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EICHLER-SHIMURA ISOMORPHISM AND GROUP COHOMOLOGY ON ARITHMETIC GROUPS

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INTRODUCTION

The Eichler-Shimura isomorphism establishes a bijection between the space of modular forms and certain cohomology groups with coefficients in a space of polynomials. More precisely, let $k \geq 2$ be an integer and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, then we have the following isomorphism of Hecke modules

$$(0.1) \quad M_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \simeq H^1(\Gamma, V(k)^\vee),$$

where $V(k)^\vee$ is the dual of the \mathbb{C} -vector space of homogenous polynomials of degree $k-2$, $M_k(\Gamma, \mathbb{C})$ is the space of modular forms of weight k and $S_k(\Gamma, \mathbb{C}) \subset M_k(\Gamma, \mathbb{C})$ is the subspace of cuspidal modular forms (see [6, Thm. 8.4] and [4, Thm. 6.3.4]).

This isomorphism can be interpreted in geometric terms. Indeed, a modular form of weight k can be interpreted as a section of certain sheaf of differential forms on the open modular curve attached to Γ . With this in mind, the Eichler-Shimura isomorphism can be obtained comparing deRham and singular cohomology, noticing that the singular cohomology of the open modular curve is given by the group cohomology $H^\bullet(\Gamma, V(k)^\vee)$. The aim of this paper is to omit this geometric interpretation and to provide a new group cohomological interpretation.

The restriction of the Eichler-Shimura isomorphism to the spaces of cuspidal modular forms is given by the morphisms

$$\partial^\pm : S_k(\Gamma, \mathbb{C}) \longrightarrow H^1(\Gamma, V(k)^\vee),$$

where

$$\partial^\pm(f)(\gamma)(P) = \int_{z_0}^{\gamma z_0} P(1, -\tau) f(\tau) d\tau \pm \int_{z_0}^{\gamma z_0} P(1, \bar{\tau}) f(-\bar{\tau}) d(-\bar{\tau}).$$

for any z_0 in \mathcal{H} the Poincaré hyperplane, $\gamma \in \Gamma$ and $P \in V(k)$. In fact, the morphism defining the cuspidal part of (0.1) is given by

$$\begin{aligned} S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} &\longrightarrow H^1(\Gamma, V(k)^\vee) \\ (f_1, \bar{f}_2) &\longmapsto (\partial^+ + \partial^-)(f_1) + (\partial^+ - \partial^-)(f_2^*), \end{aligned}$$

where $f_2^* \in S_k(\Gamma, \mathbb{C})$ is obtained from f_2 by complex conjugating its Fourier coefficients. The image of ∂^\pm lies in the subspaces $H^1(\Gamma, V(k)^\vee)^\pm \subset H^1(\Gamma, V(k)^\vee)$

where the natural action of $\omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ normalizing Γ acts by ± 1 .

In a general setting, F is a totally real number field of degree d , G is the multiplicative group of a quaternion algebra A over F , and ϕ is a weight $\underline{k} = (k_1, \dots, k_d)$ cuspidal automorphic form of G with level $\mathcal{U} \subset G(\mathbb{A}^\infty)$ and central character $\psi : \mathbb{A}^\times / F^\times \rightarrow \mathbb{C}^\times$. We assume that $\psi_{\sigma_i}(x) = \mathrm{sign}(x)^{k_i} |x|^{\mu_i}$, for any archimedean place $\sigma_i : F \hookrightarrow \mathbb{R}$. In this scenario, by an automorphic form we mean a function on $\mathcal{H}^r \times G(\mathbb{A}^\infty) / \mathcal{U}$, where \mathcal{H} is the Poincaré upperplane and r is the cardinal of the set Σ of archimedean places where A splits, with values in $\bigotimes_{\sigma_i \in \infty \setminus \Sigma} V_{\mu_i}(k_i)^\vee$ (see §1.2 for a precise definition of $V_{\mu_i}(k_i)$), that satisfies the usual transformation laws with

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respect to the weight- k_i -actions of $G(F)$. The interesting cohomology subgroups to consider are:

$$H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^\mathcal{U}) = \bigoplus_{i=1}^n H^r(\Gamma_{g_i}, V_\psi(\underline{k})^\vee), \quad \Gamma_{g_i} = G(F)^+ \cap g_i \mathcal{U} g_i^{-1},$$

where $V_\psi(\underline{k})$ is the tensor product of the polynomial spaces $V_{\mu_i}(k_j)$ ($j = 1, \dots, d$), $G(F)^+ \subseteq G(F)$ is the subgroup of totally positive elements, $\{g_i\}_{i=1, \dots, n} \subset G(\mathbb{A}^\infty)$ is a set of representatives of the double coset space $G(F)^+ \backslash G(\mathbb{A}^\infty) / \mathcal{U}$, and $\mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^\mathcal{U} = C(G(\mathbb{A}^\infty) / \mathcal{U}, V_\psi(\underline{k})^\vee)$. Similarly as in the classical case, for any character $\varepsilon : G(F) / G(F)^+ \rightarrow \pm 1$ we can define a morphism

$$\partial^\varepsilon : S_{\underline{k}}(\mathcal{U}, \psi) \rightarrow H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^\mathcal{U})(\varepsilon),$$

from the set $S_{\underline{k}}(\mathcal{U}, \psi)$ of automorphic cuspforms of weight \underline{k} , level \mathcal{U} and central character ψ , to the ε -isotypical component of $H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^\mathcal{U})$. Such a map is given by:

$$\partial^\varepsilon \phi = \sum_{\gamma \in G(F) / G(F)^+} \varepsilon(\gamma) \partial \phi^\gamma,$$

where $\partial \phi \in H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))$ is the class of the cocycle

$$(G(F)^+)^r \ni (g_1, g_2, \dots, g_r) \mapsto \int_{\tau_1}^{g_1 \tau_1} \dots \int_{g_1 \dots g_{r-1} \tau_r}^{g_1 \dots g_r \tau_r} P_\Sigma(1, -\underline{z}) \langle \phi(\underline{z}, g), P^\Sigma \rangle d\underline{z},$$

with $P_\Sigma \in \bigotimes_{j=1}^r V_{\mu_j}(k_j)$, $P^\Sigma \in \bigotimes_{j=r+1}^d V_{\mu_j}(k_j)$ and $\underline{z} = (z_1, \dots, z_r)$, $(\tau_1, \dots, \tau_r) \in \mathcal{H}^r$. Our result will provide a group cohomological interpretation to the morphisms ∂^ε , for any character ε .

