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Title:	<i>Riesz Potential and Maximal Function for Dunkl transform</i>
Journal Information:	<i>Potential Analysis,</i>
Author(s):	Gorbachev D.V., Ivanov V.I., Tikhonov S.Y..
Volume, pages:	<i>1 Springer</i> , DOI:[10.1007/s11118-020-09867-z]



Riesz Potential and Maximal Function for Dunkl transform

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Received: 8 May 2018 / Accepted: 9 July 2020 / Published online: 22 July 2020
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Abstract

We study weighted (L^p, L^q) -boundedness properties of Riesz potentials and fractional maximal functions for the Dunkl transform. In particular, we obtain the weighted Hardy–Littlewood–Sobolev type inequality and weighted weak (L^1, L^q) estimate. We find a sharp constant in the weighted L^p -inequality, generalizing the results of W. Beckner and S. Samko.

Keywords Dunkl transform · Generalized translation operator · Convolution · Riesz potential

Mathematics Subject Classification (2010) 42B10 · 33C45 · 33C52

1 Introduction

Let \mathbb{R}^d be the real Euclidean space of d dimensions equipped with a scalar product $\langle x, y \rangle$ and a norm $|x| = \sqrt{\langle x, x \rangle}$. Let $d\mu(x) = (2\pi)^{-d/2} dx$ be the normalized Lebesgue measure,

The first and the second authors were supported by the Russian Science Foundation under grant 18-11-00199. The third author was partially supported by MTM 2014-59174-P, 2014 SGR 289, and by the CERCA Programme of the Generalitat de Catalunya.

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$L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, be the Lebesgue space with the norm $\|f\|_p = (\int_{\mathbb{R}^d} |f|^p d\mu)^{1/p}$, and $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space. The Fourier transform is given by

$$\mathcal{F}(f)(y) = \int_{\mathbb{R}^d} f(x) e^{-i\langle x, y \rangle} d\mu(x).$$

Throughout the paper, we will assume that $A \lesssim B$ means that $A \leq CB$ with a constant C depending only on nonessential parameters. For $p \geq 1$, $p' = \frac{p}{p-1}$ is the Hölder conjugate and χ_E is the characteristic function of a set E .

The Riesz potential operator or fractional integral I_α is defined by

$$I_\alpha f(x) = (\gamma_\alpha)^{-1} \int_{\mathbb{R}^d} f(y) |x - y|^{\alpha-d} d\mu(y) = (\gamma_\alpha)^{-1} \int_{\mathbb{R}^d} \tau^{-y} f(x) |y|^{\alpha-d} d\mu(y),$$

where $0 < \alpha < d$, $\gamma_\alpha = 2^{\alpha-d/2} \Gamma(\alpha/2) / \Gamma((d-\alpha)/2)$, and $\tau^y f(x) = f(x+y)$ is the translation operator. Such operator was first investigated by O. Frostman [7]. Several important properties of the potential were obtained by M. Riesz [18].

The weighted (L^p, L^q) -boundedness of Riesz potentials is given by the following Stein–Weiss inequality

$$\| |x|^{-\gamma} I_\alpha f(x) \|_q \leq \mathbf{c}(\alpha, \beta, \gamma, p, q, d) \| |x|^\beta f(x) \|_p \quad (1.1)$$

with the sharp constant $\mathbf{c}(\alpha, \beta, \gamma, p, q, d)$ and $1 < p \leq q < \infty$. Sufficient conditions for the finiteness of $\mathbf{c}(\alpha, \beta, \gamma, p, q, d)$ are well known.

Theorem 1.1 Let $d \in \mathbb{N}$, $1 \leq p \leq q < \infty$, $\gamma < \frac{d}{q}$, $\gamma + \beta \geq 0$, $0 < \alpha < d$, and $\alpha - \gamma - \beta = d(\frac{1}{p} - \frac{1}{q})$.

(a) If $1 < p \leq q < \infty$ and $\beta < \frac{d}{p'}$, then $\mathbf{c}(\alpha, \beta, \gamma, p, q, d) < \infty$.

(b) If $p = 1$, $1 < q < \infty$, $\beta \leq 0$, then, for $f \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda > 0$,

$$\int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} |I_\alpha f(x)| > \lambda\}} d\mu(x) \lesssim \left(\frac{\| |x|^\beta f(x) \|_1}{\lambda} \right)^q.$$

The part (a) in Theorem 1.1 was proved by G.H. Hardy and J.E. Littlewood [12] for $d = 1$, S. Sobolev [26] for $d > 1$ and $\gamma = \beta = 0$, E.M. Stein and G. Weiss [27] in the general case. The conditions for weak boundedness can be found in [9, 25].

The sharp constant $\mathbf{c}(\alpha, 0, 0, p, q, d)$ in the non-weighted Sobolev inequality was calculated by E.H. Lieb [15] in any of the following cases: (1) $q = p'$, $1 < p < 2$, (2) $q = 2$, $1 < p < 2$, (3) $p = 2$, $2 < q < \infty$. Moreover, in these cases there exist maximizing functions. In the weighted Hardy–Littlewood–Sobolev inequality the constant $\mathbf{c}(\alpha, \beta, \gamma, p, q, d)$ is known only for $q = p$.

Theorem 1.2 If $d \in \mathbb{N}$, $1 < p < \infty$, $\gamma < \frac{d}{p}$, $\beta < \frac{d}{p'}$, $\alpha > 0$, and $\gamma = \alpha - \beta$, then

$$\mathbf{c}(\alpha, \beta, \gamma, p, p, d) = 2^{-\alpha} \frac{\Gamma\left(\frac{1}{2}\left(\frac{d}{p} - \alpha + \beta\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{d}{p'} - \beta\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{d}{p'} + \alpha - \beta\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{d}{p} + \beta\right)\right)}.$$

Theorem 1.2 was proved by I.W. Herbst [14] for $\beta = 0$ and W. Beckner [4] and S. Samko [24] in the general case.

For $\alpha \in \mathbb{R}$, we define the Riesz potential in the distributional sense. Let Φ be the Lizorkin space [16], [23, p. 39], that is, a subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ which consists of

functions orthogonal to all polynomials:

$$\int_{\mathbb{R}^d} x^n f(x) d\mu(x) = 0, \quad n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d.$$

The subspace Φ is invariant with respect to the operator I_α and its inverse $I_\alpha^{-1} = (-\Delta)^{\alpha/2}$:

$$I_\alpha(\Phi) = (-\Delta)^{\alpha/2}(\Phi) = \Phi,$$

where Δ is the Laplacian. Note that Φ is dense in $L_p(\mathbb{R}^d, |x|^{\beta p} d\mu)$ for $1 < p < \infty$ and $\beta \in (-d/p, d/p')$ [23, p. 41].

It is worth mentioning that the Stein–Weisz inequality (1.1) on Φ is equivalent to the Hardy–Rellich inequality

$$\| |x|^{-\gamma} f(x) \|_q \leq \mathbf{c}(\alpha, \beta, \gamma, p, q, d) \| |x|^\beta (-\Delta)^{\alpha/2} f(x) \|_p.$$

Let $D_j f(x)$ be the usual partial derivative with respect to a variable x_j , $j = 1, \dots, d$, $D = (D_1, \dots, D_d)$, $D^n f(x) = \prod_{j=1}^d D_j^{n_j} f(x)$, $n \in \mathbb{Z}_+^d$. The subspace

$$\Psi = \{ \mathcal{F}(f) : f \in \Phi \} = \{ f \in \mathcal{S}(\mathbb{R}^d) : D^n f(0) = 0, \quad n \in \mathbb{Z}_+^d \}$$

is invariant with respect to the operator $\mathcal{F}(I_\alpha)$ and $\mathcal{F}((-\Delta)^{\alpha/2})$:

$$\mathcal{F}(I_\alpha)(\Psi) = \mathcal{F}((-\Delta)^{\alpha/2})(\Psi) = \Psi.$$

For a distribution $f \in \Phi'$ and $\alpha \in \mathbb{R}$ we set

$$I_\alpha f = \mathcal{F}^{-1} | \cdot |^{-\alpha} \mathcal{F}(f), \quad (-\Delta)^{\alpha/2} f = \mathcal{F}^{-1} | \cdot |^\alpha \mathcal{F}(f).$$

If $\varphi \in \Phi$, then

$$\langle I_\alpha f, \varphi \rangle = \langle f, \mathcal{F} | \cdot |^{-\alpha} \mathcal{F}^{-1}(\varphi) \rangle, \quad \langle (-\Delta)^{\alpha/2} f, \varphi \rangle = \langle f, \mathcal{F} | \cdot |^\alpha \mathcal{F}^{-1}(\varphi) \rangle.$$

One of the generalizations of the Fourier transform is the Dunkl transform \mathcal{F}_k (see [6, 21]). Our main goal in this paper is to prove analogues of Theorems 1.1 and 1.2 for the Riesz potential associated with the Dunkl transform. We shall call it *the D-Riesz potential*.

Let a finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ be a root system, let R_+ be positive subsystem of R , let $G(R) \subset O(d)$ be finite reflection group, generated by reflections $\{\sigma_a : a \in R\}$, where σ_a is a reflection with respect to hyperplane $\langle a, x \rangle = 0$, let $k : R \rightarrow \mathbb{R}_+$ be G -invariant multiplicity function. Recall that a finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system, if

$$R \cap \mathbb{R}a = \{a, -a\} \quad \text{and} \quad \sigma_a R = R \quad \text{for all } a \in R.$$

Let

$$v_k(x) = \prod_{a \in R_+} |\langle a, x \rangle|^{2k(a)}$$

be the Dunkl weight. The normalized Macdonald–Metha–Selberg constant is given by

$$c_k^{-1} = \int_{\mathbb{R}^d} e^{-|x|^2/2} v_k(x) dx.$$

Let $L^p(\mathbb{R}^d, d\mu_k)$ be the space of complex-valued Lebesgue measurable functions f such that

$$\|f\|_{p, d\mu_k} = \left(\int_{\mathbb{R}^d} |f|^p d\mu_k \right)^{1/p} < \infty,$$

where $d\mu_k(x) = c_k v_k(x) dx$ is the Dunkl measure. Assume that

$$T_j f(x) = D_j f(x) + \sum_{a \in R_+} k(a) \langle a, e_j \rangle \frac{f(x) - f(\sigma_a x)}{\langle a, x \rangle} \quad (1.2)$$

are differential-differences Dunkl operators, $j = 1, \dots, d$, and $\Delta_k = \sum_{j=1}^d T_j^2$ is the Dunkl Laplacian [10].

The Dunkl kernel $E_k(x, y)$ is a unique solution of the system

$$T_j f(x) = y_j f(x), \quad j = 1, \dots, d, \quad f(0) = 1.$$

Let $e_k(x, y) = E_k(x, iy)$. It plays the role of a generalized exponential function. Its properties are similar to those of the classical exponential function $e^{i\langle x, y \rangle}$. Several basic properties follow from the integral representation given by M. Rösler [20]

$$e_k(x, y) = \int_{\mathbb{R}^d} e^{i\langle \xi, y \rangle} d\mu_x^k(\xi), \quad (1.3)$$

where μ_x^k is a probability Borel measure, whose support is contained in $\text{co}(\{gx : g \in G(R)\})$ the convex hull of the G -orbit of x in \mathbb{R}^d . In particular, $|e_k(x, y)| \leq 1$ and $\text{supp } \mu_x^k \subset B_{|x|}$, where B_r is the Euclidean ball of radius r centered at 0.

