



CENTRE DE RECERCA MATEMÀTICA

Title:	<i>Wiener–Beurling spaces and their properties</i>
Journal Information:	<i>Bulletin des Sciences Mathématiques,</i>
Author(s):	Nursultanov E., Tikhonov S..
Volume, pages:	159 1, DOI:[10.1016/j.bulsci.2019.102825]

Wiener–Beurling spaces and their properties

Erlan Nursultanov and Sergey Tikhonov

ABSTRACT. In this paper, we introduce the Wiener–Beurling spaces and study their main properties (embeddings, duality, interpolation). We also discuss the product and convolution operators as well as various average operators in the Wiener–Beurling spaces.

1. Introduction

The Wiener algebra $A = A^1$ of absolutely convergent Fourier series on the circle \mathbb{T}^1 (see, e.g., [Ka]) and its ℓ^β -versions, the β -Wiener spaces A^β , are defined as follows

$$A^\beta = A^\beta(\mathbb{T}^1) = \left\{ f \in L^1(\mathbb{T}^1) : \|f\|_{A^\beta(\mathbb{T}^1)} = \left(\sum_{j=-\infty}^{\infty} |\widehat{f}(j)|^\beta \right)^{1/\beta} < \infty \right\}.$$

Beurling [Be] introduced the spaces A^* , nowadays called the Beurling space, to study contraction properties of functions. By definition,

$$A^*(\mathbb{T}^1) = \left\{ f \in L^1(\mathbb{T}^1) : \|f\|_{A^*(\mathbb{T}^1)} = \sum_{j=0}^{\infty} \sup_{j \leq |k|} |\widehat{f}(k)| < \infty \right\}.$$

Important properties of the Wiener and Beurling spaces and their comparison can be found in [BLT, Ka]. In the case of function on \mathbb{R} the Wiener algebra was studied in [LST].

Recently [NRT], the β -Beurling spaces have been introduced:

$$A^{*,\beta} = A^{*,\beta}(\mathbb{T}^1) = \left\{ f \in L^1(\mathbb{T}^1) : \|f\|_{A^{*,\beta}(\mathbb{T}^1)} = \left(\sum_{s=0}^{\infty} 2^s \sup_{2^s \leq |k|} |\widehat{f}(k)|^\beta \right)^{1/\beta} < \infty \right\}.$$

It is easy to see that $A^{*,1} = A^*$. Both Wiener and Beurling function spaces play an important role in harmonic analysis, in particular, in the summability theory and in the Fourier synthesis (see, e.g., the texts [Ben], [SW, Theorems 1.25 and 1.16], and [TB, Theorem 8.1.3, Ch. 6]).

The main goal of this paper is to introduce a more general scale of spaces, which we call the Wiener–Beurling spaces, and investigate their basic properties.

2000 *Mathematics Subject Classification.* Primary 46E30; Secondary 46A45, 44A35, 46B70.

Key words and phrases. Wiener–Beurling spaces, Embeddings, Convolution, Interpolation, Duality.

This research was partially supported by MTM 2017-87409-P, 2017 SGR 358, the CERCA Programme of the Generalitat de Catalunya, and Ministry of Education and Science of the Republic of Kazakhstan (0112RK02176, 0112RK00608).

As usual, we will write $F_1 \asymp F_2$ if $c^{-1}F_1 \leq F_2 \leq cF_1$ for some positive constant c independent of essential parameters. Moreover, $F_1 \lesssim F_2$ stands for $F_1 \leq cF_2$. We will assume that $\frac{1}{p} + \frac{1}{p'} = 1$ for any $p \in (0, \infty]$.

DEFINITION 1.1. Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. We define the space $WB_{p,q}^\alpha$ to be the collection of formal trigonometric series

$$f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}, \quad x \in \mathbb{T},$$

such that

$$(1.1) \quad \|f\|_{WB_{p,q}^\alpha} = \left(\sum_{k=0}^{\infty} \left[2^{\alpha k} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |c_m|^p \right)^{1/p} \right]^q \right)^{1/q} < \infty,$$

with the usual modification for $p = \infty$ or/and $q = \infty$. Moreover, $\|f\|_{WB_{p,q}^\alpha}$ is a (quasi-) norm (a norm for $1 \leq p, q \leq \infty$).

Several remarks are in order.

(i) A similar definition (with specific values of the parameters α, p , and q) has first appeared in the paper by D. Borwein [Bo] devoted to summability problems. Various applications related to some special cases of Wiener–Beurling norm can be found in, e.g., [AM2, Ben, He, Ke, LL, Li, Sl, Su].

