q-p)/q} \int_{-1}^1 |P_n|^p (\omega u)^{p/q}, \end{aligned}$$

which gives (8.9).

Let now $0 < p < \infty$ and $u \in A^*$. Let $v_n^{(i)}(\cos t)$ be the n -th partial Fourier sum of $\omega_i(\cos t) \in \Omega$, $i = 1, \dots, s$. Then, by Corollary 5.3 changing variables gives

$$\|P_n \omega u\|_{L_\infty[-1,1]} \asymp \|P_n v_{K_n}^{(1)} \cdots v_{K_n}^{(s)} u\|_{L_\infty[-1,1]},$$

provided that $u(\cos t)|\sin t|$ is an A^* weight on \mathbb{T} . The latter holds from (5.14).

Moreover, since $u^p \in A^* \subset A_\infty$, $p > 1$, Corollary 5.3 implies that

$$\int_{-1}^1 |P_n|^p (\omega u)^p \asymp \int_{-1}^1 |P_n|^p |v_{K_n}^{(1)}|^p \cdots |v_{K_n}^{(s)}|^p u^p.$$

Then (8.10) for $1 \leq p < \infty$ follows from

$$\|P_n\|_{L_\infty(u)} \leq C(p, u) n^{2/p} \|P_n\|_{L_p(u^p)}, \quad u \in A^*, \quad 1 \leq p < \infty;$$

see [MT, (7.31)]. The case $0 < p < 1$ can be treated as in the proof of Corollary 8.1. □

9. Necessary conditions for weighted Bernstein inequality

We will use the following properties of the Chebyshev polynomials \mathcal{T}_n defined by $\mathcal{T}_n(\cos t) = \cos nt$:

$$(9.1) \quad |\mathcal{T}_n(x)| \leq 1, \quad |x| \leq 1;$$

$$(9.2) \quad \mathcal{T}_n(x) \text{ is increasing on } (1, \infty);$$

$$(9.3) \quad \mathcal{T}_n(x) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right) \quad \text{for every } x \in \mathbb{R} \setminus (-1, 1).$$

The last identity readily implies that

$$(9.4) \quad \mathcal{T}_n\left(1 + \frac{1}{n^2}\right) \leq C_1, \quad n \in \mathbb{N},$$

and

$$(9.5) \quad \frac{\mathcal{T}'_n(x)}{\mathcal{T}_n(x)} \geq \frac{1}{4} \frac{n}{\sqrt{x^2 - 1}}, \quad x > 1 + \frac{1}{n^2}, \quad n \in \mathbb{N}.$$

To prove the main theorems of this section we need two auxiliary results.

LEMMA 9.1. *Let ξ be a negative increasing continuous function on $(0, \epsilon)$, for some $\epsilon > 0$, and such that $\xi(0+) = -\infty$ and, for each $r \in (0, 1)$,*

$$(9.6) \quad \frac{\xi(rx)}{\xi(x)} \rightarrow \infty \quad \text{as } x \rightarrow 0+.$$

Then for each positive sequences h_n such that $h_n \rightarrow 0$ as $n \rightarrow \infty$ there exists a positive sequence $\beta_n \rightarrow 0$ as $n \rightarrow \infty$ such that, for each $r \in (0, 1)$,

$$\inf_{x \in (0, h_n)} \frac{\xi(rx)}{\xi(x)} \beta_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

PROOF. Fix a positive sequence $h_n \rightarrow 0$. By (9.6), there exists an increasing sequence of positive integers $n(k)$ such that for each $n > n(k)$

$$(9.7) \quad \inf_{x \in (0, h_n)} \frac{\xi((1 - 1/k)x)}{\xi(x)} > k^2.$$

Put $\beta_n = 1/k$ for $n \in [n(k) + 1, n(k + 1)]$. Fix $r \in (0, 1)$. Consider a positive integer K such that $1 - 1/K > r$ and $h_n < \epsilon$ for $n > n(K)$. Applying monotonicity of ξ and (9.7), we get that

$$\inf_{x \in (0, h_n)} \frac{\xi(rx)}{\xi(x)} \beta_n > \inf_{x \in (0, h_n)} \frac{\xi((1 - 1/K)x)}{\xi(x)} \frac{1}{K} > K, \quad n \in [n(K) + 1, n(K + 1)].$$

This establishes the statement of the lemma. □

The proof of the next lemma is a trivial corollary of the mean value theorem.