For $f \in L^1(\mathbb{R}^d, d\mu_k)$, the Dunkl transform is defined by the equality

$$\mathcal{F}_k(f)(y) = \int_{\mathbb{R}^d} f(x) \overline{e_k(x, y)} d\mu_k(x).$$

If $k \equiv 0$, then \mathcal{F}_0 is the Fourier transform \mathcal{F} . We note that $\mathcal{F}_k(e^{-|\cdot|^2/2})(y) = e^{-|y|^2/2}$ and $\mathcal{F}_k^{-1}(f)(x) = \mathcal{F}_k(f)(-x)$. The Dunkl transform is isometry in $\mathcal{S}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d, d\mu_k)$ and $\|f\|_{2, d\mu_k} = \|\mathcal{F}_k(f)\|_{2, d\mu_k}$.

M. Rösler [19] defined the generalized translation operator τ^y , $y \in \mathbb{R}^d$, on $L^2(\mathbb{R}^d, d\mu_k)$ by equality

$$\mathcal{F}_k(\tau^y f)(z) = e_k(y, z) \mathcal{F}_k(f)(z),$$

or

$$\tau^y f(x) = \int_{\mathbb{R}^d} e_k(y, z) e_k(x, z) \mathcal{F}_k(f)(z) d\mu_k(z). \quad (1.4)$$

It acts from $L^2(\mathbb{R}^d, d\mu_k)$ to $L^2(\mathbb{R}^d, d\mu_k)$ and $\|\tau^y\|_{2 \rightarrow 2} = 1$.

If $k \equiv 0$, then $\tau^y f(x) = f(x + y)$. If $f \in \mathcal{S}(\mathbb{R}^d)$, then $\tau^y f(x) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ and equality (1.4) holds pointwise. K. Trimèche extended τ^y on $C^\infty(\mathbb{R}^d)$ [30]. For example, $\tau^y 1 = 1$. In general, τ^y is not positive operator and the question of its L_p -boundedness remains open.

First, we define the D-Riesz potential for distributions. Let

$$\Phi_k = \left\{ f \in \mathcal{S}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^n f(x) d\mu_k(x) = 0, \quad n \in \mathbb{Z}_+^d \right\}$$

be the weighted Lizorkin space,

$$\Psi_k = \{\mathcal{F}_k(f) : f \in \Phi_k\}.$$

For $\alpha \in \mathbb{R}$, we define the D-Riesz potential on Φ_k by equality

$$I_\alpha^k f = \mathcal{F}_k^{-1} |\cdot|^{-\alpha} \mathcal{F}_k(f).$$

In Section 6, we will prove that $\Psi_k = \Psi$ (see Theorem 6.1) and then $I_\alpha^k(\Phi_k) = \Phi_k$ and $\mathcal{F}_k(I_\alpha^k(\Psi_k)) = \Psi_k$. Therefore, we can define the D-Riesz potential I_α^k for $f \in \Phi'_k$ and $\alpha \in \mathbb{R}$ by the same equality $I_\alpha^k f = \mathcal{F}_k^{-1} |\cdot|^{-\alpha} \mathcal{F}_k(f)$ as follows

$$\langle I_\alpha f, \varphi \rangle = \langle f, \mathcal{F}_k |\cdot|^{-\alpha} \mathcal{F}_k^{-1}(\varphi) \rangle, \quad \varphi \in \Phi_k.$$

We will also prove (see Theorem 6.3) that Φ_k is dense in $L_p(\mathbb{R}^d, |x|^{\beta p} d\mu_k)$ for $1 < p < \infty$ and $\beta \in (-d_k/p, d_k/p')$, where

$$d_k = 2\lambda_k + 2, \quad \lambda_k = \frac{d}{2} - 1 + \sum_{a \in R_+} k(a). \quad (1.5)$$

S. Thangavelu and Y. Xu defined [29] the D-Riesz potential on Schwartz space as follows

$$I_\alpha^k f(x) = (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^d} \tau^{-y} f(x) |y|^{\alpha-d_k} d\mu_k(y), \quad (1.6)$$

where $0 < \alpha < d_k$ and $\gamma_\alpha^k = 2^{\alpha-d_k/2} \Gamma(\alpha/2) / \Gamma((d_k - \alpha)/2)$.

We are interested in the Stein–Weiss inequality for the D-Riesz potential

$$\left\| |x|^{-\gamma} I_\alpha^k f(x) \right\|_{q, d\mu_k} \leq \mathbf{c}_k(\alpha, \beta, \gamma, p, q, d) \left\| |x|^\beta f(x) \right\|_{p, d\mu_k}, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (1.7)$$

with the sharp constant $\mathbf{c}_k(\alpha, \beta, \gamma, p, q, d)$ and $1 < p \leq q < \infty$. On Φ_k , it is equivalent to the Hardy–Rellich type inequality

$$\left\| |x|^{-\gamma} f(x) \right\|_{q, d\mu_k} \leq \mathbf{c}_k(\alpha, \beta, \gamma, p, q, d) \left\| |x|^\beta (-\Delta_k)^{\alpha/2} f(x) \right\|_{p, d\mu_k}.$$

Our main results read as follows.

Theorem 1.3 *If $d \in \mathbb{N}$, $1 < p < \infty$, $\gamma < \frac{d_k}{p}$, $\beta < \frac{d_k}{p'}$, $\alpha > 0$, and $\alpha = \gamma + \beta$, then*

$$\begin{aligned} \mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) &= 2^{-\alpha} \frac{\Gamma\left(\frac{1}{2}\left(\frac{d_k}{p} - \gamma\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{d_k}{p'} - \beta\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{d_k}{p'} + \gamma\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{d_k}{p} + \beta\right)\right)} \\ &= \mathbf{c}(\alpha, \beta, \gamma, p, p, d_k). \end{aligned}$$

Theorem 1.4 *Let $d \in \mathbb{N}$, $1 \leq p \leq q < \infty$, $\gamma < \frac{d_k}{q}$, $\gamma + \beta \geq 0$, $0 < \alpha < d_k$, and $\alpha - \gamma - \beta = d_k(\frac{1}{p} - \frac{1}{q})$.*

- (a) *If $1 < p \leq q < \infty$ and $\beta < \frac{d_k}{p'}$, then $\mathbf{c}_k(\alpha, \beta, \gamma, p, q, d) < \infty$.*
- (b) *If $p = 1$, $1 < q < \infty$, $\beta \leq 0$, and $\lambda > 0$, then*

$$\int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} |I_\alpha^k f(x)| > \lambda\}} d\mu_k(x) \lesssim \left(\left\| |x|^\beta f(x) \right\|_{1, d\mu_k} / \lambda \right)^q, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

For $k \equiv 0$, Theorems 1.3 and 1.4 become Theorems 1.1 and 1.2, therefore it is enough to consider the case $k \not\equiv 0$, i.e., $\lambda_k = \frac{d}{2} - 1 + \sum_{a \in R_+} k(a) > -1/2$ and $d_k = 2\lambda_k + 2 > 1$. It is clear that d_k plays the role of the generalized dimension of the space $(\mathbb{R}^d, d\mu_k)$.

For the reflection group \mathbb{Z}_2^d and $\gamma = \beta = 0$, Theorem 1.4 was proved in [29]. For arbitrary reflection group G and $\gamma = \beta = 0$, it was proved by S. Hassani, S. Mustapha and M. Sifi [13]. Following an idea from [29], we have recently given another proof in [11]. Regarding the weighted setting, part (a) was proved in [1] in the case $q = p$ under more restrictive conditions $1 < p < \infty$, $0 < \gamma < \frac{d_k}{p}$, $0 < \beta < \frac{d_k}{p'}$, and $\alpha > 0$.

To estimate the L^p -norm of operator I_α^k , S. Thangavelu and Y. Xu [29] used the maximal function, defined for $f \in \mathcal{S}(\mathbb{R}^d)$ as follows

$$M^k f(x) = \sup_{r>0} \frac{\left| \int_{\mathbb{R}^d} \tau^{-y} f(x) \chi_{B_r}(y) d\mu_k(y) \right|}{\int_{B_r} d\mu_k},$$

where $B_r = \{x : |x| \leq r\}$. They proved the strong L^p -boundedness of M^k for $1 < p < \infty$ and the weak boundedness for $p = 1$ [28].

We will use Theorem 1.4 to obtain weighted boundedness of the fractional maximal function $M_\alpha^k f$, $0 \leq \alpha < d_k$, given by

$$\begin{aligned} M_\alpha^k f(x) &= \sup_{r>0} r^{\alpha-d_k} \left| \int_{\mathbb{R}^d} \tau^{-y} f(x) \chi_{B_r}(y) d\mu_k(y) \right| \\ &= \sup_{r>0} r^{\alpha-d_k} \left| \int_{\mathbb{R}^d} f(x) \tau^{-y} \chi_{B_r}(x) d\mu_k(y) \right|. \end{aligned}$$

If $\alpha = 0$, then M_0^k coincides with M^k up to a constant. Since τ^y is a positive operator on radial functions [22, 28], and using

$$M_\alpha^k f(x) \leq M_\alpha^k |f|(x) \lesssim I_\alpha^k |f|(x),$$

Theorem 1.4 implies the boundedness conditions of the fractional maximal function.

Theorem 1.5 Let $d \in \mathbb{N}$, $1 \leq p \leq q < \infty$, $\gamma < \frac{d_k}{q}$, $\gamma + \beta \geq 0$, $0 < \alpha < d_k$, $\alpha - \gamma - \beta = d_k(\frac{1}{p} - \frac{1}{q})$, and $f \in \mathcal{S}(\mathbb{R}^d)$.

(a) If $1 < p \leq q < \infty$ and $\beta < \frac{d_k}{p'}$, then

$$\left\| |x|^{-\gamma} M_\alpha^k f(x) \right\|_{q, d\mu_k} \lesssim \left\| |x|^\beta f(x) \right\|_{p, d\mu_k}.$$

(b) If $p = 1$, $1 < q < \infty$, $\beta \leq 0$, and $\lambda > 0$, then

$$\int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} |M_\alpha^k f(x)| > \lambda\}} d\mu_k(x) \lesssim \left(\left\| |x|^\beta f(x) \right\|_{1, d\mu_k} / \lambda \right)^q.$$

In the case $\gamma = \beta = 0$ Theorem 1.5 was proved in [13].