(ii) Clearly, $\|f\|_{WB_{p,q}^\alpha}$ can be equivalently defined by $\left(\sum_{k=0}^{\infty} \left[\gamma_k^\alpha \left(\sum_{[\gamma_k] \leq |m| < \gamma_{k+1}} |c_m|^p \right)^{1/p} \right]^q \right)^{1/q}$, where $0 < \gamma_0 < 1$ such that $1 < \gamma < \gamma_{k+1}/\gamma_k < \gamma^* < \infty$.

(iii) The parameter α in definition (1.1) can be seen as the smoothness parameter since $f \in WB_{p,q}^{\alpha+\beta}$ with $f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}$ if and only if $f^{(\beta)} \in WB_{p,q}^\alpha$, where $f^{(\beta)}$ is the fractional derivative of f .

EXAMPLES. It is clear that both Wiener and Beurling spaces are particular cases of the Wiener–Beurling spaces, namely, $WB_{\beta,\beta}^0 = A^\beta$ and $WB_{\infty,\beta}^{1/\beta} = A^{*,\beta}$. The latter follows from the fact that

$$\|f\|_{A^{*,\beta}} \asymp \left(\sum_{s=0}^{\infty} 2^s \sup_{2^s \leq |k| < 2^{s+1}} |c_k|^\beta \right)^{1/\beta}.$$

It is also easy to see that, for $1 < q \leq \infty$, we have

$$\|f\|_{WB_{1,q}^{\frac{1}{q}-1}} \asymp \|\{c_n\}\|_{ces(q)},$$

where the discrete Cesàro spaces $ces(q)$ is given by

$$ces(q) = \left\{ \{c_m\}_{m=0}^{\infty}, c_m \in \mathbb{R} : \|\{c_n\}\|_{ces(q)} = \left(\sum_{s=0}^{\infty} \left[\frac{1}{2s+1} \sum_{m=-s}^s |c_m| \right]^q \right)^{1/q} < \infty \right\}.$$

Indeed, for $\alpha < 0$ Hardy's inequality gives

$$(1.2) \quad \|f\|_{WB_{p,q}^\alpha} \asymp \left(\sum_{k=0}^{\infty} 2^{\alpha k q} \left(\sum_{0 \leq |m| < 2^k} |c_m|^p \right)^{q/p} \right)^{1/q}.$$

For the case $p = 1$ and $\alpha = \frac{1}{q} - 1 < 0$ we then have

$$\|f\|_{WB_{p,q}^\alpha} \asymp \left(\sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{0 \leq |m| < k} |c_m| \right)^q \right)^{1/q} \asymp \|\{c_n\}\|_{ces(q)}.$$

Similarly, one can show that for any $0 < q < \infty$ we have

$$\|f\|_{WB_{1,q}^{\frac{1}{q}-1}} = \|\{c_n\}\|_{cop(q)},$$

where the discrete Copson spaces $cop(q)$ is given by

$$cop(q) = \left\{ \{c_m\}_{m=0}^{\infty}, c_m \in \mathbb{R} : \left(\sum_{s=0}^{\infty} \left[\sum_{m=s}^{\infty} \frac{|c_m| + |c_{-m}|}{m} \right]^q \right)^{1/q} < \infty \right\}.$$

In particular, this immediately implies the following important fact due to Bennett [**Ben**, (6.8)]: $ces(q) = cop(q)$ for $1 < q < \infty$.

For the survey of Cesàro and Copson spaces see [**AM2**].

STRUCTURE OF THE PAPER. Section 2 consists of the key embeddings between Wiener–Beurling spaces with various parameters and Lebesgue and Besov spaces. In Section 3 we prove that the Köthe dual space of $WB_{p,q}^\alpha$ is $WB_{p',q'}^{-\alpha}$ for $\alpha \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. This, in particular, extends the results on duality of Cesaro space obtained by Bennett. As a corollary, we derive that $(A^{*,\beta})' = WB_{1,\beta'}^{-1/\beta}$, cf. [**BLT**, Prop. 2] for the case $\beta = 1$.

Section 4 studies the convolution and product operator for functions from Wiener–Beurling spaces. We claim that $WB_{p,1}^\alpha$ is an algebra for $1 \leq p \leq \infty$ and $\alpha \geq \frac{1}{p'}$, significantly extending the results by Belinskii, Liflyand and Trigub [**BLT**].