LEMMA 9.2. *Let ξ be an increasing continuous function on $(0, \epsilon)$, for some $\epsilon > 0$, and such that $\xi(0+) = -\infty$. Then, for each M large enough, the equation*

$$\xi(x) = -Mx$$

has a unique solution $y(M) \in (0, \epsilon)$, which is continuous in M and decreasing to 0 as $M \rightarrow \infty$.

Now we give the following extension of Theorem 1.3.

THEOREM 9.1. *Let $\omega \in C(\mathbb{T})$ be an arbitrary weight function satisfying the following conditions:*

$$(9.8) \quad \omega(t_0) = 0, \text{ for some } t_0 \in \mathbb{T},$$

$$(9.9) \quad \omega \text{ is increasing on } (t_0, t_0 + \epsilon) \text{ and } \omega \text{ is decreasing on } (t_0 - \epsilon, t_0) \text{ for some } \epsilon > 0,$$

$$(9.10) \quad \lim_{t \rightarrow t_0} \frac{\log \omega(t_0 + r(t - t_0))}{\log \omega(t)} = \infty, \text{ for each } r \in (0, 1).$$

Then for each $0 < p \leq \infty$ there exists a sequence of trigonometric polynomials Q_n of degree at most n such that

$$\lim_{n \rightarrow \infty} \frac{\|Q'_n\|_{L_p(\omega)}}{n\|Q_n\|_{L_p(\omega)}} = \infty.$$

REMARK 9.1. (i) Note that if ω is a continuous nondoubling weight then $\omega(t_0) = 0$ for some $t_0 \in \mathbb{T}$, i.e., condition (9.8) holds. Without loss of generality we assume below that $t_0 = 0$ and $\|\omega\|_{C(\mathbb{T})} \leq 1$.

(ii) Condition (9.9) is assumed to simplify the proof. The principal condition is (9.10), which implies that ω goes to 0 fast enough as $t \rightarrow 0$. Condition (9.10) can be equivalently written as follows: for each $r \in (0, 1)$,

$$\lim_{t \rightarrow 0} \frac{\log \omega(rt)}{\log \omega(t)} \text{ exists or equal } \infty,$$

and, for some $r^* \in (0, 1)$,

$$\lim_{t \rightarrow 0} \frac{\log \omega(r^*t)}{\log \omega(t)} = \infty.$$

EXAMPLE. A typical example of weights satisfying conditions of Theorem 9.1 is

$$\omega_\alpha^*(t) = \exp(-F(g(t))),$$

where

$$F(x) = \exp(|x|^{-\alpha}), \quad \alpha > 0,$$

and g is an analytic function, $g(t) : \mathbb{T} \rightarrow [-1, 1]$, $g(0) = 0$. Although $\omega_\alpha^* \in C^\infty(\mathbb{T})$, the result of Theorem 1.1 is not true for this kind of functions.

PROOF OF THEOREM 9.1. Our proof is in five steps. First, we will prove the theorem for $p = \infty$ (steps 1–4).

Step 1. Recall that $t_0 = 0$ and $\|\omega\|_{C(\mathbb{T})} \leq 1$. We choose Q_n as follows:

$$Q_n(t) := \mathcal{T}_n(1 + a_n^2 - \sin^2 t),$$

where $a_n \rightarrow 0$ is a positive sequence depending on ω to be chosen later. For each $n \in \mathbb{N}$, we denote by b_n any point on \mathbb{T} such that

$$\|Q_n \omega\|_{C(\mathbb{T})} = |Q_n(b_n) \omega(b_n)|.$$

Without loss of generality we may assume that $b_n \in (0, \pi)$. Suppose that the sequence $\{a_n\}$ is such that

$$(9.11) \quad \lim_{n \rightarrow \infty} Q_n(b_n) \omega(b_n) = \infty,$$

and

$$(9.12) \quad b_n = a_n(1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

Then (9.4) and (9.11) imply

$$(9.13) \quad 1 + a_n^2 - \sin^2 b_n > 1 + \frac{1}{n^2}$$

for n large enough.