The paper is organized as follows. In the next section, we obtain the sharp inequalities for Mellin convolution and investigate the following representation of the Riesz potential

$$I_\alpha^k f(x) = \int_{\mathbb{R}^d} f(y) \Phi(x, y) d\mu_k(y)$$

and basic properties of the kernel

$$\Phi(x, y) = \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} \tau^{-y} (e^{-s|\cdot|^{d_k}})^2(x) ds, \quad (x, y) \neq (0, 0).$$

In Section 3, we prove sharp (L_p, L_p) Hardy's inequalities with weights for the averaging operator $Hf(x) = \int_{|y| \leq |x|} f(y) d\mu_k(y)$. In the classical setting ($k = 0$), this result was proved by M. Christ and L. Grafakos [5] and Z.W. Fu, L. Grafakos, S.Z. Lu and F.Y. Zhao [8]. Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4 correspondingly. We finish with Section 6, which contains some important properties of the spaces Φ_k and Ψ_k .

2 Notations and Auxiliary Statements

Set as usual $\mathbb{R}_+ = [0, \infty)$, $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ and $x = rx' \in \mathbb{R}^d$, $r = |x| \in \mathbb{R}_+$, $x' \in \mathbb{S}^{d-1}$. Let

$$dv(r) = r^{-1} dr, \quad dv_\lambda(r) = b_\lambda r^{2\lambda+1} dr, \quad b_\lambda^{-1} = 2^\lambda \Gamma(\lambda + 1), \quad \lambda \geq -1/2,$$

be the measures on \mathbb{R}_+ ,

$$d\sigma_k(x') = a_k v_k(x') dx'$$

be the probability measure on \mathbb{S}^{d-1} , and

$$d\mu_k(x) = c_k v_k(x) dx, \quad dm_k(x) = dv(r) d\sigma_k(x')$$

be the measures on \mathbb{R}^d .

Note that

$$d\mu_k(x) = dv_{\lambda_k}(r) d\sigma_k(x') = b_{\lambda_k} |x|^{d_k} dm_k(x), \quad (2.1)$$

where λ_k and d_k are defined in Eq. 1.5.

Let $L^p(X, d\mu)$, $1 \leq p \leq \infty$, be the Banach space with the norm

$$\|f\|_{p,d\mu} = \begin{cases} \left(\int_X |f|^p d\mu \right)^{1/p}, & p < \infty, \\ \sup_{\text{vrai}_X} |f|, & p = \infty. \end{cases}$$

Depending on the context, we assume that $L^p = L^p(X, d\mu)$ and $\|f\|_p = \|f\|_{p,d\mu}$.

2.1 Convolution Inequalities

The Mellin convolution is given by

$$A_g f(r) = (f * g)(r) = \int_0^\infty f(r/t) g(t) dv(t).$$

We will frequently use the fact that

$$(f * g)(r) = (g * f)(r). \quad (2.2)$$

Lemma 2.1 Let $L^p = L^p(\mathbb{R}_+, dv)$, $1 \leq p \leq \infty$. If $f \in L^p$, $h \in L^{p'}$, $g \in L^1$, then

$$\|f * g\|_p \leq \|g\|_1 \|f\|_p,$$

or

$$\left| \int_0^\infty \int_0^\infty h(r) f(t) g(r/t) dv(t) dv(r) \right| \leq \|g\|_1 \|h\|_{p'} \|f\|_p.$$

If $g \geq 0$, then

$$\|A_g\|_{p \rightarrow p} = \|g\|_1, \quad (2.3)$$

or

$$\sup_{\|f\|_p \leq 1} \sup_{\|h\|_{p'} \leq 1} \left| \int_0^\infty h(r) A_g f(r) dv(r) \right| = \|g\|_1.$$

Proof For the classical convolution on the \mathbb{R} , see, e.g., [14]. We sketch the proof here for completeness of the exposition. For $1 < p < \infty$, using Hölder's inequality, we obtain

$$\left| \int_0^\infty f(r/t) g(t) dv(t) \right| \leq \left(\int_0^\infty |f(r/t)|^p |g(t)| dv(t) \right)^{1/p} \left(\int_0^\infty |g(t)| dv(t) \right)^{1/p'}$$

and

$$\begin{aligned} \|f * g\|_p &\leq \left(\int_0^\infty \int_0^\infty |f(r/t)|^p |g(t)| dv(t) dv(r) \right)^{1/p} \|g\|_1^{1/p'} \\ &= \left(\int_0^\infty |g(t)| \int_0^\infty |f(r/t)|^p dv(r) dv(t) \right)^{1/p} \|g\|_1^{1/p'} = \|f\|_p \|g\|_1. \end{aligned}$$

Let $g \geq 0$. If $p = 1$, $f \in L^1$, $f \geq 0$, then

$$\begin{aligned}\|A_g f\|_1 &= \int_0^\infty \int_0^\infty f(r/t)g(t) dv(t) dv(r) \\ &= \int_0^\infty f(r) dv(r) \int_0^\infty g(t) dv(t) = \|g\|_1 \|f\|_1,\end{aligned}$$

which gives Eq. 2.3. If $p = \infty$, we define $f = \chi_{[\lambda, 1/\lambda]}$, $0 < \lambda < 1$. Then $\|f\|_\infty = 1$ and, for $r \in [1, 2]$,

$$\begin{aligned}A_g f(r) &= \int_0^\infty f(t)g(r/t) dv(t) = \int_\lambda^{1/\lambda} g(r/t) dv(t) \\ &= \int_{r\lambda}^{r/\lambda} g(t) dv(t) \geq \int_{2\lambda}^{1/\lambda} g(t) dv(t) \rightarrow \|g\|_1, \quad \lambda \rightarrow 0.\end{aligned}$$

If $1 < p < \infty$, $f = (2\lambda)^{-1/p} \chi_{[e^{-\lambda}, e^\lambda]}$, and $h = (2\lambda)^{-1/p'} \chi_{[e^{-\lambda}, e^\lambda]}$, then $\|f\|_p = \|h\|_{p'} = 1$ and by the Lebesgue dominated convergence theorem

$$\begin{aligned}\|A_g\|_{p \rightarrow p} &\geq \lim_{\lambda \rightarrow \infty} \left\{ (2\lambda)^{-1} \int_{e^{-\lambda}}^{e^\lambda} \int_{e^{-\lambda}}^{e^\lambda} g(r/t) dv(r) dv(t) \right\} \\ &= \lim_{\lambda \rightarrow \infty} \left\{ (2\lambda)^{-1} \int_{e^{-\lambda}}^{e^\lambda} \int_{e^{-\lambda}/t}^{e^\lambda/t} g(r) dv(r) dv(t) \right\} \\ &= \lim_{\lambda \rightarrow \infty} (2\lambda)^{-1} \left\{ \int_{e^{-2\lambda}}^1 \int_{e^{-\lambda}/r}^{e^\lambda} \frac{dt}{t} g(r) \frac{dr}{r} + \int_1^{e^{2\lambda}} \int_{e^{-\lambda}}^{e^\lambda/r} \frac{dt}{t} g(r) \frac{dr}{r} \right\} \\ &= \lim_{\lambda \rightarrow \infty} (2\lambda)^{-1} \left\{ \int_{e^{-2\lambda}}^1 g(r)(2\lambda + \ln r) \frac{dr}{r} + \int_1^{e^{2\lambda}} g(r)(2\lambda - \ln r) \frac{dr}{r} \right\} \\ &= \int_0^\infty g(r) \frac{dr}{r} = \|g\|_1.\end{aligned}$$

□

2.2 A Representation of the Riesz Potential

We will use the following representation (see [1]), which is different from the definition (1.6):

$$I_\alpha^k f(x) = \int_{\mathbb{R}^d} f(y) \Phi(x, y) d\mu_k(y), \quad (2.4)$$

where

$$\Phi(x, y) = \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} \tau^{-y} (e^{-s|\%cdot|^2})(x) ds, \quad (x, y) \neq (0, 0). \quad (2.5)$$

To verify Eq. 2.4, we first remark that the convolution

$$(f * _k g)(x) = \int_{\mathbb{R}^d} \tau^{-y} f(x) g(y) d\mu_k(y)$$

is commutative, i.e., $(f * _k g)(x) = (g * _k f)(x)$. Indeed, we have the following

Lemma 2.2 If $f \in \mathcal{S}(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d, d\mu_k)$, and $f_t(x) = f(tx)$, then

$$\int_{\mathbb{R}^d} \tau^{-y} f(x) g(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) \tau^{-y} g(x) d\mu_k(y), \quad (2.6)$$

$$\mathcal{F}_k(f_t)(z) = \frac{1}{t^{d_k}} \mathcal{F}_k(f)\left(\frac{z}{t}\right), \quad \tau^y(f_t)(x) = \tau^{ty} f(tx). \quad (2.7)$$

Relation (2.6) has been recently proved in [11]. Equalities (2.7) can be verified by simple calculations.

Remark 2.1 It is worth mentioning that if the convolution is defined by

$$(f *_k g)(x) = \int_{\mathbb{R}^d} \tau^x f(y) g(y) d\mu_k(y)$$

(see [22]), it is not commutative:

$$\int_{\mathbb{R}^d} \tau^x f(y) g(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) \tau^{-x} g(y) d\mu_k(y).$$

Completing the proof of Eq. 2.4, we use Eqs. 1.6, 2.6 and the fact that (see [29])

$$\frac{1}{|y|^{d_k-\alpha}} = \frac{1}{\Gamma((d_k-\alpha)/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s|y|^2} ds, \quad (2.8)$$

to obtain

$$\begin{aligned} I_\alpha^k f(x) &= (\gamma_k^\alpha)^{-1} \int_{\mathbb{R}^d} \tau^{-y} f(x) |y|^{\alpha-d_k} d\mu_k(y) \\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_{\mathbb{R}^d} \tau^{-y} f(x) \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s|y|^2} ds d\mu_k(y) \\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} \int_{\mathbb{R}^d} \tau^{-y} f(x) e^{-s|y|^2} d\mu_k(y) ds \\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_{\mathbb{R}^d} f(y) \int_0^\infty s^{(d_k-\alpha)/2-1} \tau^{-y} (e^{-s|\cdot|^2})(x) ds d\mu_k(y). \end{aligned}$$

The interchange of the order of integration is legitimate, since, for any $x \in \mathbb{R}^d$, the iterated integral

$$\int_{\mathbb{R}^d} |\tau^{-y} f(x)| \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s|y|^2} ds d\mu_k(y)$$

converges, where we have used the fact that $\tau^y f(x) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ whenever $f \in \mathcal{S}(\mathbb{R}^d)$.

2.3 Properties of the Kernel $\Phi(x, y)$

We will need the following notation. Let $\lambda \geq -1/2$, $J_\lambda(t)$ be the classical Bessel function of degree λ and

$$j_\lambda(t) = 2^\lambda \Gamma(\lambda + 1) t^{-\lambda} J_\lambda(t)$$

be the normalized Bessel function. The Hankel transform is defined as follows

$$\mathcal{H}_\lambda(f_0)(r) = \int_0^\infty f_0(t) j_\lambda(rt) dv_\lambda(t), \quad r \in \mathbb{R}_+.$$

It is a unitary operator in $L^2(\mathbb{R}_+, dv_\lambda)$ and $\mathcal{H}_\lambda^{-1} = \mathcal{H}_\lambda$ [3, Chap. 7]. If $\lambda = \lambda_k$, the Hankel transform is a restriction of the Dunkl transform on radial functions. Recall that we assume that $\lambda_k > -1/2$.