In Section 5 two interpolation theorems of Wiener–Beurling spaces are obtained. We prove that $(WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q} = WB_{p,q}^\alpha$ for $0 < p, q, q_0, q_1 \leq \infty$, $-\infty < \alpha_1 < \alpha_0 < \infty$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $0 < \theta < 1$. In the diagonal case, i.e., $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we derive that $(WB_{p_0,q_0}^{\alpha_0}, WB_{p_1,q_1}^{\alpha_1})_{\theta,q} = WB_{q,q}^\alpha$ for $0 < p_0, p_1 \leq \infty$, $0 < q, q_0, q_1 < \infty$, and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $0 < \theta < 1$. In particular, we obtain the interpolation properties of Cesaro and Copson spaces, investigated recently by Astashkin and Maligranda [**AM1**, **AM2**], as well as properties of the $A_{p,q}$ spaces studied by Lerner and Liflyand [**LL**]. Recall that the $\|\{c_m\}\|_{A_{p,q}}$ norm is given by $\left(\sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{k \leq |m| < 2k} |c_m|^p \right)^{q/p} \right)^{1/q}$.

Finally, in Section 6 we obtain Bernstein and Nikol'skii type polynomial inequalities in Wiener–Beurling spaces. Moreover, we study the Fourier series with averages of Fourier coefficients substantially improving the remarkable result by Bennett [**Ben**, Theorem 20.31] for Cesàro and Copson spaces.

2. Embedding properties of Wiener–Beurling spaces

2.1. Embeddings between Wiener–Beurling and Lebesgue spaces. We start with the simple embeddings, which follow from the properties of l^p spaces and Hölder's inequality. Let $\alpha \in \mathbb{R}$, $\varepsilon > 0$, $0 < p \leq \infty$, and $0 < q, q_1 \leq \infty$. Then

- (1) $WB_{p,q}^\alpha \hookrightarrow WB_{p,q+\varepsilon}^\alpha$,
- (2) $WB_{p,q}^\alpha \hookrightarrow WB_{p,q_1}^{\alpha+\varepsilon}$, $0 < p \leq \infty, 0 < q, q_1 \leq \infty$,
- (3) $WB_{p,q}^\alpha \hookrightarrow WB_{p+\varepsilon,q}^\alpha$,

(4) if $-\infty < \alpha_1 < \alpha_2 < \infty$ and $\alpha_1 + \frac{1}{p_1} = \alpha_2 + \frac{1}{p_2}$, then

$$(2.1) \quad WB_{p_2, q}^{\alpha_2} \hookrightarrow WB_{p_1, q}^{\alpha_1}.$$

The latter implies that $WB_{p, \beta}^{\alpha} \hookrightarrow A^{\beta}$ with $\frac{1}{\beta} = \alpha + \frac{1}{p}$.

Let us now recall Pitt's inequality for Fourier coefficients [**Pi**, **St**]:

$$\|f\|_{WB_{q, q}^{-\beta}} \asymp \left(\sum_{n \in \mathbb{Z}} \left((|n| + 1)^{-\beta} |\widehat{f}_n| \right)^q \right)^{1/q} \leq C \left(\int_{\mathbb{T}} (|x|^{\alpha} f(x))^p dx \right)^{1/p}$$

provided

$$1 < p \leq q < \infty, \quad 0 \leq \alpha < \frac{1}{p'}, \quad \beta = \alpha + \frac{1}{p} + \frac{1}{q} - 1 \geq 0.$$

The reverse inequality is given as follows: suppose

$$1 < p \leq q < \infty, \quad 0 \leq \alpha < \frac{1}{q}, \quad \beta = \alpha - \frac{1}{p} - \frac{1}{q} + 1 \geq 0,$$

then

$$\left(\int_{\mathbb{T}} (|x|^{-\alpha} |f(x)|)^q dx \right)^{1/q} \leq C \left(\sum_{n \in \mathbb{Z}} \left((|n| + 1)^{\beta} |\widehat{f}_n| \right)^p \right)^{1/p} \asymp \|f\|_{WB_{p, p}^{\beta}}.$$

In particular, we derive that $L_p \hookrightarrow WB_{p, p}^{\frac{1}{p'} - \frac{1}{p}}$ for $1 < p \leq 2$ and $WB_{p, p}^{\frac{1}{p'} - \frac{1}{p}} \hookrightarrow L_p$ for $2 \leq p < \infty$.