Hence,

$$\begin{aligned} \frac{\|Q'_n \omega\|_{C(\mathbb{T})}}{n\|Q_n \omega\|_{C(\mathbb{T})}} &\geq \frac{|Q'_n(b_n)\omega(b_n)|}{nQ_n(b_n)\omega(b_n)} = \frac{\mathcal{T}'_n(1+a_n^2-\sin^2 b_n)|\sin 2b_n|}{n\mathcal{T}_n(1+a_n^2-\sin^2 b_n)} \\ &\geq \frac{|\sin 2b_n|}{4\sqrt{(1+a_n^2-\sin^2 b_n)^2-1}}, \end{aligned}$$

where in the last inequality we used (9.5). Finally, taking into account (9.12), we obtain

$$\lim_{n \rightarrow \infty} \frac{\|Q'_n \omega\|_{C(\mathbb{T})}}{n\|Q_n \omega\|_{C(\mathbb{T})}} = \infty,$$

which is the statement of the theorem in the case $p = \infty$.

Step 2. Let us now focus on the search of the sequence a_n which satisfies (9.11) and (9.12). Note that if we take sequences $a_n \rightarrow 0$ and $\lambda_n \rightarrow 1$ such that

$$(9.14) \quad \mathcal{T}_n(1+a_n^2-\sin^2(\lambda_n a_n))\omega(\lambda_n a_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

and, for each $r \in (0, 1)$,

$$(9.15) \quad \mathcal{T}_n(1+a_n^2)\omega(ra_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

then a_n satisfies (9.11) and (9.12). Indeed, condition (9.14) immediately implies (9.11), so (9.13) holds as well, and hence

$$\limsup_{n \rightarrow \infty} \frac{b_n}{a_n} \leq 1.$$

If

$$\liminf_{n \rightarrow \infty} \frac{b_n}{a_n} < r < 1,$$

then, applying (9.2) and (9.9), we have

$$Q_n(b_n)\omega(b_n) \leq \mathcal{T}_n(1+a_n^2)\omega(ra_n)$$

for infinitely many $n \in \mathbb{N}$. This inequality together with (9.15) contradicts (9.11). So,

$$\liminf_{n \rightarrow \infty} \frac{b_n}{a_n} \geq 1 \quad \text{and therefore,} \quad \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = 1,$$

which is (9.12).

Step 3. Let us set

$$\xi := \log \omega.$$

Taking logarithm in the both sides of (9.14) and (9.15), and applying (9.3) we get that if $\{a_n\}$ and $\{\lambda_n\}$ satisfy

$$n \log \left(1 + a_n^2 - \sin^2(\lambda_n a_n) + \sqrt{(1 + a_n^2 - \sin^2(\lambda_n a_n))^2 - 1} \right) + \xi(\lambda_n a_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

and, for each $r \in (0, 1)$,

$$n \log \left(1 + a_n^2 + \sqrt{(1 + a_n^2)^2 - 1} \right) + \xi(ra_n) \rightarrow -\infty \quad \text{as } n \rightarrow \infty,$$

then $\{a_n\}$ and $\{\lambda_n\}$ satisfy (9.14) and (9.15) as well. Finally, since $\log(1+t+\sqrt{(1+t)^2-1}) \sim \sqrt{2t}$ as $t \rightarrow 0$ it is enough to choose $a_n \rightarrow 0$ and $\lambda_n \rightarrow 1$ such that

$$(9.16) \quad n\lambda_n a_n \sqrt{1 - \lambda_n^2} + \xi(\lambda_n a_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

and, for each $r \in (0, 1)$,

$$(9.17) \quad 2na_n + \xi(ra_n) \rightarrow -\infty \quad \text{as } n \rightarrow \infty.$$

Step 4. Now we are in a position to choose $\{a_n\}$ and $\{\lambda_n\}$. For n large enough, let h_n be a unique solution of the equation

$$\xi(x) = -n^{1/2}x,$$

provided by Lemma 9.2. It follows from Lemma 9.1 that there exists a sequence $\{\lambda_n\}$ which goes to 1 slow enough such that

$$\sqrt{1 - \lambda_n^2} > n^{-1/3}$$

and, for each $r \in (0, 1)$,

$$\inf_{t \in (0, 2h_n)} \frac{\xi(rt)}{\xi(t)} \sqrt{1 - \lambda_n^2} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Moreover, for each $r \in (0, 1)$ and $r_1 \in (r, 1)$, we have