For $\lambda > -1/2$, let us consider the Gegenbauer-type translation operator (see, e.g., [17])

$$G^s f_0(r) = c_\lambda \int_0^\pi f_0(\sqrt{r^2 + s^2 - 2rs \cos \varphi}) \sin^{2\lambda} \varphi d\varphi, \quad (2.9)$$

where $c_\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(1/2)\Gamma(\lambda+1/2)}$. If $f_0 \in \mathcal{S}(\mathbb{R}_+)$, then

$$G^s f_0(r) = \int_0^\infty j_\lambda(rt) j_\lambda(st) \mathcal{H}_\lambda(f_0)(t) dv_\lambda(t). \quad (2.10)$$

We will also need the following partial case of the Funk–Hecke formula [31]

$$\int_{\mathbb{S}^{d-1}} e_k(x, ty') d\sigma_k(y') = j_{\lambda_k}(t|x|). \quad (2.11)$$

Let $x = rx'$, $y = ty'$, $r, t > \mathbb{R}_+$, and $x', y' \in \mathbb{S}^{d-1}$.

Lemma 2.3 *The kernel $\Phi(x, y)$ satisfies the following properties*

- (1) $\Phi(x, y) = \Phi(y, x)$;
- (2) $\Phi(rx', ty') = r^{\alpha-d_k} \Phi(x', (t/r)y')$;
- (3) $\int_{\mathbb{S}^{d-1}} \Phi(rx', ty') d\sigma_k(x') = \Phi_0(r, t)$, where

$$\Phi_0(r, t) := (\gamma_\alpha^k)^{-1} c_{\lambda_k} \int_0^\pi (r^2 + t^2 - 2rt \cos \varphi)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi d\varphi;$$

- (4) $\Phi(x, y) = (\gamma_\alpha^k)^{-1} \tau^{-y}(| \cdot |^{\alpha-d_k})(x)$ or, equivalently,

$$\Phi(x, y) = (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{(\alpha-d_k)/2} d\mu_x^k(\eta),$$

where μ_x^k is a probability measure from Eq. 1.3.

Proof Recall that $E_k(x, y)$ is the Dunkl kernel. Using $E_k(\lambda x, y) = E_k(x, \lambda y)$, $\lambda \in \mathbb{C}$, we have from [19, Sec. 4.9] that

$$\int_{\mathbb{R}^d} e_k(x, z) e_k(-y, z) e^{-|z|^2/2} d\mu_k(z) = e^{-\frac{|x|^2 + |y|^2}{2}} E_k(x, y).$$

This, Eq. 2.7, and the fact that $\mathcal{F}_k(e^{-|\cdot|^2/2})(y) = e^{-|x|^2/2}$ imply that

$$\tau^{-y}(e^{-s|\cdot|^2})(x) = e^{-s(|x|^2 + |y|^2)} E_k(\sqrt{2s} x, \sqrt{2s} y).$$

Since $E_k(x, y) = E_k(y, x)$, the property (1) follows and, moreover,

$$\Phi(x, y) = \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s(|x|^2 + |y|^2)} E_k(\sqrt{2s} x, \sqrt{2s} y) ds.$$

Changing variables $s \rightarrow u/r^2$, we obtain the property (2):

$$\begin{aligned} \Phi(rx', ty') &= r^{\alpha-d_k} \int_0^\infty u^{(d_k-\alpha)/2-1} e^{-u(1+(t/r)^2)} E_k(\sqrt{2u} x', \sqrt{2u} (t/r)y') du \\ &= r^{\alpha-d_k} \Phi(x', (t/r)y'). \end{aligned}$$

Since, by Eq. 2.7 and Eq. 1.4, we have

$$\tau^{-ty'}(e^{-s|\cdot|^{2\alpha}})(rx') = \int_{\mathbb{R}^d} e_k(\sqrt{2s}rx', z)e_k(-\sqrt{2s}ty', z)e^{-|z|^2/2} d\mu_k(z),$$

then taking into account Eqs. 2.1, 2.9, 2.10, and 2.11, we obtain

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \tau^{-ty'}(e^{-s|\cdot|^{2\alpha}})(rx') d\sigma_k(x') \\ &= \int_{\mathbb{R}^d} e_k(-\sqrt{2s}ty', z)e^{-|z|^2/2} \int_{\mathbb{S}^{d-1}} e_k(\sqrt{2s}rx', z) d\sigma_k(x') d\mu_k(z) \\ &= \int_0^\infty j_{\lambda_k}(\sqrt{2s}ru)e^{-u^2/2} \int_{\mathbb{S}^{d-1}} e_k(-\sqrt{2s}ty', uz') d\sigma_k(z') dv_{\lambda_k}(u) \\ &= \int_0^\infty j_{\lambda_k}(\sqrt{2s}ru)j_{\lambda_k}(\sqrt{2s}tu)e^{-u^2/2} dv_{\lambda_k}(u) \\ &= c_{\lambda_k} \int_0^\pi e^{-s(r^2+t^2-2rt \cos \varphi)} \sin^{d_k-2} \varphi d\varphi. \end{aligned}$$

This and Eq. 2.5 imply that

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \Phi(rx', ty') d\sigma_k(x') \\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} c_{\lambda_k} \int_0^\infty s^{(d_k-\alpha)/2-1} \int_0^\pi e^{-s(r^2+t^2-2rt \cos \varphi)} \sin^{d_k-2} \varphi d\varphi ds. \end{aligned}$$

Finally, applying Eq. 2.8 gives

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \Phi(rx', ty') d\sigma_k(x') \\ &= (\gamma_\alpha^k)^{-1} c_{\lambda_k} \int_0^\pi (r^2+t^2-2rt \cos \varphi)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi d\varphi = \Phi_0(r, t), \end{aligned}$$

i.e., the property (3) follows.

Let us prove the property (4). Since for radial functions $f(x) = f_0(|x|) \in \mathcal{S}(\mathbb{R}^d)$ [22, 28]

$$\tau^{-y} f(x) = \int_{\mathbb{R}^d} f_0\left(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta \rangle}\right) d\mu_x^k(\eta),$$

where μ_x^k is the probability measure in Eq. 1.3, we derive

$$\tau^{-y}(e^{-s|\cdot|^{2\alpha}})(x) = \int_{\mathbb{R}^d} e^{-s(|x|^2 + |y|^2 - 2\langle y, \eta \rangle)} d\mu_x^k(\eta)$$

and

$$\begin{aligned} \Phi(x, y) &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_0^\infty s^{(d_k-\alpha)/2-1} \int_{\mathbb{R}^d} e^{-s(|x|^2 + |y|^2 - 2\langle y, \eta \rangle)} d\mu_x^k(\eta) ds \\ &= \frac{2^{d_k/2-\alpha}}{\Gamma(\alpha/2)} \int_{\mathbb{R}^d} \int_0^\infty s^{(d_k-\alpha)/2-1} e^{-s(|x|^2 + |y|^2 - 2\langle y, \eta \rangle)} ds d\mu_x^k(\eta) \\ &= (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{(\alpha-d_k)/2} d\mu_x^k(\eta), \end{aligned}$$

where we have used the Tonelli–Fubini Theorem for nonnegative functions. \square

3 Sharp Hardy's Inequalities

Define the Hardy and Bellman operators as follows

$$Hf(x) = \int_{|y| \leq |x|} f(y) d\mu_k(y)$$

and

$$Bf(x) = \int_{|y| \geq |x|} f(y) d\mu_k(y).$$

Let $1 \leq p \leq \infty$. We are interested in the weighted Hardy inequalities of the form

$$\| |x|^{-a} Hf(x) \|_{p, d\mu_k} \leq \mathbf{c}_k^H(a, b, p, d) \| |x|^b f(x) \|_{p, d\mu_k} \quad (3.1)$$

and

$$\| |x|^{-a} Bf(x) \|_{p, d\mu_k} \leq \mathbf{c}_k^B(a, b, p, d) \| |x|^b f(x) \|_{p, d\mu_k} \quad (3.2)$$

with the sharp constants $\mathbf{c}_k^H(a, b, p, d)$ and $\mathbf{c}_k^B(a, b, p, d)$.

In the classical setting ($k \equiv 0$), the sharp constants were calculated by M. Christ and L. Grafakos [5] in the non-weighted case ($b = 0, a = d$) and later by Z.W. Fu, L. Grafakos, S.Z. Lu and F.Y. Zhao [8] in the general case. We extend these results for the Dunkl setting. Recall that

$$\lambda_k = \frac{d}{2} - 1 + \sum_{a \in R_+} k(a), \quad d_k = 2\lambda_k + 2, \quad b_{\lambda_k} = \frac{1}{2^{\lambda_k} \Gamma(\lambda_k + 1)}.$$

Theorem 3.1 *Let $d \in \mathbb{N}$ and $1 \leq p \leq \infty$. Inequality (3.1) holds with $\mathbf{c}_k^H(a, b, p, d) < \infty$ if and only if $\frac{a}{p'} > \frac{b}{p}$ and $a + b = d_k$. Moreover,*

$$\mathbf{c}_{\lambda_k}^H(a, b, p, d) = \frac{b_{\lambda_k}}{\frac{a}{p'} - \frac{b}{p}}.$$

Proof Assume that $\frac{a}{p'} > \frac{b}{p}$ and $a + b = d_k$. We consider

$$\tilde{H}f(x) = \int_{|y| \leq |x|} |y|^{d_k/p' - b} f(y) dm_k(y).$$

According to Eq. 2.1, inequality (3.1) is equivalent to the following estimate

$$b_{\lambda_k} \| |x|^{-a+d_k/p} \tilde{H}f(x) \|_{p, dm_k} \leq \mathbf{c}_k^H(a, b, p, d) \| f \|_{p, dm_k}.$$

If $x = rx'$, $y = ty'$, then changing variables $y \rightarrow (r/t)y'$ yields

$$\tilde{H}f(x) = r^{d_k/p' - b} \int_{\mathbb{R}^d} f((r/t)y') g_0(t) dm_k(ty'),$$

where

$$g_0(t) = t^{b-d_k/p'} \chi_{[1, \infty)}(t).$$

Hence, by Eq. 2.2, we have

$$|x|^{-a+d_k/p} \tilde{H}f(x) = \int_{\mathbb{R}^d} f((r/t)y') g_0(t) dm_k(ty') = \int_{\mathbb{R}^d} f(ty') g_0(r/t) dm_k(ty').$$

Let us consider the integral

$$\begin{aligned} J &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(rx') f(ty') g_0(r/t) dm_k(x) dm_k(y) \\ &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty h(rx') f(ty') g_0(r/t) dv(t) dv(r) d\sigma_k(x') d\sigma_k(y'). \end{aligned}$$