However, the latter two embeddings can be strengthened using the results of Kellogg [**Ke**]. Namely, we have

(5) if $1 < p \leq 2$, then

$$L_p \hookrightarrow WB_{p', 2}^0,$$

(6) if $p \geq 2$, then

$$WB_{p', 2}^0 \hookrightarrow L_p.$$

The next theorem discusses the well-known Wiener's problem on functions with positive Fourier coefficients. Recall a space of functions X is called *solid* if it satisfies the following property: For every $f = \sum c_n e^{in\theta}$ in X , if another function $g = \sum d_n e^{in\theta}$ satisfies $|d_n| \leq |c_n|$ for every n , then g is also in X .

THEOREM 2.1. [**ATT**] *Let X be a space of functions. If $L^p \hookrightarrow X$ and X is solid, then*

$$L_{loc+}^p \hookrightarrow X,$$

where

$$L_{loc+}^p = \left\{ f : \text{all } \widehat{f}(n) \geq 0 \text{ and } \int_{-\delta}^{\delta} |f|^p dx < \infty \text{ for some } \delta = \delta(f) > 0 \right\}.$$

The space $WB_{p, q}^{\alpha}$ is clearly solid, which, together with embedding (5), implies that

(5') if $1 < p \leq 2$, then

$$L_{loc+}^p \hookrightarrow WB_{p', 2}^0.$$

2.2. Embeddings between Besov and Wiener–Beurling spaces. We define the Besov space $B_{p,q}^\alpha$, $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, to be the collection of formal trigonometric series $f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}$ such that

$$\|f\|_{B_{p,q}^\alpha} = \left(\sum_{k=0}^{\infty} 2^{\alpha q k} \left\| \sum_{[2^{k-1}] \leq |m| < 2^k - 1} c_m e^{imx} \right\|_p^q \right)^{1/q} < \infty.$$

It is well known (see, e.g., [Ni]) that for $1 < p < \infty$ this definition is equivalent to the classical definitions of the Besov space given in terms of moduli of smoothness or Fourier-analytical decompositions. In light of Nikol'skii's inequality for trigonometric polynomials it is easy to see the following (Jawerth–Franke-type) embedding: if $-\infty < \alpha_1 < \alpha_2 < \infty$, $0 < p_2 < p_1 \leq \infty$, and $\alpha_1 - \frac{1}{p_1} = \alpha_2 - \frac{1}{p_2}$, then $B_{p_2,q}^{\alpha_2} \hookrightarrow B_{p_1,q}^{\alpha_1}$, cf. embedding (4) for Wiener–Beurling spaces.

To compare Besov and Wiener–Beurling spaces, we start with the classical Bernstein theorem and its generalization by Szasz [Sz]: if $f \in B_{2,p}^{1/p-1/2}$, $1 \leq p \leq 2$, then $\{\hat{f}_n\} \in l_p$. In fact, using the embedding properties of Wiener–Beurling spaces (see (2.1)), this result trivially follows from Parseval's identity:

$$B_{2,p}^{1/p-1/2} = WB_{2,p}^{1/p-1/2} \hookrightarrow WB_{p,p}^0 = A^p.$$

Further, the Hausdorff–Young and Nikol'skii's inequalities immediately give

$$B_{p,q}^\alpha \hookrightarrow \begin{cases} WB_{\infty,q}^{\alpha+1-\frac{1}{p}}, & 0 < p \leq 1; \\ WB_{p',q}^\alpha, & 1 < p \leq 2; \\ WB_{2,q}^\alpha, & 2 < p \leq \infty; \end{cases}$$

and

$$WB_{p,q}^\alpha \hookrightarrow \begin{cases} B_{\infty,q}^\alpha, & 0 < p \leq 1; \\ B_{p',q}^\alpha, & 1 < p \leq 2; \\ B_{2,q}^{\alpha+\frac{1}{p}-\frac{1}{2}}, & 2 < p \leq \infty. \end{cases}$$

Pitt's inequalities imply the following embeddings

$$B_{p,q}^{\alpha+\beta} \hookrightarrow WB_{r,q}^\alpha, \quad 1 < p \leq r \leq p', \quad \beta = \frac{1}{p} + \frac{1}{r} - 1 \geq 0, \quad 0 < q \leq \infty, \quad \forall \alpha \in \mathbb{R}$$

and

$$WB_{p,q}^{\alpha+\beta} \hookrightarrow B_{r,q}^\beta, \quad 1 < r' \leq p \leq r < \infty, \quad \alpha = 1 - \frac{1}{p} - \frac{1}{r} \geq 0, \quad 0 < q \leq \infty, \quad \forall \beta \in \mathbb{R}.$$

Sharpness of these embeddings can be seen with the help of functions with lacunary and monotone Fourier coefficients. For trigonometric series with lacunary coefficients, i.e., $f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}$ such that $c_k = 0$ for $|k| \neq n_s$ with $n_{s+1}/n_s \geq \lambda > 1$, we have

$$\|f\|_{WB_{p_1,q}^\alpha} \asymp \|f\|_{B_{p_2,q}^\alpha}, \quad \alpha \in \mathbb{R}, \quad 0 < p_1, p_2, q \leq \infty.$$

Now we consider series with monotone coefficients.