$$\begin{aligned} \inf_{t \in (0, h_n/\lambda_n)} \frac{\xi(rt)}{\xi(\lambda_n t)} \sqrt{1 - \lambda_n^2} &= \inf_{t \in (0, h_n)} \frac{\xi(rt/\lambda_n)}{\xi(t)} \sqrt{1 - \lambda_n^2} \\ &\geq \inf_{t \in (0, h_n/\lambda_n)} \frac{\xi(rt/\lambda_n)}{\xi(t)} \sqrt{1 - \lambda_n^2} \\ (9.18) \quad &\geq \inf_{t \in (0, h_n/\lambda_n)} \frac{\xi(r_1 t)}{\xi(t)} \sqrt{1 - \lambda_n^2} \rightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Put $a_n := z_n/\lambda_n$, where z_n is a unique solution of the equation

$$(9.19) \quad \xi(z) = -\frac{1}{2}nz\sqrt{1 - \lambda_n^2},$$

provided by Lemma 9.2. Then, Lemma 9.2 implies that $z_n \rightarrow 0$, and hence $a_n \rightarrow 0$. Therefore,

$$n\lambda_n a_n \sqrt{1 - \lambda_n^2} + \xi(\lambda_n a_n) = -\xi(\lambda_n a_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

i.e., (9.16) holds.

On the other hand, Lemma 9.2 together with the condition $\frac{1}{2}n\sqrt{1 - \lambda_n^2} > n^{1/2}$ for n large enough implies that $z_n = a_n\lambda_n < h_n$. Thus, (9.18) yields

$$\lim_{n \rightarrow \infty} \frac{\xi(ra_n)}{\xi(\lambda_n a_n)} \sqrt{1 - \lambda_n^2} = \infty.$$

Moreover, (9.19) implies

$$\xi(\lambda_n a_n) = -\frac{1}{2}n\lambda_n a_n \sqrt{1 - \lambda_n^2}.$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{\xi(ra_n)}{n\lambda_n a_n/2} = -\infty,$$

which gives (9.17).

Thus, the sequence $\{a_n\}$ satisfies (9.16) and (9.17) and therefore, (9.11) and (9.12), which concludes the proof of Theorem 9.1 in the case $p = \infty$.

Step 5. The proof for the case $0 < p < \infty$ follows the same lines as the one for the case $p = \infty$. We again choose the polynomial Q_n as

$$Q_n(t) := \mathcal{T}_n(1 + a_n^2 - \sin^2 t),$$

where $a_n = a_n(\omega, p) \rightarrow 0$ is a positive sequence to be chosen later. Similarly to Steps 1 and 2, it is enough to find a sequence $\{a_n\}$ such that $a_n \rightarrow 0$ as $n \rightarrow \infty$, for each $r \in (0, 1)$,

$$|\mathcal{T}_n(1 + a_n^2)|^p \omega(ra_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$\int_{\mathbb{T}} |\mathcal{T}_n(1 + a_n^2 - \sin^2 t)|^p \omega(t) dt \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

The latter holds if for some sequence $\{\lambda_n\}$ such that $\lambda_n \rightarrow 1-$ one has

$$\begin{aligned} &\int_{(2\lambda_n - 1)a_n}^{\lambda_n a_n} |\mathcal{T}_n(1 + a_n^2 - \sin^2 t)|^p \omega(t) dt \\ &\geq |\mathcal{T}_n(1 + a_n^2 - \sin^2(\lambda_n a_n))|^p \omega((2\lambda_n - 1)a_n)(1 - \lambda_n)a_n \rightarrow \infty. \end{aligned}$$

Similarly to the Step 3 (cf. (9.16) and (9.17)) it is enough to choose sequences $\{\lambda_n\}$ and $\{a_n\}$ such that

$$(9.20) \quad pn\lambda_n a_n \sqrt{1 - \lambda_n^2} + \log(1 - \lambda_n) + \log a_n + \xi((2\lambda_n - 1)a_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

and for each $r \in (0, 1)$

$$(9.21) \quad 2pna_n + \xi(ra_n) \rightarrow -\infty \quad \text{as } n \rightarrow \infty.$$

Similarly to the Step 4 one can choose sequences $\{\lambda_n\}$ and $\{a_n\}$ satisfying

$$(9.22) \quad \xi((2\lambda_n - 1)a_n) = -pn\lambda_n a_n(1 - \lambda_n^2)$$

and (9.21). Finally, note that (9.22) together with $\lim_{n \rightarrow \infty} \omega(a_n)/a_n = 0$ implies (9.20). \square

The next theorem (cf. Theorem 1.2) is the main negative result in the paper providing a necessary condition for the weighted Bernstein inequality to hold.