Let $F_\infty \simeq \mathbb{R}^d$ be the product of the archimedean completions of F , let \mathcal{G}_∞ be the Lie algebra of $G(F_\infty)$ and let $K_\infty \subseteq G(F_\infty)$ be a maximal compact subgroup. Then the $(\mathcal{G}_\infty, K_\infty)$ -module generated by ϕ is isomorphic to $D_\psi(\underline{k})$, the tensor product of discrete series of weight k_j at archimedean places in Σ and polynomial spaces $V_{\mu_j}(k_j)$ at archimedean places not in Σ . This implies that any $\phi \in S_{\underline{k}}(\mathcal{U}, \psi)$ provides an element

$$\phi \in H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^\mathcal{U}); \quad \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^\mathcal{U} := \text{Hom}_{(\mathcal{G}_\infty, K_\infty)}(D_\psi(\underline{k}), \mathcal{A}^\mathcal{U}),$$

where $\mathcal{A}^\mathcal{U}$ is the $(\mathcal{G}_\infty, K_\infty)$ -module of smooth admissible functions $f : G(\mathbb{A}) / \mathcal{U} \rightarrow \mathbb{C}$. Our main result (Theorem 2.4) can be rewritten as follows:

Theorem 0.1. *There exists an exact sequence of $G(F)$ -modules*

$$0 \rightarrow \mathcal{A}(V_\psi(\underline{k})(\varepsilon), \mathbb{C})^\mathcal{U} \rightarrow \mathcal{A}(I_1^\varepsilon(\underline{k}), \mathbb{C})^\mathcal{U} \rightarrow \mathcal{A}(I_2^\varepsilon(\underline{k}), \mathbb{C})^\mathcal{U} \rightarrow \dots \rightarrow \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^\mathcal{U} \rightarrow 0,$$

such that, up to an explicit constant, the morphism ∂^ε is given by the corresponding connection morphism

$$H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^\mathcal{U}) \rightarrow H^r(G(F), \mathcal{A}(V_\psi(\underline{k})(\varepsilon), \mathbb{C})^\mathcal{U}) \simeq H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^\mathcal{U})(\varepsilon).$$

We obtain the above exact sequence from extensions of the $(\mathcal{G}_{\sigma_i}, K_{\sigma_i})$ -modules of discrete series $D_{\mu_i}(k_i)$ at every place $\sigma_i \in \Sigma$. The archimedean local nature of these connection morphisms ∂^ε implies that the $G(\mathbb{A}^\infty)$ -representation generated by $\partial^\varepsilon \phi$ coincides with the restriction to $G(\mathbb{A}^\infty)$ of π_ϕ , the automorphic representation attached to ϕ .

The image of a cuspidal automorphic representation π_ϕ through the morphisms ∂^ε is used in many papers to give a group cohomological construction of cyclotomic and anti-cyclotomic p -adic L -functions and Stickelberger elements attached to quadratic extensions of a totally real number field (see for instance [7], [5] and [1]). The explicit form of ∂^ε given in Theorem 2.4 provides the interpolation properties of these objects.

Another application is the construction of Stark-Heegner points. By means of the connection morphisms $\partial^{\varepsilon^{\pm 1}}$, where $(\varepsilon^+, \varepsilon^-)$ is a well chosen pair of characters,

one can construct a complex torus $\mathbb{C}^{[L:\mathbb{Q}]} / \Lambda$ attached to a weight 2 automorphic representation π_ϕ with field of coefficients L . It is conjectured that such complex torus coincides with the abelian variety of GL_2 -type attached to π_ϕ . In [3], we use the cohomological description of $\partial^{\varepsilon^{\pm 1}}$ to construct Stark-Heegner points in the complex torus, that we conjecture to be global points in the corresponding abelian variety. Such points are conjecturally defined over class fields of quadratic extensions of F and satisfy explicit reciprocity laws.

Notation. Throughout this paper, we will denote by $\int_{S^1} d\theta = \int_{\mathrm{SO}(2)} d\theta$ the Haar measure of $S^1 = \mathrm{SO}(2)$ such that $\mathrm{vol}(S^1) = \pi$.

Let F be a number field. For any place v of F , we denote by F_v its completion at v . Given a finite set of places S of F , we denote by F_S the product of completions at every place in S . We denote by F_∞ the product of completions at every archimedean place. Similarly, for any subset Σ of archimedean places, $F_{\infty \setminus \Sigma}$ will be the product of completions at every archimedean place not in Σ . We denote by \mathbb{A} the ring of adeles of F . For any set S of places of F , we write \mathbb{A}^S for the ring adeles outside S . Consistent with this notation, we denote by \mathbb{A}^∞ the ring of finite adeles of F .

1. DISCRETE SERIES

1.1. Finite dimensional representations. Let A be a quaternion algebra defined over a local field F . Let K/F be an extension where A splits. For any natural number $k \in \mathbb{N}$, let $\mathcal{P}_{k-2}^K \simeq \mathrm{Sym}^{k-2}(K^2)$ be the finite K -vector space of homogeneous polynomials of degree $k-2$. We have a well defined action of $\mathrm{GL}_2(K)$ on \mathcal{P}_{k-2}^K given by

$$(1.2) \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} * P \right) (x, y) := P \left((x, y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = P(ax + cy, bx + dy).$$

If we fix an embedding

$$\iota : A \longrightarrow A \otimes_F K \simeq M_2(K),$$

then \mathcal{P}_{k-2}^K is equipped with an action of A^\times .

We denote by $\det : A \rightarrow F$ the reduced norm of A , and let us consider

$$A^+ = \{a \in A : \det(a) \in F^2\}.$$

We write $V(k)_K = \mathcal{P}_{k-2}^K \otimes \det^{\frac{2-k}{2}}$ with the natural action of A^+ . It is clear that the centre of A^\times acts trivially on $V(k)_K$. Notice that, if k is even, the action of A^+ on $V(k)_K$ extends to a natural action of A .

1.2. Discrete series and exact sequences. Assume that $F = \mathbb{R}$ and $A = M_2(\mathbb{R})$. Let $\mathcal{GL}_2(\mathbb{R})$ be the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$. For any $k \in \mathbb{Z}$ and $\mu \in \mathbb{C}$, we define $I_\mu(k)$ as the $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -module of smooth admissible vectors in

$$\left\{ f : \mathrm{GL}_2(\mathbb{R})^+ \rightarrow \mathbb{C} : f \left(\begin{pmatrix} t_1 & x \\ & t_2 \end{pmatrix} g \right) = \mathrm{sign}(t_1)^k (t_1 t_2)^{\frac{\mu}{2}} \left(\frac{t_1}{t_2} \right)^{\frac{k}{2}} f(g) \right\}.$$

Write $V_\mu(k) = V(k)_\mathbb{C} \otimes \det^{\frac{\mu}{2}}$. If we assume that $k \geq 2$, we have the well defined morphism of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules

$$\iota : V_\mu(k) \longrightarrow I_\mu(2-k); \quad \iota(P) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{\frac{2-k+\mu}{2}} P(c, d).$$

Moreover, we have a $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -invariant pairing (see [2, §2])

$$\langle \cdot, \cdot \rangle_I : I_\mu(k) \times I_{-\mu}(2-k) \longrightarrow \mathbb{C}; \quad (f, g) \longmapsto \int_{\mathrm{SO}(2)} f(\theta) g(\theta) d\theta,$$

providing the morphism of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules

$$\bar{\varphi} : I_\mu(k) \longrightarrow V_{-\mu}(k)^\vee; \quad \langle \bar{\varphi}(f), P \rangle_V := \langle f, \iota(P) \rangle_I.$$