THEOREM 2.2. *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, and $0 < q, q_1 \leq \infty$. Let $f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}$ be such that $c_k \leq c_m$ for $|m| \leq |k|$, then*

$$\|f\|_{WB_{p,q}^\alpha} \asymp \|f\|_{B_{p',q}^\alpha}.$$

PROOF. Since the coefficients c_k are decreasing, then by Hardy–Littlewood theorem on monotone Fourier coefficients [Zy, V.2, XII, §6], we have that

$$\begin{aligned} \|f\|_{B_{p',q}^\alpha} &= \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} \left\| \sum_{[2^{k-1}] \leq |m| < 2^k} c_m e^{imx} \right\|_{L_{p'}} \right)^q \right)^{1/q} \\ &\asymp \left(\sum_{k=0}^{\infty} 2^{\alpha q k} \left(\sum_{m=1}^{2^{k-1}} \left(m^{1/p} (|c_{2^{k-1}+m}| + |c_{-2^{k-1}-m}|) \right)^{p'} \frac{1}{m} \right)^{q/p'} \right)^{1/q} \\ &\asymp \left(\sum_{k=0}^{\infty} 2^{(\alpha+1/p)qk} (|c_{2^k}| + |c_{-2^k}|)^q \right)^{1/q}, \end{aligned}$$

where in the last step we have used Hardy's inequality.

On the other hand, the monotonicity of coefficients implies that

$$\begin{aligned} \|f\|_{WB_{p,q}^\alpha} &= \left(\sum_{k=0}^{\infty} 2^{\alpha q k} \left(\sum_{[2^{k-1}] \leq |m| < 2^k-1} |c_m|^p \right)^{q/p} \right)^{1/q} \\ &\asymp \left(\sum_{k=0}^{\infty} 2^{(\alpha+1/p)qk} (|c_{2^k}| + |c_{-2^k}|)^q \right)^{1/q}, \end{aligned}$$

completing the proof. \square

REMARK 2.1. Analyzing the proof of this theorem, we observe that the same result holds under less restrictive conditions on the coefficients $\{c_n\}$. In particular, monotonicity can be assumed only on each dyadic block (that is, $\{c_n\}_{2^{k-1} \leq |n| < 2^k}$ is decreasing or increasing) or, what is more, monotonicity can be replaced by the general monotonicity, see, e.g., [Ti].

3. Duality of Wiener-Beurling space

The problem to describe the dual of the Cesaro space $ces(q)$, which is a particular case of the Wiener-Beurling space, has a long history (see, e.g., [Ben, Sect. 2]). Bennett proved that, for $1 < q < \infty$,

$$(3.1) \quad (ces(q))' = \left\{ \{c_n\} : \sum_n \sup_{k \geq n} |c_k|^{q'} < \infty \right\},$$

where the Köthe dual of a space E is given by $E' = \left\{ \{d_n\} : \sum_n |c_n d_n| < \infty \text{ for all } \{c_n\} \in E \right\}$. We extend this result to the Wiener-Beurling space.

THEOREM 3.1. *Let $\alpha \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. We have*

$$(WB_{p,q}^\alpha)' = WB_{p',q'}^{-\alpha},$$

where $\|f\|_{X'} := \sup_{\|g\|_X=1} \int_{\mathbb{R}} f(x) \overline{g(x)} dx$.

In particular, if $p = 1$, $\alpha = \frac{1}{q} - 1$, and $1 < q < \infty$ we recover (3.1). For the Beurling space $A^{*,1}$ (the case $p = \infty$ and $\alpha = q = 1$), this result was proved in [BLT, Proposition 2].