THEOREM 9.2. *Let $\omega \in C(\mathbb{T})$ be an arbitrary weight function satisfying (9.8), (9.9), and the following condition:*

$$(9.23) \quad \limsup_{t \rightarrow t_0} \frac{\log \omega(t_0 + r(t - t_0))}{\log \omega(t)} = \infty \quad \text{for each } r \in (0, 1).$$

Then for each $0 < p \leq \infty$ there exists a sequence of positive integers $K_n \rightarrow \infty$ as $n \rightarrow \infty$, and a sequence of trigonometric polynomials Q_n of degree at most K_n such that

$$\lim_{n \rightarrow \infty} \frac{\|Q'_n\|_{L_p(\omega)}}{K_n \|Q_n\|_{L_p(\omega)}} = \infty.$$

REMARK 9.2. If condition (9.23) holds for some $r \in (0, 1)$, then it also holds for any $r \in (0, 1)$.

PROOF. Without loss of generality we assume below that $t_0 = 0$ and $\|\omega\|_{C(\mathbb{T})} \leq 1$. We will prove the theorem only for the case $p = \infty$. The case $0 < p < \infty$ is similar (see the proof of Theorem 9.1, Step 5). Define Q_n as follows:

$$Q_n(t) := \mathcal{T}_{K_n}(1 + a_n^2 - \sin^2 t),$$

where K_n and $a_n \rightarrow 0$ to be chosen later. Put $\xi := \log \omega$. Now proceeding step by step the proof of Theorem 9.1 up to (9.16) and (9.17) one can see that it is enough to choose $a_n \rightarrow 0$, an increasing sequence of integers $\{K_n\}$, and $\lambda_n \rightarrow 1-$ such that

$$(9.24) \quad K_n \lambda_n a_n \sqrt{1 - \lambda_n^2} + \xi(\lambda_n a_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

and, for each $r \in (0, 1)$,

$$(9.25) \quad 2K_n a_n + \xi(ra_n) \rightarrow -\infty \quad \text{as } n \rightarrow \infty.$$

Since

$$\limsup_{t \rightarrow 0} \frac{\xi(rt)}{\xi(t)} = \infty \quad \text{for each } r \in (0, 1),$$

there exists decreasing positive sequence c_n such that $c_n \rightarrow 0$ as $n \rightarrow \infty$, and

$$(9.26) \quad \frac{\xi((1 - 1/n)c_n)}{\xi(c_n)} > n^2.$$

Put

$$\lambda_n := 1 - 1/n, \quad a_n := c_n/\lambda_n, \quad \text{and} \quad K_n := 2 \left[\frac{-\xi(c_n)}{\lambda_n a_n \sqrt{1 - \lambda_n^2}} \right].$$

Since $\lim_{t \rightarrow 0} \xi(t) = -\infty$, then $K_n \rightarrow \infty$ as $n \rightarrow \infty$, and hence (9.24) holds.

To complete the proof, take an arbitrary $r \in (0, 1)$. Since $r < \lambda_n^2$ for n large enough, by monotonicity of ξ ,

$$2K_n a_n + \xi(ra_n) < 2K_n a_n + \xi((1 - 1/n)c_n).$$

Thus, by (9.26)

$$2K_n a_n + \xi(ra_n) < 2K_n a_n + n^2 \xi(c_n) \leq \frac{-4\xi(c_n)}{\lambda_n \sqrt{1 - \lambda_n^2}} + n^2 \xi(c_n) \rightarrow -\infty$$

as $n \rightarrow \infty$. This proves (9.25). \square

The next theorem shows an essential difference between Theorems 9.1 and 9.2 in the case when the weight satisfies (9.23) but not (9.10). In this case Bernstein's inequality may hold for some subsequence of integers $\{K_n\}$ but not for all $n \in \mathbb{N}$. For simplicity we consider only the case $p = \infty$ and $t_0 = 0$.