Composing with the natural $\mathrm{GL}_2(\mathbb{R})^+$ -morphism

$$(1.3) \quad V_{-\mu}(k)^\vee \longrightarrow V_\mu(k); \quad F \longmapsto P_F(x, y) = \langle F, (Yx - Xy)^{k-2} \rangle_{V_{(x, y)}},$$

we obtain a map

$$\begin{aligned} \varphi : I_\mu(k) &\longrightarrow V_\mu(k); \\ \varphi(f)(x, y) &= \langle \bar{\varphi}(f), (Yx - Xy)^{k-2} \rangle_{V_{(x, y)}} = \int_{S^1} f(\theta) (y \sin \theta + x \cos \theta)^{k-2} d\theta \end{aligned}$$

Remark 1.1. Notice that we have the symmetry

$$\begin{aligned} \langle F, P_G \rangle_V &= \langle F, \langle G, (Yx - Xy)^{k-2} \rangle_{V_{(x, y)}} \rangle_{V_{(x, y)}} \\ &= (-1)^k \langle G, \langle F, (Xy - Yx)^{k-2} \rangle_{V_{(x, y)}} \rangle_{V_{(x, y)}} = (-1)^k \langle G, P_F \rangle_V, \end{aligned}$$

for all $F, G \in V_\mu(k)^\vee$.

The kernel of φ is $D_\mu(k)$ the *Discrete Series* $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module of weight k and central character $x \mapsto \mathrm{sign}(x)^k |x|^\mu$. This definition implies that $D_\mu(k)$ lies in the following exact sequence of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules:

$$(1.4) \quad 0 \longrightarrow D_\mu(k) \longrightarrow I_\mu(k) \xrightarrow{\varphi} V_\mu(k) \longrightarrow 0.$$

Since any $g \in \mathrm{GL}_2(\mathbb{R})^+$ can be written uniquely as $g = u \cdot \tau(x, y) \cdot \kappa(\theta)$, where

$$u \in \mathbb{R}^+, \quad \tau(x, y) = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \in B, \quad \kappa(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \mathrm{SO}(2),$$

we have that

$$I_\mu(k) = \bigoplus_{t \equiv k \pmod{2}} \mathbb{C} f_t; \quad f_t(u \cdot \tau(x, y) \cdot \kappa(\theta)) = u^\mu y^{\frac{k}{2}} e^{ti\theta}.$$

The $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -module structure of $I_s(k)$ can be described as follows: Let $L, R \in \mathcal{GL}_2(\mathbb{R})$ be the *Maass differential operators* defined in [2, §2.2]

$$L = e^{-2i\theta} \left(-iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2i} \frac{\partial}{\partial \theta} \right), \quad R = e^{2i\theta} \left(iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta} \right).$$

Then, the $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -module $I_s(k)$ is characterized by the relations:

$$(1.5) \quad Rf_t = \left(\frac{k+t}{2} \right) f_{t+2}; \quad Lf_t = \left(\frac{k-t}{2} \right) f_{t-2};$$

$$(1.6) \quad \kappa(\theta)f_t = e^{ti\theta} f_t; \quad uf_t = u^\mu f_t,$$

for any $\kappa(\theta) \in \mathrm{SO}(2)$ and $u \in \mathbb{R}^+ \subset \mathrm{GL}_2(\mathbb{R})^+$.

Write $z = x + iy$ and $\bar{z} = x - iy$. For $n \in \{0, 1, \dots, k-2\}$, let us consider the elements $P_n \in V_\mu(k)$, $P_n(x, y) = z^n \bar{z}^{k-2-n}$. It is clear that $\{P_n\}_{n=0, \dots, k-2}$ is a basis for the \mathbb{C} -vector space $V_\mu(k)$. We compute that

$$\begin{aligned} \varphi(f)(x, y) &= \int_{S^1} f(\theta) ((2i^{-1})(z - \bar{z}) \sin \theta + 2^{-1}(z + \bar{z}) \cos \theta)^{k-2} d\theta \\ &= \int_{S^1} 2^{2-k} f(\theta) (ze^{-i\theta} + \bar{z}e^{i\theta})^{k-2} d\theta \\ &= 2^{2-k} \sum_{n=0}^{k-2} \binom{k-2}{n} P_n(x, y) \int_{S^1} f(\theta) e^{-i(2n-k+2)\theta} d\theta \end{aligned}$$

By orthogonality, we deduce that $\varphi(f_{2n-k+2}) = 2^{2-k} \pi \binom{k-2}{n} P_n(x, y)$.

Since $\kappa(\theta)P_n = e^{(2n-k+2)i\theta}P_n$, the morphism of \mathbb{C} -vector spaces

$$(1.7) \quad s : V_\mu(k) \longrightarrow I_\mu(k); \quad s(P_n) = \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} f_{2n-k+2},$$

defines a section of φ as $\mathrm{SO}(2)\mathbb{R}^+$ -modules.

Remark 1.2. Since (1.3) is an isomorphism, we can define a non-degenerate $\mathrm{GL}_2(\mathbb{R})^+$ -invariant bilinear pairing $V_\mu(k) \times V_{-\mu}(k) \rightarrow \mathbb{C}$

$$\langle P_F, Q \rangle = \langle F, Q \rangle_V, \quad F \in V_{-\mu}(k)_{\mathbb{C}}^{\vee}, \quad Q \in V_{-\mu}(k),$$

which is symmetric or antisymmetric depending on the parity of k , by Remark 1.1. Moreover, by the definition of (1.3),

$$P(s, t) = \langle P, Q_{s,t} \rangle, \quad Q_{s,t}(x, y) = (ys - xt)^{k-2}.$$

In particular,

$$(1.8) \quad P(-1, i) = i^{2-k} \langle P, P_0 \rangle = i^{k-2} \langle P_0, P \rangle.$$

Since s is a section of φ , we compute

$$(1.9) \quad \langle P, Q \rangle = \langle \bar{\varphi}(s(P)), Q \rangle_V = \langle s(P), \iota(Q) \rangle_I,$$

for all $P, Q \in V_\mu(k)$.

1.3. The $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module of discrete series. We want to give structure of $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module to $I_\mu(k)$. Hence, we have to define the action of $\omega = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in O(2) \setminus \mathrm{SO}(2)$. That is to say, we have to define $\omega \in \mathrm{End}(I_\mu(k))$ such that

$$(i) \quad \omega f_t \in \mathbb{C}f_{-t}; \quad (ii) \quad \omega^2 = 1; \quad (iii) \quad \omega R = L\omega.$$

If we write $\omega f_t = \lambda(t)f_{-t}$, condition (ii) implies that $\lambda(t)\lambda(-t) = 1$. Moreover, condition (iii) implies that $\lambda(t) = \lambda(t+2)$. We obtain two possible $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module structures for $I_\mu(k)$: Letting $\lambda(t) = 1$ for all $t \equiv k \pmod{2}$, or letting $\lambda(t) = -1$ for all $t \equiv k \pmod{2}$. Write $I_\mu(k)^\pm$ for the $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module such that $\omega f_t = \pm f_{-t}$, respectively.