PROOF. For $f = \sum_n \xi_n e^{inx}$ and $g = \sum_{n \in \mathbb{Z}} c_n e^{inx}$, we have

$$\|f\|_{(WB_{p,q}^\alpha)'} := \sup_{\|g\|_{WB_{p,q}^\alpha}=1} \int_{\mathbb{R}} f(x) \overline{g(x)} dx = \sup_{\|g\|_{WB_{p,q}^\alpha}=1} \sum_{n \in \mathbb{Z}} \xi_n \overline{c_n}.$$

Then Hölder's inequality implies

$$\|f\|_{(WB_{p,q}^\alpha)'} \leq \sup_{\|g\|_{WB_{p,q}^\alpha}=1} \|f\|_{WB_{p',q'}^{-\alpha}} \|g\|_{WB_{p,q}^\alpha} = \|f\|_{WB_{p',q'}^{-\alpha}}.$$

To prove the reverse, for a given function $f = \sum_n \xi_n e^{inx} \in WB_{p',q'}^{-\alpha}$, we define $g = \sum_{n \in \mathbb{Z}} c_n e^{inx}$ with

$$c_n = d_k \xi_n |\xi_n|^{p'-1} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |\xi_m|^{p'} \right)^{-1/p} \quad \text{for } [2^{k-1}] \leq |n| < 2^k, \quad k \in \mathbb{Z}_+,$$

$$d_k = 2^{-\alpha k q'} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |\xi_m|^{p'} \right)^{\frac{q'-1}{p'}} \left(\|f\|_{WB_{p',q'}^{-\alpha}} \right)^{-\frac{q'}{q}}.$$

Then we have

$$\begin{aligned} \|g\|_{WB_{p,q}^\alpha} &= \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} d_k \frac{\left(\sum_{[2^{k-1}] \leq |n| < 2^k} |\xi_n|^{(p'-1)p} \right)^{1/p}}{\left(\sum_{[2^{k-1}] \leq |n| < 2^k} |\xi_n|^{p'} \right)^{1/p}} \right)^q \right)^{1/q} = \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} d_k \right)^q \right)^{1/q} \\ &= \left(\sum_{k=0}^{\infty} \left(2^{\alpha k} 2^{-\alpha k q'} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |\xi_m|^{p'} \right)^{\frac{q'-1}{p'}} \right)^q \right)^{1/q} \left(\|f\|_{WB_{p',q'}^{-\alpha}} \right)^{-\frac{q'}{q}} = 1. \end{aligned}$$

Thus,

$$\|f\|_{(WB_{p,q}^\alpha)'} \geq \sum_{n \in \mathbb{Z}} \xi_n \overline{c_n} = \|f\|_{WB_{p',q'}^{-\alpha}},$$

where the last equality follows by the choice of c_n with the help of the Parseval identity. \square

4. Convolution and product of function from Wiener-Beurling spaces

For functions $f = \sum_k c_k e^{ikx}$ and $g = \sum_k d_k e^{ikx}$, we define their convolution $f * g$ and product $f \cdot g$ as follows:

$$f * g = \sum_k c_k d_k e^{ikx}, \quad f \cdot g = \sum_k \sum_{\nu} c_{\nu} d_{k-\nu} e^{ikx}.$$

Holder's inequality immediately yields the following result for the convolution of functions in the Beurling-Wiener space.

THEOREM 4.1. *Let $\alpha, \alpha_i \in \mathbb{R}$ and $0 < p, p_i, q, q_i \leq \infty$ for $i = 1, 2$. Let*

$$\alpha = \alpha_1 + \alpha_2, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

If $f \in WB_{p_1, q_1}^{\alpha_1}$ and $g \in WB_{p_2, q_2}^{\alpha_2}$, then $f * g \in WB_{p, q}^{\alpha}$ and

$$\|f * g\|_{WB_{p, q}^{\alpha}} \leq \|f\|_{WB_{p_1, q_1}^{\alpha_1}} \|g\|_{WB_{p_2, q_2}^{\alpha_2}}.$$

This result is closely connected to the multiplier theory of H^p spaces; see [Ke, Sl].

The product of functions from the Beurling–Wiener space is treated by the following result.

THEOREM 4.2. *Let $1 \leq p \leq \infty$, $\alpha > 0$, and $\bar{\alpha} = \max(\frac{1}{p}, \alpha)$. Then*

$$\|f \cdot g\|_{WB_{p, 1}^{\alpha}} \lesssim \|f\|_{WB_{p, 1}^{\bar{\alpha}}} \|g\|_{WB_{p, 1}^{\bar{\alpha}}}.$$