Using Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned} |J| &\leq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left(\int_0^\infty |h(rx')|^{p'} dv(r) \right)^{1/p'} \left(\int_0^\infty |f(ty')|^p dv(t) \right)^{1/p} \\ &\quad \int_0^\infty g_0(r/t) dv(t) d\sigma_k(x') d\sigma_k(y') \\ &\leq \|g_0\|_1 \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty |h(rx')|^{p'} dv(r) d\sigma_k(x') d\sigma_k(y') \right)^{1/p'} \\ &\quad \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty |f(ty')|^p dv(t) d\sigma_k(x') d\sigma_k(y') \right)^{1/p} \\ &= \|g_0\|_1 \|h\|_{p', dm_k} \|f\|_{p, dm_k}. \end{aligned}$$

Hence,

$$\mathbf{c}_k^H(a, b, p, d) \leq b_{\lambda_k} \|g_0\|_1 = b_{\lambda_k} \int_1^\infty t^{b-d_k/p'} \frac{dt}{t} = \frac{b_{\lambda_k}}{\frac{a}{p'} - \frac{b}{p}}.$$

Considering radial functions $f(x) = f_0(|x|) = f_0(r)$, we note that

$$\tilde{H}f(x) = r^{d_k/p'-b} \int_{\mathbb{R}^d} f(r/t) g_0(t) \frac{dt}{t}$$

and

$$|x|^{a-d_k/p} \tilde{H}f(x) = \int_{\mathbb{R}^d} f(r/t) g_0(t) \frac{dt}{t}.$$

Thus, Lemma 2.1 yields that

$$\mathbf{c}_k^H(a, b, p, d) = b_{\lambda_k} \|g_0\|_1 = \frac{b_{\lambda_k}}{\frac{a}{p'} - \frac{b}{p}}.$$

Note that, in particular, this implies that the condition $\frac{a}{p'} > \frac{b}{p}$ is necessary for $\mathbf{c}_k^H(a, b, p, d) < \infty$ to hold. Moreover, if $f_t(x) = f(tx)$, then

$$Hf_t(x) = t^{-d_k} (Hf)_t(x), \quad \left\| |x|^b f_t(x) \right\|_{p, d\mu_k} = t^{-b-d_k/p} \left\| |x|^b f(x) \right\|_{p, d\mu_k}$$

and inequality Eq. 3.1 can be written as

$$t^{-d_k(1+1/p)+a} \left\| |x|^{-a} Hf(x) \right\|_{p, d\mu_k} \leq t^{-b-d_k/p} \mathbf{c}_k^H(a, b, p, d) \left\| |x|^b f(x) \right\|_{p, d\mu_k},$$

which gives the condition $a + b = d_k$. \square

Similarly, we prove the sharp Hardy's inequality for Bellman transform.

Theorem 3.2 *Let $d \in \mathbb{N}$ and $1 \leq p \leq \infty$. Inequality (3.2) holds with $\mathbf{c}_k^B(a, b, p, d) < \infty$ if and only if $\frac{a}{p'} < \frac{b}{p}$ and $a + b = d_k$. Moreover,*

$$\mathbf{c}_k^B(a, b, p, d) = \frac{b_{\lambda_k}}{\frac{b}{p} - \frac{a}{p'}}.$$

Proof We only sketch the proof. Considering

$$\widetilde{B}f(x) = \int_{|y| \geq |x|} |y|^{d_k/p' - b} f(y) dm_k(y)$$

and Eq. 2.1, we rewrite inequality (3.2) as follows

$$b_{\lambda_k} \| |x|^{-a+d_k/p} \widetilde{B}f(x) \|_{p, dm_k} \leq \mathbf{c}_k^B(a, b, p, d) \|f\|_{p, dm_k}.$$

Then we have

$$\widetilde{B}f(x) = r^{d_k/p' - b} \int_{\mathbb{R}^d} f((r/t)y') g_0(t) dm_k(ty'),$$

where

$$g_0(t) = t^{b-d_k/p'} \chi_{[0,1]}(t).$$

Finally,

$$\mathbf{c}_k^B(a, b, p, d) = b_{\lambda_k} \|g_0\|_1 = b_{\lambda_k} \int_0^1 t^{b-d_k/p'} \frac{dt}{t} = \frac{b_{\lambda_k}}{\frac{b}{p} - \frac{a}{p'}}.$$

□

4 Proof of Theorem 1.3

Recall that we consider the case $k \neq 0$, $\lambda_k > -1/2$ and $d_k > 1$. Let $1 < p < \infty$, $\gamma < \frac{d_k}{p}$, $\beta < \frac{d_k}{p'}$, $\alpha > 0$, and $\alpha = \gamma + \beta$. Consider the modified operator

$$\widetilde{I}_\alpha^k f(x) = \int_{\mathbb{R}^d} f(y) |y|^{d_k/p' - \beta} \Phi(x, y) dm_k(y).$$

According to Eq. 2.1, inequality (1.7) for $q = p$ is equivalent to

$$b_{\lambda_k} \left\| |x|^{-\gamma+d_k/p} \widetilde{I}_\alpha^k f(x) \right\|_{p, dm_k} \leq \mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) \|f(x)\|_{p, dm_k}.$$

If $x = rx'$, $y = ty'$, then using the change of variables $y \rightarrow (r/t)y'$ and applying the properties (1), (2) in Lemma 2.3, we have

$$\widetilde{I}_\alpha^k f(x) = r^{-\beta+\alpha-d_k/p} \int_{\mathbb{R}^d} f((r/t)y') \Phi_1(t, x', y') dm_k(ty'),$$

where

$$\Phi_1(t, x', y') = t^{d_k/p - \alpha + \beta} \Phi(tx', y').$$

Hence, by Eq. 2.2,

$$|x|^{-\gamma+d_k/p} \widetilde{I}_\alpha^k f(x) = \int_{\mathbb{R}^d} f(ty') \Phi_1(r/t, x', y') dm_k(ty').$$

We set

$$\begin{aligned} J &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(rx') f(ty') \Phi_1(r/t, x', y') dm_k(x) dm_k(y) \\ &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty h(rx') f(ty') \Phi_1(r/t, x', y') dv(t) dv(r) d\sigma_k(x') d\sigma_k(y'). \end{aligned}$$

In light of Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
 |J| &\leq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left(\int_0^\infty |h(rx')|^{p'} dv(r) \right)^{1/p'} \left(\int_0^\infty |f(ty')|^p dv(t) \right)^{1/p} \\
 &\quad \int_0^\infty \Phi_1(t, x', y') dv(t) d\sigma_k(x') d\sigma_k(y') \\
 &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left(\int_0^\infty |h(rx')|^{p'} dv(r) \int_0^\infty \Phi_1(t, x', y') dv(t) \right)^{1/p'} \\
 &\quad \left(\int_0^\infty |f(ty')|^p dv(t) \int_0^\infty \Phi_1(t, x', y') dv(t) \right)^{1/p} d\sigma_k(x') d\sigma_k(y') \\
 &\leq \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty |h(rx')|^{p'} dv(r) \int_0^\infty \Phi_1(t, x', y') dv(t) d\sigma_k(x') d\sigma_k(y') \right)^{1/p'} \\
 &\quad \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^\infty |f(ty')|^p dv(t) \int_0^\infty \Phi_1(t, x', y') dv(t) d\sigma_k(x') d\sigma_k(y') \right)^{1/p}.
 \end{aligned}$$

Taking into account the properties (1) and (3) of Lemma 2.3, we have

$$\int_{\mathbb{S}^{d-1}} \Phi_1(t, x', y') d\sigma_k(x') = t^{d_k/p-\alpha+\beta} \int_{\mathbb{S}^{d-1}} \Phi(tx', y') d\sigma_k(x') = t^{d_k/p-\alpha+\beta} \Phi_0(t, 1)$$

and

$$\int_{\mathbb{S}^{d-1}} \Phi_1(t, x', y') d\sigma_k(y') = t^{d_k/p-\alpha+\beta} \Phi_0(1, t) = t^{d_k/p-\alpha+\beta} \Phi_0(t, 1).$$

Then, changing the order of integration implies

$$\begin{aligned}
 |J| &\leq \left(\int_0^\infty t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) dv(t) \right)^{1/p'} \left(\int_{\mathbb{S}^{d-1}} \int_0^\infty |h(rx')|^{p'} dv(r) d\sigma_k(x') \right)^{1/p'} \\
 &\quad \left(\int_0^\infty t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) dv(t) \right)^{1/p} \left(\int_{\mathbb{S}^{d-1}} \int_0^\infty |f(ty')|^p dv(t) d\sigma_k(y') \right)^{1/p} \\
 &= \int_0^\infty t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) dv(t) \|h\|_{p', dm_k} \|f\|_{p, dm_k}.
 \end{aligned}$$

Thus,

$$\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) \leq b_{\lambda_k} \int_0^\infty t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) dv(t).$$

Since for radial functions $f(x) = f_0(|x|) = f_0(r)$ we have that

$$|x|^{-\gamma+d_k/p} \tilde{I}_\alpha^k f(x) = \int_0^\infty f_0(r/t) t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) \frac{dt}{t},$$

Lemma 2.1 gives

$$\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) = b_{\lambda_k} \int_0^\infty t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) dv(t).$$

Let us now prove that the conditions $\gamma < \frac{d_k}{p}$, $\beta < \frac{d_k}{p'}$, and $\gamma + \beta = \alpha > 0$ guarantee that $\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) < \infty$. We have

$$\begin{aligned} & \mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) \\ &= (\gamma_\alpha^k)^{-1} c_{\lambda_k} b_{\lambda_k} \int_0^\infty t^{d_k/p-\alpha+\beta} \int_0^\pi \left(t^2 + 1 - 2t \cos \varphi\right)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi \, d\varphi \, dv(t) \\ &= (\gamma_\alpha^k)^{-1} c_{\lambda_k} b_{\lambda_k} \int_0^\infty \frac{t^{d_k/p-\alpha+\beta}}{(1+t^2)^{(d_k-\alpha)/2}} \int_0^\pi \left(1 - \frac{2t \cos \varphi}{1+t^2}\right)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi \, d\varphi \, dv(t). \end{aligned}$$

The integral with respect to t has singularities at $t = 0, 1, \infty$. It converges at the origin if and only if $\gamma = \alpha - \beta < \frac{d_k}{p}$. Moreover, the integral converges at ∞ if and only if $\beta < \frac{d_k}{p'}$. Concerning the point $t = 1$, we set $r := 2t/(1+t^2)$ and note that, letting $r \rightarrow 1-0$,

$$\begin{aligned} \psi(r) &:= \int_0^\pi (1-r \cos \varphi)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi \, d\varphi \\ &\asymp \int_0^1 (1-r+r\varphi^2/2)^{(\alpha-d_k)/2} \varphi^{d_k-2} \, d\varphi + 1 \\ &\asymp \int_0^{\sqrt{1-r}} (1-r)^{(\alpha-d_k)/2} \varphi^{d_k-2} \, d\varphi + \int_{\sqrt{1-r}}^1 \varphi^{\alpha-2} \, d\varphi + 1 \\ &\asymp \begin{cases} (1-r)^{\frac{\alpha-1}{2}}, & 0 < \alpha < 1, \\ -\ln(1-r), & \alpha = 1, \\ 1, & \alpha > 1. \end{cases} \end{aligned}$$

Therefore, letting $t \rightarrow 1$, we have

$$\int_0^\pi \left(1 - \frac{2t \cos \varphi}{1+t^2}\right)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi \, d\varphi \asymp \begin{cases} |1-t|^{\alpha-1}, & 0 < \alpha < 1, \\ -\ln|1-t|, & \alpha = 1, \\ 1, & \alpha > 1, \end{cases}$$

which implies that the singularity at the point $t = 1$ is integrable.