By abuse of notation, write also $V_\mu(k)$ and $V_\mu(k)_{\mathbb{R}}$ (in case $\mu \in \mathbb{R}$) for the $\mathrm{GL}_2(\mathbb{R})$ -representations

$$V_\mu(k) = V_\mu(k)_{\mathbb{C}} = \mathcal{P}_{k-2}^{\mathbb{C}} \otimes |\det|^{\frac{2-k+\mu}{2}}, \quad V_\mu(k)_{\mathbb{R}} = \mathcal{P}_{k-2}^{\mathbb{R}} \otimes |\det|^{\frac{2-k+\mu}{2}}.$$

Remark 1.3. With this $\mathrm{GL}_2(\mathbb{R})$ -module structure, the pairing $\langle \cdot, \cdot \rangle$ introduced in Remark 1.2 is not a $\mathrm{GL}_2(\mathbb{R})$ -invariant in general. In fact one can show that

$$\langle gP, Q \rangle = \mathrm{sign}(\det g)^k \langle P, g^{-1}Q \rangle.$$

Note that, for any $f \in I_\mu(k)^\pm$, we have that $\omega f(\theta) = \pm f(-\theta)$, hence we compute that,

$$\begin{aligned} \varphi(\omega f)(x, y) &= \int_{S^1} \omega f(\theta) (x \cos \theta + y \sin \theta)^{k-2} d\theta \\ &= \pm \int_{S^1} f(-\theta) (x \cos \theta + y \sin \theta)^{k-2} d\theta \\ &= \pm \int_{S^1} f(\theta) (x \cos \theta - y \sin \theta)^{k-2} d\theta = \pm \omega(\varphi(f))(x, y) \end{aligned}$$

This implies that the exact sequence of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules (1.4) provides the exact sequences of $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -modules

$$(1.10) \quad 0 \longrightarrow D_\mu(k) \xrightarrow{\iota} I_\mu(k)^+ \longrightarrow V_\mu(k) \longrightarrow 0,$$

$$(1.11) \quad 0 \longrightarrow D_\mu(k) \xrightarrow{\iota \circ I} I_\mu(k)^- \longrightarrow V_\mu(k)(\varepsilon) \longrightarrow 0,$$

where $\varepsilon : \mathrm{GL}_2(\mathbb{R}) \rightarrow \pm 1$ is the character given by $\varepsilon(g) = \mathrm{sign} \det(g)$, $D_\mu(k)$ is the $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module with fixed action of ω given by $\omega f(\theta) = f(\theta)$, and I is the automorphism of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules

$$I : D_\mu(k) \longrightarrow D_\mu(k) : \quad I(f_t) = \mathrm{sign}(t)f_t.$$

Note that $\iota \circ I$ is a monomorphism of $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -modules because $I(\omega f) = -\omega(I(f))$.

1.4. Matrix coefficients. Let us consider $A(\mathbb{C})$ the $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -module of admissible C^∞ functions $f : \mathrm{GL}_2(\mathbb{R})^+ \rightarrow \mathbb{C}$. For any $f_0 \in I_{-\mu}(2-k)$, I claim that

$$\varphi_{f_0} : I_\mu(k) \longrightarrow A(\mathbb{C}), \quad \varphi_{f_0}(f)(g_\infty) = \langle g_\infty f, f_0 \rangle_I, \quad g_\infty \in \mathrm{GL}_2(\mathbb{R})^+,$$

provides a well defines morphism of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules. Indeed, for any element of the Lie algebra $G \in \mathcal{GL}_2(\mathbb{R})$,

$$\begin{aligned} G\varphi_{f_0}(f)(g_\infty) &= \frac{d}{dt}(\varphi_{f_0}(g_\infty \exp(tG)))|_{t=0} = \frac{d}{dt}(\langle g_\infty \exp(tG)f, f_0 \rangle_I)|_{t=0} \\ &= \varphi_{f_0}(Gf)(g_\infty). \end{aligned}$$

1.5. \mathbb{R} -structures of Discrete series. As we can see in [2, §2.2], R and L are not in $\mathcal{GL}_2(\mathbb{R})$, they are *Caley transformations* in $\mathcal{GL}_2(\mathbb{C})$ of elements in $\mathcal{GL}_2(\mathbb{R})$. In fact, $\mathcal{GL}_2(\mathbb{R})$ is generated by

$$\begin{aligned} R + L &= -2y \sin(2\theta) \frac{\partial}{\partial x} + 2y \cos(2\theta) \frac{\partial}{\partial y} + \sin(2\theta) \frac{\partial}{\partial \theta}; & u \frac{\partial}{\partial u}; \\ i(R - L) &= -2y \cos(2\theta) \frac{\partial}{\partial x} - 2y \sin(2\theta) \frac{\partial}{\partial y} + \cos(2\theta) \frac{\partial}{\partial \theta}; & \text{and } \frac{\partial}{\partial \theta}. \end{aligned}$$

If we define $h_t := f_t + f_{-t} \in I_\mu(k)^\pm$ and $g_t := i(f_t - f_{-t}) \in I_\mu(k)^\pm$, it is easy to compute that

$$\begin{aligned} (R + L)h_t &= \left(\frac{k+t}{2}\right) h_{t+2} + \left(\frac{k-t}{2}\right) h_{t-2}, & \frac{\partial}{\partial \theta} h_t &= -tg_t, \\ i(R - L)h_t &= \left(\frac{k+t}{2}\right) g_{t+2} - \left(\frac{k-t}{2}\right) g_{t-2}, & \omega h_t &= \pm h_t \\ \kappa(\theta)h_t &= \cos(t\theta)h_t - \sin(t\theta)g_t, & \omega g_t &= \mp g_t. \end{aligned}$$

Hence the \mathbb{R} -vector space $I_\mu(k)_\mathbb{R}^\pm \subset I_\mu(k)^\pm$ generated by h_t and g_t defines a $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module over \mathbb{R} .