PROOF. Let $f = \sum_k c_k e^{ikx}$ and $g = \sum_k d_k e^{ikx}$. First, we note that

$$\begin{aligned} \|f \cdot g\|_{WB_{p, 1}^{\alpha}} &= \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{[2^{k-1}] \leq |n| < 2^k} \left| \sum_{\nu \in \mathbb{Z}} c_{\nu} d_{n-\nu} \right|^p \right)^{1/p} \\ &\leq \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu \in \mathbb{Z}} c_{\nu} d_{n-\nu} \right|^p \right)^{1/p} + \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=-[2^{k-1}]}^{-2^k+1} \left| \sum_{\nu \in \mathbb{Z}} c_{\nu} d_{n-\nu} \right|^p \right)^{1/p} =: I_1 + I_2. \end{aligned}$$

We will estimate only the first term, the second one can be treated similarly. We have

$$\begin{aligned} I_1 &= \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu=-\infty}^{\frac{n}{2}} c_{\nu} d_{n-\nu} + \sum_{\nu=\frac{n}{2}}^{\infty} c_{\nu} d_{n-\nu} \right|^p \right)^{1/p} \\ &\leq \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu=-\infty}^{\frac{n}{2}} c_{\nu} d_{n-\nu} \right|^p \right)^{1/p} + \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu=-\infty}^{\frac{n}{2}} c_{n-\nu} d_{\nu} \right|^p \right)^{1/p} =: I_{11} + I_{12}. \end{aligned}$$

Changing variable and using Minkowski's inequality, we derive

$$\begin{aligned} &\left(\sum_{n=[2^{k-1}]}^{2^k-1} \left| \sum_{\nu \leq \frac{n}{2}} c_{\nu} d_{n-\nu} \right|^p \right)^{1/p} \\ &\leq \sum_{\nu=-\infty}^{2^{k-2}-1} |c_{\nu}| \left(\sum_{n=[2^{k-1}]}^{2^k-1} |d_{n-\nu}|^p \right)^{1/p} + \sum_{\nu=2^{k-2}}^{2^{k-1}-1} |c_{\nu}| \left(\sum_{n=2\nu}^{2^k-1} |d_{n-\nu}|^p \right)^{1/p} \\ &\leq \sum_{\nu=-\infty}^{2^{k-2}-1} |c_{\nu}| \left(\sum_{\xi \geq 2^{k-2}} |d_{\xi}|^p \right)^{1/p} + \sum_{\nu=2^{k-2}}^{2^{k-1}-1} |c_{\nu}| \left(\sum_{\xi=2^{k-2}}^{2^k} |d_{\xi}|^p \right)^{1/p} \\ &\leq \sum_{\nu \in \mathbb{Z}} |c_{\nu}| \left(\sum_{\xi \geq 2^{k-2}} |d_{\xi}|^p \right)^{1/p}. \end{aligned}$$

Thus we have

$$I_{11} \leq \sum_{\nu \in \mathbb{Z}} |c_{\nu}| \sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{n \geq [2^{k-2}]} |d_n|^p \right)^{1/p}.$$

Since, for $\alpha > 0$, taking into account Hardy's inequality, we have

$$\sum_{k=0}^{\infty} 2^{k\alpha} \left(\sum_{[2^{k-2}] \leq n} |d_n|^p \right)^{1/p} \lesssim \|g\|_{WB_{p,1}^\alpha}$$

and

$$\sum_{\nu \in \mathbb{Z}} |c_\nu| \leq \sum_{n=0}^{\infty} \left(\sum_{n=[2^{k-1}]}^{2^k-1} |c_\nu|^p \right)^{1/p} 2^{\frac{k}{p'}} \leq \|f\|_{WB_{p,1}^{\frac{1}{p'}}},$$

we finally have

$$I_{11} \lesssim \|f\|_{WB_{p,1}^{1/p'}} \|g\|_{WB_{p,1}^\alpha} \leq \|f\|_{WB_{p,1}^{\bar{\alpha}}} \|g\|_{WB_{p,1}^{\bar{\alpha}}}.$$

To conclude the proof, we note that I_{12} and I_2 can be estimated similarly. \square

COROLLARY 4.1. *Let $1 \leq p \leq \infty$, $\alpha \geq \frac{1}{p'}$. Then $f \cdot g \in WB_{p,1}^\alpha$ provided that $f, g \in WB_{p,1}^\alpha$, i.e., $WB_{p,1}^\alpha$ is an algebra.*

When $p = \infty$ and $\alpha = q = 1$ this result was obtained in [BLT, Proposition 1]. We also mention the recent paper [Ri], where the author studied the boundedness of the convolution operator from the Cesàro space $ces(q)$ into $ces(q)$.