It remains to calculate the integral $\int_0^\infty t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) \, dv(t)$. Let $t \neq 1$, $r = 2t/(1+t^2)$. The series

$$\begin{aligned} (1-r \cos \varphi)^{(\alpha-d_k)/2} &= \Gamma\left(\frac{\alpha-d_k}{2} + 1\right) \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(n+1)\Gamma\left(\frac{\alpha-d_k}{2} + 1 - n\right)} r^n \cos^n \varphi \\ &= \frac{1}{\Gamma\left(\frac{d_k-\alpha}{2}\right)} \sum_{n=0}^\infty \frac{(-1)^n \Gamma\left(\frac{d_k-\alpha}{2} + n\right)}{\Gamma(n+1)} r^n \cos^n \varphi \end{aligned}$$

converges uniformly on $[0, \pi]$ and

$$\begin{aligned}\psi(r) &= \int_0^\pi (1 - r \cos \varphi)^{(\alpha-d_k)/2} \sin^{d_k-2} \varphi \, d\varphi \\ &= \frac{1}{\Gamma\left(\frac{d_k-\alpha}{2}\right)} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{d_k-\alpha}{2} + 2m\right)}{\Gamma(2m+1)} r^{2m} \int_0^\pi \cos^{2m} \varphi \sin^{d_k-2} \varphi \, d\varphi \\ &= \frac{1}{\Gamma\left(\frac{d_k-\alpha}{2}\right)} \sum_{m=0}^{\infty} \frac{\Gamma\left(m + \frac{1}{2}\right) \Gamma\left(\frac{d_k-\alpha}{2} + 2m\right) \Gamma\left(\frac{d_k-1}{2}\right)}{\Gamma(2m+1) \Gamma\left(\frac{d_k}{2} + m\right)} r^{2m}.\end{aligned}$$

Since a positive series can be integrated term-by-term, it follows that

$$\begin{aligned}\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) &= b_{\lambda_k} \int_0^\infty t^{d_k/p-\alpha+\beta} \Phi_0(t, 1) \, dv(t) \\ &= \frac{(\gamma_\alpha^k)^{-1} c_{\lambda_k} b_{\lambda_k}}{\Gamma\left(\frac{d_k-\alpha}{2}\right)} \sum_{m=0}^{\infty} \frac{2^{2m} \Gamma\left(m + \frac{1}{2}\right) \Gamma\left(\frac{d_k-\alpha}{2} + 2m\right) \Gamma\left(\frac{d_k-1}{2}\right)}{\Gamma(2m+1) \Gamma\left(\frac{d_k}{2} + m\right)} \int_0^\infty \frac{t^{d_k/p-\alpha+\beta+2m-1}}{(1+t^2)^{(d_k-\alpha)/2+2m}} \, dt.\end{aligned}$$

Taking into account that

$$\int_0^\infty \frac{t^{d_k/p-\alpha+\beta+2m-1}}{(1+t^2)^{(d_k-\alpha)/2+2m}} \, dt = \frac{\Gamma\left(\frac{d_k}{2p} + \frac{\beta-\alpha}{2} + m\right) \Gamma\left(\frac{d_k}{2p'} - \frac{\beta}{2} + m\right)}{2\Gamma\left(\frac{d_k-\alpha}{2} + 2m\right)}$$

and

$$\gamma_\alpha^k = \frac{2^{\alpha-d_k/2} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{d_k-\alpha}{2}\right)}, \quad c_{\lambda_k} = \frac{\Gamma\left(\frac{d_k}{2}\right)}{\Gamma(1/2) \Gamma\left(\frac{d_k-1}{2}\right)}, \quad b_{\lambda_k} = \frac{1}{2^{d_k/2-1} \Gamma\left(\frac{d_k}{2}\right)},$$

we arrive at

$$\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) = \frac{2^{-\alpha}}{\Gamma(\alpha/2)} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{d_k}{2p} + \frac{\beta-\alpha}{2} + m\right) \Gamma\left(\frac{d_k}{2p'} - \frac{\beta}{2} + m\right)}{\Gamma(m+1) \Gamma\left(\frac{d_k}{2} + m\right)}.$$

Letting

$$a = \frac{d_k}{2p} + \frac{\beta-\alpha}{2}, \quad b = \frac{d_k}{2p'} - \frac{\beta}{2}, \quad c = \frac{d_k}{2},$$

we write

$$\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) = \frac{2^{-\alpha}}{\Gamma(\alpha/2)} \sum_{m=0}^{\infty} \frac{\Gamma(a+m) \Gamma(b+m)}{\Gamma(1+m) \Gamma(c+m)}.$$

Using now the hypergeometric function [2, Ch. II]

$$F(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m! (c)_m} z^m, \quad (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)},$$

we obtain that

$$\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) = \frac{2^{-\alpha}}{\Gamma(\alpha/2)} \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} F(a, b; c; 1).$$

Finally, since [2, Sect. 2.8, (46)]

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c \neq 0, -1, -2, \dots, \quad c > a+b,$$

we have

$$\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) = \frac{2^{-\alpha}\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(\alpha/2)\Gamma(c-a)\Gamma(c-b)},$$

where

$$\begin{aligned} c-a-b &= \frac{d_k}{2} - \left(\frac{d_k}{2p} + \frac{\beta-\alpha}{2} + \frac{d_k}{2p'} - \frac{\beta}{2} \right) = \frac{\alpha}{2}, \\ c-a &= \frac{d_k}{2} - \left(\frac{d_k}{2p} + \frac{\beta-\alpha}{2} \right) = \frac{d_k}{2p'} + \frac{\alpha-\beta}{2}, \\ c-b &= \frac{d_k}{2} - \left(\frac{d_k}{2p'} - \frac{\beta}{2} \right) = \frac{d_k}{2p} + \frac{\beta}{2}, \end{aligned}$$

or, equivalently,

$$\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) = 2^{-\alpha} \frac{\Gamma(\frac{1}{2}(\frac{d_k}{p} - \gamma))\Gamma(\frac{1}{2}(\frac{d_k}{p'} - \beta))}{\Gamma(\frac{1}{2}(\frac{d_k}{p'} + \gamma))\Gamma(\frac{1}{2}(\frac{d_k}{p} + \beta))}.$$

Remark 4.1 It is clear that the condition $\alpha = \gamma + \beta$ is necessary for $\mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) < \infty$ to hold. Indeed, setting $f_t(x) = f(tx)$, we have

$$\mathcal{F}_k(f_t)(z) = t^{-d_k} \mathcal{F}_k(f)\left(\frac{z}{t}\right), \quad \tau^y f_t(x) = \tau^{ty} f(tx), \quad I_\alpha^k f_t(x) = t^{-\alpha} (I_\alpha^k f)_t(x),$$

$$\| |x|^\beta f_t(x) \|_{p, d\mu_k} = t^{-\beta-d_k/p} \| |x|^\beta f(x) \|_{p, d\mu_k}.$$

Writing inequality Eq. 1.7 with $q = p$ as follows

$$t^{\gamma-\alpha-d_k/p} \| |x|^{-\gamma} I_\alpha^k f(x) \|_{p, d\mu_k} \leq t^{-\beta-d_k/p} \mathbf{c}_k(\alpha, \beta, \gamma, p, p, d) \| |x|^\beta f(x) \|_{p, d\mu_k}$$

implies $\alpha = \gamma + \beta$.

5 Proof of Theorem 1.4

Part (a) Let $1 < p < q < \infty$, $\gamma < \frac{d_k}{q}$, $\beta < \frac{d_k}{p'}$, $\gamma + \beta \geq 0$, $0 < \alpha < d_k$, and $\alpha - \gamma - \beta = d_k(\frac{1}{p} - \frac{1}{q})$. Note that the case $q = p$ was studied in Theorem 1.3. We will use the representation of the kernel $\Phi(x, y)$ given in Lemma 2.3 and then essentially follow the ideas of [27].

We write

$$\tilde{I}_\alpha^k f(x) = \int_{\mathbb{R}^d} f(y) |y|^{-\beta} \Phi_\alpha(x, y) d\mu_k(y),$$

where

$$f \in \mathcal{S}(\mathbb{R}^d), \quad \Phi_\alpha(x, y) = \int_{\mathbb{R}^d} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{(\alpha-d_k)/2} d\mu_x^k(\eta),$$

and

$$\text{supp } \mu_x^k \subset B_{|x|} = \{\eta : |\eta| \leq |x|\}.$$

We define

$$J := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) g(x) \frac{\Phi_\alpha(x, y)}{|x|^\gamma |y|^\beta} d\mu_k(y) d\mu_k(x).$$

It is sufficient to prove the inequality

$$J \lesssim \|f\|_{p,d\mu_k} \|g\|_{q',d\mu_k} \quad (5.1)$$

for $f, g \geq 0$.

Recall that in the case $1 < p < q < \infty$, $\gamma = \beta = 0$, and $\alpha = d_k(\frac{1}{p} - \frac{1}{q})$ inequality Eq. 5.1 holds (see [1, 11]). Let

$$\mathbb{R}^d \times \mathbb{R}^d = E_1 \sqcup E_2 \sqcup E_3,$$

where

$$E_1 = \{(x, y) : 2^{-1}|y| < |x| < 2|y|\},$$

$$E_2 = \{(x, y) : |x| \leq 2^{-1}|y|\},$$

$$E_3 = \{(x, y) : |y| \leq 2^{-1}|x|\}.$$

Then

$$J = \iint_{E_1} + \iint_{E_2} + \iint_{E_3} = J_1 + J_2 + J_3.$$

Estimate of J_1 . If $(x, y) \in E_1$, using $|\eta| \leq |x|$, then by conditions $\alpha - \beta - \gamma = d_k(\frac{1}{p} - \frac{1}{q})$, $\gamma + \beta \geq 0$ we have

$$\begin{aligned} (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{\frac{\gamma+\beta}{2}} &\leq (|x|^2 + 4|x|^2 + 2|x||y|)^{\frac{\gamma+\beta}{2}} \\ &\lesssim |x|^{\gamma+\beta} \lesssim |x|^\gamma |y|^\beta \end{aligned}$$

and

$$\begin{aligned} \frac{(|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{\frac{\alpha-d_k}{2}}}{|x|^\gamma |y|^\beta} &\lesssim (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{\frac{\alpha-\beta-\gamma-d_k}{2}} \\ &= (|x|^2 + |y|^2 - 2\langle y, \eta \rangle)^{d_k(\frac{1}{p}-\frac{1}{q})-d_k)/2}. \end{aligned}$$