We check that the morphisms $\varphi : I_\mu(k)^+ \rightarrow V_\mu(k)_\mathbb{C}$ and $\varphi : I_\mu(k)^- \rightarrow V_\mu(k)(\varepsilon)_\mathbb{C}$ descend to morphisms of $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -modules over \mathbb{R}

$$\varphi^+ : I_\mu(k)_\mathbb{R}^+ \longrightarrow V_\mu(k)_\mathbb{R}, \quad \varphi^- : I_\mu(k)_\mathbb{R}^- \longrightarrow V_\mu(k)_\mathbb{R}(\varepsilon).$$

Hence the kernel $D_\mu(k)_\mathbb{R} \subset D_\mu(k)$ of φ^+ defines a $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -module over \mathbb{R} , generated by h_k , such that $D_\mu(k)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = D_\mu(k)$. Nevertheless, the automorphism of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules $I : D_\mu(k) \rightarrow D_\mu(k)$ does not descend to an automorphism of $(\mathcal{GL}_2(\mathbb{R}), \mathrm{SO}(2))$ -modules over \mathbb{R} since $I(h_t) = -\mathrm{sign}(t)ig_t$. In fact,

$$I(D_\mu(k)_\mathbb{R}) = iD_\mu(k)_\mathbb{R} \subset D_\mu(k).$$

We obtain the exact sequences of $(\mathcal{GL}_2(\mathbb{R}), O(2))$ -modules over \mathbb{R}

$$(1.12) \quad 0 \longrightarrow D_\mu(k)_\mathbb{R} \xrightarrow{\iota} I_\mu(k)_\mathbb{R}^+ \longrightarrow V_\mu(k)_\mathbb{R} \longrightarrow 0,$$

$$(1.13) \quad 0 \longrightarrow D_\mu(k)_\mathbb{R} \xrightarrow{\iota \circ I} iI_\mu(k)_\mathbb{R}^- \longrightarrow iV_\mu(k)_\mathbb{R}(\varepsilon) \longrightarrow 0.$$

2. CONNECTION MORPHISMS

In this section, we assume that G is the multiplicative group of a quaternion algebra that splits at the set of archimedean places Σ . Write $r = \#\Sigma$. Let us consider the \mathbb{C} -vector space $\mathcal{A}(\mathbb{C})$ of functions $f : G(\mathbb{A}) \rightarrow \mathbb{C}$ such that:

- There exists an open compact subgroup $U \subseteq G(\mathbb{A}^\infty)$ such that $f(gU) = f(g)$, for all $g \in G(\mathbb{A})$.
- Under a fixed identification $G(F_\Sigma) \simeq \mathrm{GL}_2(\mathbb{R})^r$, $f|_{G(F_\Sigma)} \in C^\infty(\mathrm{GL}_2(\mathbb{R})^r, \mathbb{C})$.
- Fixing K_Σ , a maximal compact subgroup of $G(F_\Sigma)$ isomorphic to $O(2)^r$, we assume that any $f \in \mathcal{A}(\mathbb{C})$ is K_Σ -finite, namely, its right translates by elements of K_Σ span a finite-dimensional vector space.
- We assume that any $f \in \mathcal{A}(\mathbb{C})$ is \mathcal{Z} -finite, where \mathcal{Z} is the centre of the universal enveloping algebra of $G(F_\Sigma)$.

Write ρ for the action of $G(\mathbb{A})$ given by right translation, then $(\mathcal{A}(\mathbb{C}), \rho)$ defines a smooth $G(\mathbb{A}^\infty)$ -representation and a $(\mathcal{G}_\infty, K_\infty)$ -module, where \mathcal{G}_∞ is the Lie algebra of $G(F_\Sigma)$ and $K_\infty = K_\Sigma \times G(F_{\infty \setminus \Sigma})$. Moreover, $\mathcal{A}(\mathbb{C})$ is also equipped with the $G(F)$ -action:

$$(h \cdot f)(g) = f(h^{-1}g), \quad h \in G(F),$$

where $g \in G(\mathbb{A})$, $f \in \mathcal{A}(\mathbb{C})$. Let us fix an isomorphism $G(F_\Sigma) \simeq \mathrm{GL}_2(\mathbb{R})^r$ that maps K_Σ to $O(2)^r$ and let V be a $(\mathcal{G}_\infty, K_\infty)$ -module. We define

$$\mathcal{A}(V, \mathbb{C}) := \mathrm{Hom}_{(\mathcal{G}_\infty, K_\infty)}(V, \mathcal{A}(\mathbb{C})),$$

endowed with the natural $G(F)$ - and $G(\mathbb{A}^\infty)$ -actions.

Remark 2.1. Note that if the $(\mathcal{G}_\infty, K_\infty)$ -module V comes from a finite dimensional $G(F_\infty)$ -representation V ,

$$\mathcal{A}(V, \mathbb{C}) \simeq C(G(\mathbb{A}^\infty), \mathrm{Hom}(V, \mathbb{C})) = C(G(\mathbb{A}^\infty), V^\vee),$$

where V^\vee is seen as a $G(F)$ -module by means of the usual injection $G(F) \hookrightarrow G(F_\infty)$, and the action of $G(F)$ on $C(G(\mathbb{A}^\infty), V^\vee)$ is given by $(h * f)(g) = h(f(h^{-1}g))$.

Fix $\sigma \in \Sigma$, $\mu \in \mathbb{C}$ and let us consider $D_\mu(k)$, $V_\mu(k)$ and $V_\mu(k)(\varepsilon)$ as $(\mathcal{G}_\infty, K_\infty)$ -modules by means of the projection $G(F_\infty) \rightarrow G(F_\sigma)$. The exact sequences (1.10) and (1.11) provide the connection morphisms

$$\partial_{\sigma}^{\varepsilon_\sigma} : H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})) \rightarrow H^{i+1}(G(F), \mathcal{A}(V \otimes V_\mu(k)(\varepsilon_\sigma), \mathbb{C})),$$

for any of the two characters $\varepsilon_\sigma : G(F_\sigma)/G(F_\sigma)^+ \rightarrow \pm 1$.

Let $V_{\mathbb{R}}$ be a $(\mathcal{G}_\infty, K_\infty)$ -representation over \mathbb{R} such that $V = V_{\mathbb{R}} \otimes \mathbb{C}$. This implies that we have a well defined complex conjugation on V by conjugating on the second factor. Thus, we have a complex conjugation on $\mathcal{A}(V, \mathbb{C})$, given by

$$\mathcal{A}(V, \mathbb{C}) \ni \phi \mapsto \bar{\phi} \in \mathcal{A}(V, \mathbb{C}); \quad \bar{\phi}(v) = \overline{\phi(\bar{v})}.$$

Lemma 2.2. Assume that $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for some $(\mathcal{G}_\infty, K_\infty)$ -module $V_{\mathbb{R}}$ over \mathbb{R} , and let $\mu \in \mathbb{R}$. Then, for any $\phi \in H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C}))$, we have

$$\overline{\partial_{\sigma}^{\varepsilon_\sigma}(\phi)} = \varepsilon_\sigma(c) \cdot \partial_{\sigma}^{\varepsilon_\sigma}(\bar{\phi})$$

for any $c \in G(F_\sigma) \setminus G(F_\sigma)^+$

Proof. We denote by $\mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1} \subset \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})$ the subspaces where complex conjugation acts by ± 1 , respectively. Since exact sequences (1.10) and (1.11) descend to exact sequences (1.12) and (1.13), we obtain

$$0 \rightarrow \mathcal{A}(V \otimes V_\mu(k)(\varepsilon_\sigma), \mathbb{C})^{\pm \varepsilon_\sigma} \rightarrow \mathcal{A}(V \otimes I_\mu(k)^{\varepsilon_\sigma}, \mathbb{C})^{\pm \varepsilon_\sigma} \rightarrow \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1} \rightarrow 0.$$

Hence the connection morphism satisfies

$$\delta_{\sigma}^{\varepsilon_\sigma} (H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1})) \subseteq H^{i+1}(G(F), \mathcal{A}(V \otimes V_\mu(k)(\varepsilon_\sigma), \mathbb{C})^{\pm \varepsilon_\sigma})$$

and the result follows. \square

2.1. Explicit computation of the connection morphisms. Let us consider the section $s : V_\mu(k)_\mathbb{C} \rightarrow I_\mu(k)$ of (1.7).