THEOREM 9.3. *There exists an even weight function $\omega \in C^\infty(\mathbb{T})$ satisfying (9.8) and (9.9) and*

$$(9.27) \quad \limsup_{t \rightarrow 0} \frac{\log \omega(rt)}{\log \omega(t)} = \infty, \quad \text{for each } r \in (0, 1),$$

such that for some increasing sequence of positive integers K_n the Bernstein inequality

$$\|T'_n \omega\|_{C(\mathbb{T})} \leq CK_n \|T_n \omega\|_{C(\mathbb{T})}$$

holds for any trigonometric polynomial T_n of degree at most K_n .

PROOF. Let

$$W(x) = \frac{\int_0^{\pi x} \exp(-1/\sin^2 t) dt}{\int_0^\pi \exp(-1/\sin^2 t) dt}, \quad x \in [0, 1].$$

Define an even weight ω as follows

$$\omega(t) := \begin{cases} 1, & \text{if } t \in [\alpha_1, \pi], \\ d_n, & \text{if } t \in [\alpha_n, \frac{\alpha_{n-1}}{2}], n \geq 2, \\ d_{n+1} + (d_n - d_{n+1})W\left(\frac{2t}{\alpha_n} - 1\right), & \text{if } t \in [\frac{\alpha_n}{2}, \alpha_n], n \geq 1, \\ 0, & \text{if } t = 0, \end{cases}$$

where $d_n := \exp(-\exp(n^2))$ and $\alpha_n := d_n^2$. By construction, $\omega \in C^\infty(\mathbb{T})$. Since

$$\lim_{n \rightarrow \infty} \frac{\log \omega(\alpha_n/2)}{\log \omega(\alpha_n)} = \infty,$$

then ω satisfies (9.27).

For each $n \in \mathbb{N}$, we also define an even weight ω_n :

$$\omega_n(t) := \begin{cases} \omega(t), & \text{if } t \in [\alpha_n, \pi], \\ d_n, & \text{if } t \in [0, \alpha_n]. \end{cases}$$

Put

$$(9.28) \quad K_n := \left\lfloor \frac{1}{100\alpha_n} \right\rfloor.$$

Take a polynomial T_n of degree at most K_n . Since $\omega_n(t) \geq \omega(t)$, $t \in \mathbb{T}$, then $\|T_n \omega\|_{C(\mathbb{T})} \leq \|T_n \omega_n\|_{C(\mathbb{T})}$.

On the other hand,

$$(9.29) \quad \|T_n \omega_n\|_{C(\mathbb{T})} \leq 2 \|T_n \omega\|_{C(\mathbb{T})}.$$

Indeed, let $t_0 \in \mathbb{T}$ be a point where $|T_n \omega_n|$ attains its maximum. If $|t_0| \geq \alpha_n$, then (9.29) is obvious. If $|t_0| < \alpha_n$, then using Remez's inequality and (9.28) we get

$$(9.30) \quad \begin{aligned} \|T_n \omega_n\|_{C(\mathbb{T})} &= d_n \|T_n\|_{C(\mathbb{T})} \leq d_n \exp(8\alpha_n K_n) \max_{t \in \mathbb{T} \setminus [-\alpha_n, \alpha_n]} |T_n(t)| \\ &< 2 \max_{t \in \mathbb{T} \setminus [-\alpha_n, \alpha_n]} |T_n(t) \omega(t)| \leq 2 \|T_n \omega\|_{C(\mathbb{T})}. \end{aligned}$$

Note that by definition of ω_n we have

$$(9.31) \quad |\omega'_n(t)| \leq C \max_{1 \leq k \leq n-1} \frac{d_k}{\alpha_k} \leq C \frac{d_{n-1}}{\alpha_{n-1}}, \quad t \in \mathbb{T}.$$

Hence,

$$(9.32) \quad |\omega'_n(t)| \leq C \max_{1 \leq k \leq n-1} \frac{d_k}{d_{k+1} \alpha_k} |\omega_n(t)| \leq C \frac{d_{n-1}}{d_n \alpha_{n-1}} |\omega_n(t)|, \quad t \in \mathbb{T}.$$