Set $\tilde{\alpha} = d_k(\frac{1}{p} - \frac{1}{q})$. By Eq. 5.1 with $\gamma = \beta = 0$ and $0 < \tilde{\alpha} < d_k$, we have

$$J_1 \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)g(x) \Phi_{\tilde{\alpha}}(x, y) d\mu_k(y) d\mu_k(x) \lesssim \|f\|_{p,d\mu_k} \|g\|_{q',d\mu_k}.$$

Estimate of J_2 . If $(x, y) \in E_2$, then

$$\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta \rangle} \geq \sqrt{|x|^2 + |y|^2 - 2|x||y|} \geq |y| - |x| \geq 2^{-1}|y|,$$

therefore

$$\begin{aligned} \Phi_\alpha(x, y) &= \int_{\mathbb{R}^d} \frac{1}{(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta \rangle})^{d_k-\alpha}} d\mu_x^k(\eta) \\ &\lesssim |y|^{\alpha-d_k} \int_{\mathbb{R}^d} d\mu_x^k(\eta) = |y|^{\alpha-d_k}. \end{aligned}$$

From here and since $E_2 \subset \{(x, y) : |x| \leq |y|\}$,

$$\begin{aligned} J_2 &\lesssim \iint_{|x| \leq |y|} \frac{f(y)g(x)}{|x|^\gamma |y|^{\beta-\alpha+d_k}} d\mu_k(x) d\mu_k(y) \\ &= \int_{\mathbb{R}^d} f(y) |y|^{\alpha-\beta-d_k} \int_{|x| \leq |y|} g(x) |x|^{-\gamma} d\mu_k(x) d\mu_k(y) \\ &= \int_{\mathbb{R}^d} f(y) |y|^{\alpha-\beta-\gamma} Vg(y) d\mu_k(y), \end{aligned}$$

where

$$Vg(y) = |y|^{\gamma-d_k} \int_{|x| \leq |y|} g(x) |x|^{-\gamma} d\mu_k(x).$$

Note that

$$\begin{aligned} Vg(y) &\leq |y|^{\gamma-d_k} \left(\int_{|x| \leq |y|} |x|^{-q\gamma} d\mu_k(x) \right)^{1/q} \|g\|_{q', d\mu_k} \\ &\lesssim |y|^{\gamma-d_k} |y|^{d_k/q-\gamma} \|g\|_{q', d\mu_k} = |y|^{-d_k/q'} \|g\|_{q', d\mu_k}. \end{aligned}$$

Hence

$$|Vg(y)|^{p'-q'} |y|^{(\alpha-\beta-\gamma)p'} \lesssim |x|^{-d_k(p'-q')/q'+(\alpha-\beta-\gamma)p'} \|g\|_{q', d\mu_k}^{p'-q'}.$$

Since

$$\begin{aligned} -\frac{d_k(p'-q')}{q'} + (\alpha-\beta-\gamma)p' &= p' \left\{ \alpha-\beta-\gamma-d_k \left(\frac{1}{q'} - \frac{1}{p'} \right) \right\} \\ &= p' \left\{ \alpha-\beta-\gamma+d_k \left(\frac{1}{q} - \frac{1}{p} \right) \right\} = 0, \end{aligned}$$

it follows that

$$|Vg(y)|^{p'-q'} |y|^{(\alpha-\beta-\gamma)p'} \lesssim \|g\|_{q', d\mu_k}^{p'-q'}. \quad (5.2)$$

On the other hand, by Theorem 3.1 with $a = d_k - \gamma$, $b = \gamma$, $p = q'$, and $\frac{a}{q} > \frac{b}{q'}$ (or, equivalently, $\gamma < \frac{d_k}{q}$), we see that

$$\|Vg\|_{q', d\mu_k} \lesssim \|g\|_{q', d\mu_k}. \quad (5.3)$$

Using Eqs. 5.2 and 5.3, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |Vg(y)|^{p'} |y|^{(\alpha-\beta-\gamma)p'} d\mu_k(y) &= \int_{\mathbb{R}^d} |Vg(y)|^{q'} |Vg(y)|^{p'-q'} |y|^{(\alpha-\beta-\gamma)p'} d\mu_k(y) \\ &\lesssim \int_{\mathbb{R}^d} |Vg(y)|^{q'} d\mu_k(y) \|g\|_{q', d\mu_k}^{p'-q'} \lesssim \|g\|_{q', d\mu_k}^{p'}. \end{aligned}$$

This gives

$$J_2 \lesssim \|f\|_{p, d\mu_k} \| |y|^{\alpha-\beta-\gamma} Vg(y) \|_{p', d\mu_k} \lesssim \|f\|_{p, d\mu_k} \|g\|_{q', d\mu_k}.$$

Note that, for $p = 1$, a similar result is valid as well, i.e.,

$$J_2 \lesssim \|f\|_{1, d\mu_k} \|g\|_{q', d\mu_k}, \quad \gamma < \frac{d_k}{q}, \quad \beta \leq 0, \quad (5.4)$$

since $\alpha - \beta - \gamma = d_k/q'$ and

$$|y|^{\alpha-\beta-\gamma} Vg(y) \lesssim |y|^{\alpha-\beta-\gamma} |y|^{-d_k/q'} \|g\|_{q', d\mu_k} = \|g\|_{q', d\mu_k}.$$

Estimate of J_3 . If $(x, y) \in E_3$, we similarly have

$$\begin{aligned} \sqrt{|x|^2 + |y|^2 - 2\langle y, \eta \rangle} &\geq 2^{-1}|x|, \\ \Phi_\alpha(x, y) &= \int_{\mathbb{R}^d} \frac{1}{(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta \rangle})^{d_k-\alpha}} d\mu_x^k(\eta) \lesssim |x|^{\alpha-d_k} \end{aligned}$$

and

$$J_3 \lesssim \iint_{|y| \leq |x|} \frac{f(y)g(x)}{|x|^{\gamma-\alpha+d_k}|y|^\beta} d\mu_k(x) d\mu_k(y) = \int_{\mathbb{R}^d} g(x) |x|^{\alpha-\beta-\gamma} V f(x) d\mu_k(x),$$

where

$$Vf(x) = |x|^{\beta-d_k} \int_{|y| \leq |x|} f(y) |y|^{-\beta} d\mu_k(y).$$

Since

$$\begin{aligned} Vf(x) &\leq |x|^{\beta-d_k} \left(\int_{|y| \leq |x|} |y|^{-p'\beta} d\mu_k(y) \right)^{1/p'} \|f\|_{p, d\mu_k} \\ &\lesssim |x|^{\beta-d_k} |x|^{d_k/p' - \beta} \|f\|_{p, d\mu_k} = |x|^{-d_k/p} \|f\|_{p, d\mu_k}, \end{aligned}$$

we obtain

$$|Vf(x)|^{q-p} |x|^{(\alpha-\beta-\gamma)q} \lesssim |x|^{-d_k(q-p)/p + (\alpha-\beta-\gamma)q} \|f\|_{p, d\mu_k}^{q-p} = \|f\|_{p, d\mu_k}^{q-p}.$$

Taking into account Theorem 3.1 with $a = d_k - \beta$, $b = \beta$, $p = p$, $\frac{a}{p'} > \frac{b}{p}$ (or, $\beta < \frac{d_k}{p'}$), we obtain

$$\|Vf\|_{p, d\mu_k} \lesssim \|f\|_{p, d\mu_k},$$

which implies

$$\begin{aligned} \int_{\mathbb{R}^d} |Vf(x)|^q |x|^{(\alpha-\beta-\gamma)q} d\mu_k(x) &= \int_{\mathbb{R}^d} |Vf(x)|^p |Vf(x)|^{q-p} |x|^{(\alpha-\beta-\gamma)q} d\mu_k(x) \\ &\lesssim \int_{\mathbb{R}^d} |Vf(x)|^p d\mu_k(x) \|f\|_{p, d\mu_k}^{q-p} \lesssim \|f\|_{p, d\mu_k}^q. \end{aligned}$$

We finally have

$$J_3 \lesssim \|g\|_{q', d\mu_k} \| |x|^{\alpha-\beta-\gamma} Vf(x) \|_{q, d\mu_k} \lesssim \|f\|_{p, d\mu_k} \|g\|_{q', d\mu_k}.$$

Again, the estimate $J_3 \lesssim \|f\|_{p, d\mu_k} \|g\|_{q', d\mu_k}$ also holds for $p = 1$, $\gamma < \frac{d_k}{q}$, and $\beta < 0$.

This completes the proof of part (a).

Part (b) Let us prove the weak (L^q, L^1) -boundedness of D-Riesz potential for $1 < q < \infty$, $\gamma < \frac{d_k}{q}$, $\beta < 0$, $\gamma + \beta \geq 0$, $0 < \alpha < d_k$. We will use the notations and assumptions of part (a).

It is sufficient to prove the inequality

$$S = \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} \tilde{I}_\alpha^k f(x) > \lambda\}} d\mu_k(x) \lesssim \left(\frac{\|f\|_{1, d\mu_k}}{\lambda} \right)^q. \quad (5.5)$$

Let us consider the operators

$$A_\alpha^i f(x) = \int_{\mathbb{R}^d} f(y) |y|^{-\beta} \Phi_\alpha(x, y) \chi_{E_i}(x, y) d\mu_k(y), \quad i = 1, 2, 3.$$

We have

$$\tilde{I}_\alpha^k = \sum_{i=1}^3 A_\alpha^i,$$

and

$$S = \sum_{i=1}^3 \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} A_\alpha^i f(x) > \lambda/3\}} d\mu_k(x) = S_1 + S_2 + S_3.$$

Estimate of S_1 . Applying the estimate of J_1 and inequality (5.5) with $1 < q < \infty$, $\gamma = \beta = 0$, and $\alpha = \tilde{\alpha} = \frac{d_k}{q'}$ (see [1, 11]), we derive

$$A_\alpha^1 f(x) \lesssim \int_{\mathbb{R}^d} f(y) |y|^{-\beta} \Phi_{\tilde{\alpha}}(x, y) d\mu_k(y) = \tilde{I}_{\tilde{\alpha}}^k f(x),$$

and

$$\begin{aligned} S_1 &= \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} A_\alpha^1 f(x) > \lambda/3\}} d\mu_k(x) \\ &\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} \tilde{I}_\alpha^k f(x) \gtrsim \lambda\}} d\mu_k(x) \lesssim \left(\frac{\|f\|_{1, d\mu_k}}{\lambda} \right)^q. \end{aligned}$$

Estimate of S_2 . Applying the obtained estimate of J_2 , we get

$$A_\alpha^2 f(x) \lesssim \int_{|y| \geq |x|} |y|^{\alpha-\beta-d_k} f(y) d\mu_k(y) = B_1 f(x).$$