We compute that

$$\begin{aligned} \langle f_m, \iota P_n \rangle_I &= \int_{S^1} e^{mi\theta} P_n(-\sin \theta, \cos \theta) d\theta \\ &= \int_{S^1} i^{2n-k+2} e^{mi\theta} e^{ni\theta} e^{(n-k+2)i\theta} d\theta = \pi i^{2n-k+2} \delta(2n-k+2+m), \end{aligned}$$

where $\delta(n)$ is the Dirac delta. Thus, $g_\infty^{-1}P = \sum_{n=0}^{k-2} \alpha_n(g_\infty) P_n$, where

$$\alpha_n(g_\infty) = \frac{i^{k-2-2n}}{\pi} \langle f_{k-2-2n}, \iota(g_\infty^{-1}P) \rangle_I = \frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2-2n}, \iota(P) \rangle_I.$$

Since $V_\mu(k)$ is generated by $\{P_0, \dots, P_{k-2}\}$, we deduce that $\alpha_n = 0$ unless $n \in \{0, \dots, k-2\}$. Since matrix coefficient morphism are $(\mathcal{G}_\sigma, K_\sigma)$ -module morphisms by §1.4, we can compute on the one side

$$\begin{aligned} R\alpha_n(g_\infty) &= \frac{i^{k-2-2n}}{\pi} \langle g_\infty(Rf_{k-2-2n}), \iota(P) \rangle_I \\ &= (k-n-1) \frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2n}, \iota(P) \rangle_I = (n+1-k) \alpha_{n-1}(g_\infty), \\ L\alpha_n(g_\infty) &= \frac{i^{k-2-2n}}{\pi} \langle g_\infty(Lf_{k-2-2n}), \iota(P) \rangle_I \\ &= (n+1) \frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2n-4}, \iota(P) \rangle_I = -(n+1) \alpha_{n+1}(g_\infty). \end{aligned}$$

On the other side, we have that $s(P_n) = \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} f_{2n-k+2}$. Hence,

$$\begin{aligned} (n < k-2) \quad R s(P_n) &= \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} (n+1) f_{2n-k+4} = (k-2-n) s(P_{n+1}), \\ (n > 0) \quad L s(P_n) &= \frac{2^{k-2}}{\pi} \binom{k-2}{n}^{-1} (k-n-1) f_{2n-k} = n s(P_{n-1}). \end{aligned}$$

Assume that $\tilde{\phi} \in \mathcal{A}(V \otimes I_\mu(k), \mathbb{C})$ and the action of $(\mathcal{G}_\sigma, K_\sigma)$ on V is trivial. For any $f \in I_\mu(k)$ and $v \in V$, we will usually denote by $\tilde{\phi}_v(f)$ the expression $\tilde{\phi}(v \otimes f)$. We aim to compute

$$h_P(g_\infty) := \tilde{\phi}_v(s(g_\infty^{-1}P))(g_\infty, g), \quad g_\infty \in G(F_\sigma)^+ \simeq \mathrm{GL}_2(\mathbb{R})^+,$$

for all $g \in G(\mathbb{A}^\sigma)$, $P \in V_\mu(k)$, and $v \in V$. Since $h_P(g_\infty) = \sum_{n=0}^{k-2} \alpha_n(g_\infty) \tilde{\phi}_v(s(P_n))(g_\infty, g)$, we compute

$$\begin{aligned} Rh_P &= \sum_{n=0}^{k-2} \left((R\alpha_n) \tilde{\phi}_v(s(P_n)) + \alpha_n \tilde{\phi}_v(Rs(P_n)) \right) = \\ &= \sum_{n=0}^{k-2} (n+1-k) \alpha_{n-1} \tilde{\phi}_v(s(P_n)) + \frac{k-1}{2^{2-k}\pi} \alpha_{k-2} \tilde{\phi}_v(f_k) + \sum_{n=0}^{k-3} (k-2-n) \alpha_n \tilde{\phi}_v(s(P_{n+1})) \\ &= \frac{k-1}{\pi} 2^{k-2} \left(\alpha_{k-2} \tilde{\phi}_v(f_k) - \alpha_{-1} \tilde{\phi}_v(f_{2-k}) \right) = \frac{k-1}{\pi} 2^{k-2} \alpha_{k-2} \tilde{\phi}_v(f_k), \\ Lh_P &= \sum_{n=0}^{k-2} \left((L\alpha_n) \tilde{\phi}_v(s(P_n)) + \alpha_n \tilde{\phi}_v(Ls(P_n)) \right) = \\ &= \sum_{n=0}^{k-2} (-n-1) \alpha_{n+1} \tilde{\phi}_v(s(P_n)) + \frac{k-1}{2^{2-k}\pi} \alpha_0 \tilde{\phi}_v(f_{-k}) + \sum_{n=1}^{k-2} n \alpha_n \tilde{\phi}_v(s(P_{n-1})) \\ &= \frac{k-1}{\pi} 2^{k-2} \left(\alpha_0 \tilde{\phi}_v(f_{-k}) - \alpha_{k-1} \tilde{\phi}_v(f_{k-2}) \right) = \frac{k-1}{\pi} 2^{k-2} \alpha_0 \tilde{\phi}_v(f_{-k}), \end{aligned}$$

Notice that $R = e^{2i\theta} 2iy(\frac{\partial}{\partial \tau} - \frac{1}{4y} \frac{\partial}{\partial \theta})$ and $L = -e^{-2i\theta} 2iy(\frac{\partial}{\partial \bar{\tau}} - \frac{1}{4y} \frac{\partial}{\partial \bar{\theta}})$ with $\tau = x + iy$. Since s is a morphism of $\mathrm{SO}(2)\mathbb{R}^+$ -modules, h_P is a function of $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2)\mathbb{R}^+ \simeq \mathcal{H}$, thus h_P is a function on τ and $\bar{\tau}$. Let us compute $\frac{\partial h_P}{\partial \tau}$ and $\frac{\partial h_P}{\partial \bar{\tau}}$: By (1.9) and (1.8),