Moreover, we have

$$(9.33) \quad |\omega_n''(t)| \leq C \max_{1 \leq k \leq n-1} \frac{d_k}{\alpha_k^2} \leq C \frac{d_{n-1}}{\alpha_{n-1}^2}, \quad t \in \mathbb{T}.$$

Since $\omega_n \in C^\infty(\mathbb{T})$ then by Jackson's theorem there exists a trigonometric polynomial Q_n of degree K_n such that

$$\|\omega_n - Q_n\| \leq C \frac{\|\omega_n'\|_{C(\mathbb{T})}}{K_n}$$

and

$$\|\omega_n' - Q_n'\| \leq C \frac{\|\omega_n''\|_{C(\mathbb{T})}}{K_n}.$$

Thus, (9.31) and (9.33) yield that

$$(9.34) \quad \|\omega_n - Q_n\| \leq C \frac{d_{n-1}\alpha_n}{\alpha_{n-1}} \leq \frac{d_n}{2}$$

and

$$(9.35) \quad \|\omega_n' - Q_n'\| \leq C \frac{d_{n-1}\alpha_n}{\alpha_{n-1}^2} \leq K_n d_n$$

for n large enough. Now by (9.34) we get

$$\begin{aligned} \|T_n' \omega\|_{C(\mathbb{T})} &\leq \|T_n' \omega_n\|_{C(\mathbb{T})} \leq \|T_n' Q_n\|_{C(\mathbb{T})} + \|T_n'\|_{C(\mathbb{T})} \|\omega_n - Q_n\|_{C(\mathbb{T})} \\ &\leq \|T_n' Q_n\|_{C(\mathbb{T})} + \frac{d_n}{2} \|T_n'\|_{C(\mathbb{T})} \leq \|T_n' Q_n\|_{C(\mathbb{T})} + \frac{1}{2} \|T_n' \omega_n\|_{C(\mathbb{T})}. \end{aligned}$$

Therefore,

$$\|T_n' \omega\|_{C(\mathbb{T})} \leq \|T_n' \omega_n\|_{C(\mathbb{T})} \leq 2 \|T_n' Q_n\|_{C(\mathbb{T})}.$$

Similarly applying the inequality

$$\|T_n' \omega_n\|_{C(\mathbb{T})} \geq \|T_n' Q_n\|_{C(\mathbb{T})} - \|T_n'\|_{C(\mathbb{T})} \|\omega_n - Q_n\|_{C(\mathbb{T})},$$

we get

$$(9.36) \quad \|T_n Q_n\|_{C(\mathbb{T})} \leq 2 \|T_n \omega_n\|_{C(\mathbb{T})}.$$

Thus,

$$\|T_n' \omega\|_{C(\mathbb{T})} \leq 2 \|T_n' Q_n\|_{C(\mathbb{T})} \leq 2 \|(T_n Q_n)'\|_{C(\mathbb{T})} + 2 \|T_n Q_n'\|_{C(\mathbb{T})} =: I_1 + I_2.$$

By Bernstein's inequality for the polynomials and (9.34) we have

$$I_1 \leq CK_n \|T_n Q_n\|_{C(\mathbb{T})} \leq 4CK_n \|T_n \omega\|_{C(\mathbb{T})}.$$

Regarding I_2 , we first note that

$$I_2 \leq 2 \|T_n \omega_n'\|_{C(\mathbb{T})} + 2 \|T_n\|_{C(\mathbb{T})} \|\omega_n' - Q_n'\|_{C(\mathbb{T})} =: I_{21} + I_{22}.$$

By (9.32) and (9.30) we get

$$I_{21} \leq C \frac{d_{n-1}}{d_n \alpha_{n-1}} \|T_n \omega_n\|_{C(\mathbb{T})} < CK_n \|T_n \omega\|_{C(\mathbb{T})}.$$

Moreover, (9.35) and (9.30) imply

$$I_{22} \leq 2K_n d_n \|T_n\|_{C(\mathbb{T})} \leq 2K_n \|T_n \omega_n\|_{C(\mathbb{T})} \leq 4K_n \|T_n \omega\|_{C(\mathbb{T})}$$

for n large enough. Hence, for any $n \in \mathbb{N}$,

$$\|T_n' \omega\|_{C(\mathbb{T})} \leq CK_n \|T_n \omega\|_{C(\mathbb{T})}.$$

□

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