Since

$$\begin{aligned} &\int_{\mathbb{R}^d} g(x) |x|^{-\gamma} \int_{|y| \geq |x|} |y|^{\alpha-\beta-d_k} f(y) d\mu_k(y) d\mu_k(x) \\ &= \int_{\mathbb{R}^d} f(y) |y|^{\alpha-\beta-d_k} \int_{|x| \leq |y|} g(x) |x|^{-\gamma} d\mu_k(x) d\mu_k(y), \end{aligned}$$

in light of Eq. 5.4 with $p = 1$, $\gamma < \frac{d_k}{q}$, and $\beta \leq 0$, we have

$$\| |x|^{-\gamma} B_1 f(x) \|_{q, d\mu_k} \lesssim \|f\|_{1, d\mu_k}.$$

Hence,

$$\begin{aligned} S_2 &= \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} A_\alpha^2 f(x) > \lambda/3\}} d\mu_k(x) \\ &\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} B_1 f(x) \gtrsim \lambda\}} d\mu_k(x) \lesssim \left(\frac{\|f\|_{1, d\mu_k}}{\lambda} \right)^q. \end{aligned}$$

Estimate of S_3 . Applying the estimate of J_3 , we obtain

$$A_\alpha^3 f(x) \lesssim |x|^{\alpha-d_k} \int_{|y| \leq |x|} |y|^{-\beta} f(y) d\mu_k(y) = H_1 f(x).$$

Using the estimate $J_3 \lesssim \|f\|_{1, d\mu_k} \|g\|_{q', d\mu_k}$ with $\gamma < \frac{d_k}{q}$ and $\beta < 0$ yields

$$\| |x|^{-\gamma} H_1 f(x) \|_{q, d\mu_k} \lesssim \|f\|_{1, d\mu_k}.$$

Thus,

$$\begin{aligned} S_3 &= \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} A_\alpha^3 f(x) > \lambda/3\}} d\mu_k(x) \\ &\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-\gamma} H_1 f(x) \gtrsim \lambda\}} d\mu_k(x) \lesssim \left(\frac{\|f\|_{1, d\mu_k}}{\lambda} \right)^q. \end{aligned} \quad (5.6)$$

If $\beta = 0$, $\alpha - \gamma = d_k(1 - \frac{1}{q})$, then

$$|x|^{-\gamma} H_1 f(x) = |x|^{\alpha-\gamma-d_k} \int_{|y| \leq |x|} f(y) d\mu_k(y) = |x|^{-d_k/q} \int_{|y| \leq |x|} f(y) d\mu_k(y).$$

Since inequality (5.6) is homogeneous, we can assume that $\|f\|_{1,d\mu_k} = 1$. Therefore,

$$\begin{aligned} S_3 &\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-d_k/q} \int_{|y| \leq |x|} f(y) d\mu_k(y) \gtrsim \lambda\}} d\mu_k(x) \\ &\lesssim \int_{\{x \in \mathbb{R}^d : |x|^{-d_k/q} \gtrsim \lambda\}} d\mu_k(x) \lesssim \lambda^{-q} = \left(\frac{\|f\|_{1,d\mu_k}}{\lambda} \right)^q, \end{aligned}$$

completing the proof.

Let us mention that for the so-called B-Riesz potentials the results that are similar to Theorem 1.4 were established in [9].

6 Properties of the Spaces Φ_k and Ψ_k

Recall that T_j , $j = 1, \dots, d$, are differential-differences Dunkl operators given by (1.2), $T^n = \prod_{j=1}^d T_j^{n_j}$, $n \in \mathbb{Z}_+^d$,

$$\Phi_k = \left\{ f \in \mathcal{S}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^n f(x) d\mu_k(x) = 0, \quad n \in \mathbb{Z}_+^d \right\},$$

and

$$\Psi_k = \{\mathcal{F}_k(f) : f \in \Phi_k\}.$$

If $f \in \mathcal{S}(\mathbb{R}^d)$, then from the definition of the generalized exponential function $e_k(x, y)$ and

$$f(x) = \int_{\mathbb{R}^d} e_k(x, y) \mathcal{F}_k(f)(y) d\mu_k(y),$$

we obtain that

$$T^n f(x) = i^n \int_{\mathbb{R}^d} y^n e_k(x, y) \mathcal{F}_k(f)(y) d\mu_k(y), \quad n \in \mathbb{Z}_+^d,$$

and

$$T^n f(0) = i^n \int_{\mathbb{R}^d} y^n \mathcal{F}_k(f)(y) d\mu_k(y).$$

Therefore,

$$\Psi_k = \left\{ f \in \mathcal{S}(\mathbb{R}^d) : T^n f(0) = 0, \quad n \in \mathbb{Z}_+^d \right\}.$$

Note that in the classical case ($k \equiv 0$) we have

$$\Psi = \Psi_0 = \{\mathcal{F}(f) : f \in \Phi\} = \left\{ f \in \mathcal{S}(\mathbb{R}^d) : D^n f(0) = 0, \quad n \in \mathbb{Z}_+^d \right\}.$$

Theorem 6.1 We have $\Psi_k = \Psi$.

Proof Let $f \in \Psi$, $D = (D_1, \dots, D_d)$, and let $\partial_a f(x) = \langle Df(x), \frac{a}{|a|} \rangle$ be the directional derivative with respect to a vector a . Taking into consideration

$$\frac{f(x) - f(\sigma_a x)}{\langle a, x \rangle} = \frac{2}{|a|} \int_0^1 \partial_a f \left(x - \frac{2t \langle a, x \rangle}{|a|^2} a \right) dt$$

and

$$T_j f(x) = D_j f(x) + \sum_{a \in R_+} \frac{2k(a) \langle a, e_j \rangle}{|a|} \int_0^1 \partial_a f \left(x - \frac{2t \langle a, x \rangle}{|a|^2} a \right) dt \quad (6.1)$$

we obtain that $T_j f(0) = 0$, $j = 1, \dots, d$. By Eq. 6.1, we derive that $\Psi \subset \Psi_k$. In addition, if $D^n f(0) = 0$ for $|n| = \sum_{j=1}^d n_j \leq m$, then $T^n f(0) = 0$ for $|n| \leq m$.

Let $m \in \mathbb{Z}_+$ and $f \in \Psi_k$ be a real function. Using the Taylor formula, we write

$$f(x) = p(x) + r(x),$$

where $p(x)$ is a polynomial of degree $\deg p \leq m$, and $D^n r(0) = 0$ for $|n| \leq m$. Since $T^n f(0) = T^n r(0) = 0$ for $|n| \leq m$, it follows that $T^n p(0) = 0$ for $|n| \leq m$ and, in particular, $p(0) = 0$. By [21],

$$0 = p(T)p(0) = \int_{\mathbb{R}^d} \left(e^{-\Delta_k/2} p(x) \right)^2 e^{-|x|^2/2} d\mu_k(x)$$

and $e^{-\Delta_k/2} p(x) = 0$. Since $e^{-\Delta_k/2}$ is a bijective operator on the set of all polynomials [21], we obtain that $p(x) \equiv 0$, and $D^n f(0) = D^n p(0) = 0$ for $|n| \leq m$. Thus, $\Psi_k \subset \Psi$. \square

Theorem 6.1 immediately implies the following

Corollary 6.2 *We have $I_\alpha^k(\Phi_k) = \Phi_k$ and $\mathcal{F}_k(I_\alpha^k)(\Psi_k) = \Psi_k$.*

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Using the positive L_p -bounded generalized translation operator

$$\mathcal{T}^t f(x) = \int_{\mathbb{S}^{d-1}} \tau^{ty'} f(x) d\sigma_k(y')$$

Gorbachev et al. [11], we can write the D-Riesz potential and the convolution with a radial function $g_0(|y|)$ as follows

$$I_\alpha^k f(x) = (\gamma_\alpha^k)^{-1} \int_0^\infty \mathcal{T}^t f(x) t^{\alpha-d_k} dv_{\lambda_k}(t) \quad (6.2)$$

and

$$\int_{\mathbb{R}^d} \tau^{-y} f(x) g_0(|y|) d\mu_k(y) = \int_0^\infty \mathcal{T}^t f(x) g_0(t) dv_{\lambda_k}(t). \quad (6.3)$$

The proof of the following result is based on Theorem 1.3.

Theorem 6.3 *If $1 < p < \infty$, $-\frac{d_k}{p} < \beta < \frac{d_k}{p'}$, then Φ_k is dense in $L^p(\mathbb{R}^d, |x|^{\beta p} d\mu_k)$.*

Proof Let $\eta \in \mathcal{S}(\mathbb{R}^d)$ be such that $\eta(x) = 1$ if $|x| \leq 1$, $\eta(x) > 0$ if $|x| < 2$, and $\eta(x) = 0$ if $|x| \geq 2$. We can assume that $f \in \mathcal{S}(\mathbb{R}^d)$.

Set

$$\psi_0(|y|) = \mathcal{F}_k(\eta)(y), \quad \psi_{0N}(|y|) = \frac{1}{N^{d_k}} \psi_0\left(\frac{|y|}{N}\right) = \mathcal{F}_k(\eta(N \cdot))(y),$$

$$\varphi_N(x) = f(x) - \int_{\mathbb{R}^d} \tau^{-y} f(x) \psi_{0N}(|y|) d\mu_k(y).$$

Since (see [11])

$$\mathcal{F}_k(\varphi_N)(z) = (1 - \eta(Nz))\mathcal{F}_k(f)(z) \in \Psi_k,$$

it follows that $\varphi_N \in \Phi_k$ and, by Eq. 6.3,

$$\| |x|^\beta (f(x) - \varphi_N(x)) \|_{p, d\mu_k} = \left\| |x|^\beta \int_0^\infty \mathcal{T}^t f(x) \psi_{0N}(t) dv_{\lambda_k}(t) \right\|_{p, d\mu_k}. \quad (6.4)$$

For any $\alpha \in (0, d_k)$, we have

$$|\psi_0(t)| \lesssim t^{\alpha-d_k} \quad \text{and} \quad |\psi_{0N}(t)| \lesssim N^{-\alpha} t^{\alpha-d_k}.$$

Hence, by positivity of the operator T^t and Eq. 6.2,

$$\begin{aligned} \left| \int_0^\infty T^t f(x) \psi_{0N}(t) dv_{\lambda_k}(t) \right| &\leq \int_0^\infty T^t |f|(x) |\psi_{0N}(t)| dv_{\lambda_k}(t) \\ &\lesssim N^{-\alpha} \int_0^\infty T^t |f|(x) t^{\alpha-d_k} dv_{\lambda_k}(t) = N^{-\alpha} I_\alpha^k |f|(x). \end{aligned}$$

This, Eq. 6.4, and Theorem 1.3 imply

$$\begin{aligned} \| |x|^\beta (f(x) - \varphi_N(x)) \|_{p, d\mu_k} &\lesssim N^{-\alpha} \| |x|^\beta I_\alpha^k |f|(x) \|_{p, d\mu_k} \\ &\lesssim N^{-\alpha} \| |x|^\delta f(x) \|_{p, d\mu_k} \lesssim N^{-\alpha}, \end{aligned}$$

where $\alpha > 0$ is chosen so that $\delta = \alpha + \beta < \frac{d_k}{p}$. \square

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