$$\begin{aligned} \frac{\partial h_P}{\partial \tau}(\tau, \bar{\tau}) &= \frac{y^{-1} e^{-4\pi i \theta}}{2i} R(h_P) = \frac{(k-1)}{2\pi i y e^{2i\theta}} i^{2-k} \langle g_\infty P_0, P \rangle \tilde{\phi}_v(f_k) \\ &= \frac{(k-1)}{2\pi i y e^{2i\theta}} (g_\infty^{-1} P)(1, -i) \tilde{\phi}_v(f_k) = \frac{(k-1)}{2\pi i} P(1, -\tau) \frac{\tilde{\phi}_v(f_k)}{f_k}(\tau, \bar{\tau}, g), \\ \frac{\partial h_P}{\partial \bar{\tau}}(\tau, \bar{\tau}) &= \frac{-y^{-1} e^{2i\theta}}{2i} L(h_P) = \frac{(1-k)}{2\pi i y e^{-2i\theta}} i^{k-2} \langle g_\infty \omega P_0, P \rangle \tilde{\phi}_v(f_{-k}) \\ &= \frac{(1-k)}{2\pi i y e^{-2i\theta}} (g_\infty^{-1} P)(1, i) \tilde{\phi}_v(f_{-k}) = \frac{(1-k)}{2\pi i} P(1, -\bar{\tau}) \frac{\tilde{\phi}_v(f_{-k})}{f_{-k}}(\tau, \bar{\tau}, g), \end{aligned}$$

by Remark 1.3, where $\tilde{\phi}_v(f_k) f_k^{-1}$ and $\tilde{\phi}_v(f_{-k}) f_{-k}^{-1}$ are seen as functions of $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2)\mathbb{R}^+ \simeq \mathcal{H}$. A similar (and classical) calculation shows that $\tilde{\phi}_v(f_k) f_k^{-1}$ and $\tilde{\phi}_v(f_{-k}) f_{-k}^{-1}$ are holomorphic and anti-holomorphic, respectively.

For any $P \in V_\mu(k)$, $g \in G(\mathbb{A}^\sigma)$, $v \in V$ and $\phi \in \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})$ the expressions

$$(2.14) \quad \omega_\phi(P, v, g)(\tau) := P(1, -\tau) \frac{\phi(v \otimes f_k)}{2\pi i f_k}(\tau, g) d\tau,$$

$$(2.15) \quad \bar{\omega}_\phi(P, v, g)(\bar{\tau}) := P(1, -\bar{\tau}) \frac{\phi(v \otimes f_{-k})}{2\pi i f_{-k}}(\bar{\tau}, g) d\bar{\tau},$$

define holomorphic and anti-holomorphic forms in \mathcal{H} , respectively. Moreover, it is easy to check that

$$\omega_\phi(P, v, g)(\gamma^{-1}\tau) = \omega_{\gamma\phi}(\gamma P, v, \gamma g)(\tau), \quad \bar{\omega}_\phi(P, v, g)(\gamma^{-1}\bar{\tau}) = \bar{\omega}_{\gamma\phi}(\gamma P, v, \gamma g)(\bar{\tau}),$$

for any $\gamma \in G(F) \cap G(F_\sigma)^+$. Assume that $c_\phi \in H^n(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C}))$ is represented by the n -cocycle $\phi : G(F)^n \rightarrow \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})$. Then $\partial_\sigma^{\epsilon_\sigma}(c_\phi)$ is represented by the $(n+1)$ -cocycle $d^n \tilde{\phi}$, where $\tilde{\phi}(\underline{\gamma}) \in \mathcal{A}(V \otimes I_\mu(k)_\mathbb{C}, \mathbb{C})$ is any preimage of $\phi(\underline{\gamma})$ for all $\underline{\gamma} \in G(F)^n$. We consider the $(n+1)$ -cocycle $\partial_\sigma^{\epsilon_\sigma}(\phi) = d^n \tilde{\phi} - d^n b$, where $b(\underline{\gamma})(g)(v \otimes P) = \tilde{\phi}(\underline{\gamma})(v \otimes s(P))(1, g)$. We compute, for all $P \in V_\mu(k)$, $v \in V$, $\underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in G(F)^n$, $\alpha \in G(F)$ and $g \in G(\mathbb{A}^\sigma)$,

$$\begin{aligned} \partial_\sigma^{\epsilon_\sigma} \phi(\alpha, \underline{\gamma})(g)(v \otimes P) &= \alpha \left((\tilde{\phi} - b)(\underline{\gamma}) \right) (v \otimes s(P))(1, g) + \\ &\quad + \sum_{i=1}^{n+1} (-1)^i (\tilde{\phi} - b)(\alpha \underline{\gamma}_i)(v \otimes s(P))(1, g), \end{aligned}$$

where $\alpha \underline{\gamma}_i = (\alpha, \gamma_1, \dots, \gamma_{i-1} \gamma_i, \dots, \gamma_n)$ for $i = 1, \dots, n$, and $\alpha \underline{\gamma}_{n+1} = (\alpha, \gamma_1, \dots, \gamma_n)$.

Since $(\tilde{\phi} - b)(\underline{\gamma})(v \otimes s(P)) = 0$ by construction, we obtain

$$\begin{aligned} \partial_\sigma^{\epsilon_\sigma} \phi(\alpha, \underline{\gamma})(g)(v \otimes P) &= \alpha \left(\tilde{\phi}(\underline{\gamma}) \right) (v \otimes s(P))(1, g) - \alpha (b(\underline{\gamma})) (v \otimes P)(g) \\ &= \tilde{\phi}(\underline{\gamma})(v \otimes s(P))(\alpha^{-1}, \alpha^{-1}g) - \tilde{\phi}(\underline{\gamma})(v \otimes s(\alpha^{-1}P))(1, \alpha^{-1}g) \end{aligned}$$

Since $\tilde{\phi}(\underline{\gamma})(v \otimes f_k) = \phi(\underline{\gamma})(v \otimes f_k)$ and $\tilde{\phi}(\underline{\gamma})(v \otimes f_{-k}) = \epsilon_\sigma(c) \phi(\underline{\gamma})(v \otimes f_{-k})$, we deduce from the above computations that, for any $\alpha \in G(F) \cap G(F_\sigma)^+$,

$$\begin{aligned} \partial_\sigma^{\epsilon_\sigma} \phi(\alpha, \underline{\gamma})(g)(v \otimes P) &= (k-1) \int_i^{\alpha^{-1}i} \omega_\phi(\underline{\gamma})(\alpha^{-1}P, v, \alpha^{-1}g) - \epsilon_\sigma(c) \bar{\omega}_\phi(\underline{\gamma})(\alpha^{-1}P, v, \alpha^{-1}g) \\ &= (1-k) \int_i^{\alpha i} \omega_{\alpha\phi}(\underline{\gamma})(P, v, g) - \epsilon_\sigma(c) \bar{\omega}_{\alpha\phi}(\underline{\gamma})(P, v, g). \end{aligned}$$

Remark 2.3. We have a well defined action of $G(F)/G(F)^+$ on $H^r(G(F)^+, M)$, for any $G(F)$ -module M , given by

$$c^\gamma(\alpha_1, \dots, \alpha_r) = \gamma(c(\gamma^{-1}\alpha_1\gamma, \dots, \gamma^{-1}\alpha_r\gamma)),$$

for $\gamma \in G(F)$, and $\alpha_i \in G(F)^+$. The image of the restriction map

$$H^r(G(F), M) \longrightarrow H^r(G(F)^+, M)$$

lies in $H^0(G(F)/G(F)^+, H^r(G(F)^+, M))$.

Let $\psi : \mathbb{A}^\times / F^\times \rightarrow \mathbb{C}^\times$ be a Hecke character such that, for any archimedean place $\sigma_i : F \hookrightarrow \mathbb{R}$, $\psi_{\sigma_i}(x) = \text{sign}(x)^{k_i} |x|^{\mu_i}$. Let $D_\psi(\underline{k})$ be the $(\mathcal{G}_\infty, K_\infty)$ -module obtained by making the tensor product of $D_{\mu_i}(k_i)$ at the place σ_i , if $\sigma_i \in \Sigma$, and $V_{\mu_j}(k_j)$ at the place σ_j , if $\sigma_j \notin \Sigma$. An element of $D_\psi(\underline{k})$ is $f_{\underline{k}} \otimes P^\Sigma$, where $f_{\underline{k}} = \bigotimes_{\sigma_i \in \Sigma} f_{k_i}$, $f_{k_i} \in D_{\mu_i}(k_i)$ are the elements defined above, and $P^\Sigma \in \bigotimes_{\sigma_i \notin \Sigma} V_{\mu_i}(k_i)$. Let $V_\psi(\underline{k})$ be the $(\mathcal{G}_\infty, K_\infty)$ -module obtained by making the tensor product of $V_{\mu_i}(k_i)$ at all the places σ_i . For any character $\epsilon : G(F_\infty)/G(F_\infty)^+ \simeq G(F)/G(F)^+ \rightarrow \pm 1$, we denote by $H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))(\epsilon)$ the ϵ -isotypical component, namely, the subspace of $H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))$ such that the action of $G(F)/G(F)^+$ is given by the character. By the above remark, the restriction map provides an isomorphism

$$H^r(G(F), \mathcal{A}(V_\psi(\underline{k})(\epsilon), \mathbb{C})) \simeq H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))(\epsilon).$$

Using the above computations, we aim to give an explicit formula for the connection morphism:

Theorem 2.4. *Let ϕ be a weight \underline{k} automorphic form of $G(\mathbb{A})$ with central character ψ . Then ϕ defines an element of $\phi \in H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C}))$. For a choice of signs at the places at infinity*

$$\epsilon : G(F_\infty)/G(F_\infty)^+ \simeq G(F)/G(F)^+ \rightarrow \pm 1,$$

the composition of the connection morphisms δ_{ϵ_σ} , for $\sigma \in \Sigma$,

$$\partial_\epsilon : H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})) \longrightarrow H^r(G(F), \mathcal{A}(V_\psi(\underline{k})(\epsilon), \mathbb{C})) \simeq H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))(\epsilon),$$

can be computed as follows:

$$\partial_\epsilon \phi = \prod_{j=1}^r (1 - k_j) \sum_{\gamma \in G(F)/G(F)^+} \epsilon(\gamma) \partial \phi^\gamma,$$

where $\partial \phi \in H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))$ is the class of the cocycle

$$(G(F)^+)^r \ni (g_1, g_2, \dots, g_r) \longmapsto \int_{\tau_1}^{g_1 \tau_1} \dots \int_{g_1 \dots g_{r-1} \tau_r}^{g_1 \dots g_r \tau_r} P_\Sigma(1, -\underline{z}) \frac{\phi(f_{\underline{k}} \otimes P^\Sigma)}{(2\pi i)^r f_{\underline{k}}}(\underline{z}, 1, g) d\underline{z},$$

for any $\underline{P} = P^\Sigma \otimes P_\Sigma \in V_{\underline{\mu}}(\underline{k})$, $\underline{z} = (z_1, \dots, z_r)$, $(\tau_1, \dots, \tau_r) \in \mathcal{H}^r$.

Proof. Let $S \subset \Sigma$ be a subset of archimedean places such that $\#S = s < r$. Assume that $\sigma = \sigma_j \in \Sigma \setminus S$ and let k be its corresponding weight and $\mu = \mu_j$. Let $V = \bigotimes_{\sigma_i \in S'} D_{\mu_i}(k_i) \otimes \bigotimes_{\sigma_i \in \Sigma \setminus (S \cup \{\sigma\})} V_{\mu_i}(k_i)(\epsilon_{\sigma_i})$, where $S' = \Sigma \setminus (S \cup \{\sigma\})$. Thus, the composition of the connection morphisms corresponding to $\sigma_i \in S$, provides a morphism

$$\delta_S : H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})) \longrightarrow H^s(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})).$$

By the previous computations, if $P \in V_\mu(k)$, $g \in G(\mathbb{A}^\sigma)$, $v \in V$,

$$\partial_\sigma^\epsilon \partial_S \phi(\alpha, \underline{\gamma})(g)(v \otimes P) = (1 - k) \int_i^{\alpha i} \omega_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g) - \epsilon_\sigma(c) \bar{\omega}_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g).$$

Notice that, letting $c \in (G(F) \cap G(F_S)^+) \setminus (G(F) \cap G(F_\sigma)^+)$, by means of the change of variables $\tau = g_\infty i \mapsto z = c\bar{\tau} = cg_\infty \omega i \in \mathcal{H}$, where $g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$, we obtain that

$$\begin{aligned} \int_i^{\alpha i} \bar{\omega}_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g) &= \int_i^{\alpha i} P(1, -\bar{\tau}) \frac{\alpha \partial_S \phi(\underline{\gamma})(v \otimes f_{-k})(g_\infty, g)}{f_{-k}(g_\infty)} d\bar{\tau} \\ &= - \int_{c(-i)}^{c\alpha(-i)} (c * P)(1, -z) \frac{c\alpha \partial_S \phi(\underline{\gamma})(v \otimes f_k)(cg_\infty \omega, cg)}{f_k(cg_\infty \omega)} dz \\ &= - \int_{\tau_0}^{c\alpha c^{-1}\tau_0} \omega_{c\alpha \partial_S \phi(\underline{\gamma})}(c * P, v, cg), \end{aligned}$$

where $\tau_0 = c(-i)$, but in fact, this last expression does not depend on the choice of τ_0 because \mathcal{H} is simply connected. Since $c\partial_S \phi(\underline{\gamma}) = \partial_S \phi(c\underline{\gamma}c^{-1})$ because $c \in G(F) \cap G(F_S)^+$, we obtain that

$$\partial_\sigma^{\epsilon_\sigma} \partial_S \phi(\alpha, \underline{\gamma}) = (1 - k)(r(\alpha, \underline{\gamma}) + \varepsilon(c)c^{-1}r(c\alpha c^{-1}, c\underline{\gamma}c^{-1})),$$

where r is the cocycle

$$r(\alpha, \underline{\gamma})(g)(v \otimes P) = \int_i^{\alpha i} \omega_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g).$$

Applying a simple induction on S we obtain the desired result. \square

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