5. Interpolation properties of Wiener–Beurling spaces

Consider a compatible couple (X_0, X_1) of Banach spaces X_0 and X_1 (see [BS, Ch. 3]). Let

$$K(f, t) := K(f, t; X_0, X_1) := \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + t\|f_1\|_{X_1}), \quad f \in X_0 + X_1$$

be the K -functional. The interpolation space $(0 < \theta < 1)$ is given by

$$(X_0, X_1)_{\theta, q} = \left\{ f \in X_0 + X_1 : \|f\|_{(X_0, X_1)_{\theta, q}} = \left(\int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

for $0 < q < \infty$, with the usual modification for $q = \infty$.

The main interpolation properties of Cesàro and Copson sequence spaces have been recently studied in [AM1, AM2]:

$$(5.1) \quad (ces(q_0), ces(q_1))_{\theta, q} = ces(q), \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 1 < q_0 < q_1 \leq \infty, \quad 0 < \theta < 1,$$

$$(5.2) \quad (cop(q_0), cop(q_1))_{\theta, q} = cop(q), \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 1 \leq q_0 < q_1 \leq \infty, \quad 0 < \theta < 1.$$

The similar result for the weighted Cesàro space was proved in [LM, Cor.5] for $q \in [1, \infty]$.

In [LL], the authors considered the $A_{p,q}$ spaces with the norm

$$\|\{c_m\}\|_{A_{p,q}} = \left(\sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{k \leq |m| < 2k} |c_m|^p \right)^{q/p} \right)^{1/q} < \infty$$

and proved that

$$(5.3) \quad (A_{p,q_0}, A_{p,q_1})_{\theta, q} = A_{p,q}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 < \theta < 1,$$

provided $1 \leq q_0 < q_1 \leq \infty$, $1 \leq p \leq \infty$ and

$$(5.4) \quad (A_{p_0,p}, A_{p_1,p})_{\theta,p} = A_{p,p} = \ell_p, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1,$$

provided $1 \leq p_0 < p_1 \leq \infty$. The $A_{p,q}$ spaces are used in various problems of Fourier analysis; see, e.g., the recent monograph [Li].

It is easy to see that $\|c_n\|_{A_{p,q}} < \infty$ if and only if $f \in WB_{p,q}^{1/q-1/p}$, where $f = \sum c_k e^{ikx}$. In fact we have that $f \in WB_{p,q}^\alpha$, $f = \sum c_k e^{ikx}$, if and only if

$$\left(\sum_{k=0}^{\infty} 2^{\alpha q k} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |c_m|^p \right)^{q/p} \right)^{1/q} \asymp \left(\sum_{k=0}^{\infty} k^{\alpha q - 1} \left(\sum_{k \leq |m| < 2k} |c_m|^p \right)^{q/p} \right)^{1/q} < \infty.$$

We extend (5.1)–(5.4) in the following

THEOREM 5.1. *Let $-\infty < \alpha_1 < \alpha_0 < \infty$.*

(i) *If $0 < p, q, q_0, q_1 \leq \infty$, then*

$$(5.5) \quad (WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q} = WB_{p,q}^\alpha,$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $0 < \theta < 1$.

(ii) *If $0 < p_0, p_1 \leq \infty$ and $0 < q, q_0, q_1 < \infty$, then*

$$(5.6) \quad (WB_{p_0,q_0}^{\alpha_0}, WB_{p_1,q_1}^{\alpha_1})_{\theta,q} = WB_{q,q}^\alpha, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $0 < \theta < 1$.

In particular, if $\alpha_0 = 1/q_0 - 1/p$ and $\alpha_1 = 1/q_1 - 1/p$ we recover (5.3) and if $q = q_0 = q_1$, $\alpha_0 = 1/q - 1/p_0$, and $\alpha_1 = 1/q - 1/p_1$ we recover (5.4) with q in place of p . Note also that (5.6) with $q = q_0 = q_1 = \infty$ becomes a well-known result ([BL, Ch. 5]).

PROOF. Let $f \in (WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q}$ be such that $f = f_0 + f_1$, where $f_0 = \sum_k d_k e^{ikx} \in WB_{p,q_0}^{\alpha_0}$, $f_1 = \sum_k \xi_k e^{ikx} \in WB_{p,q_1}^{\alpha_1}$, and $c_k = d_k + \xi_k$. Then for any $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} 2^{\alpha k} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |c_m|^p \right)^{1/p} &\lesssim 2^{(\alpha-\alpha_0)k} \left(2^{\alpha_0 k} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |d_m|^p \right)^{1/p} \right. \\ &+ \left. 2^{(\alpha_0-\alpha_1)k} 2^{\alpha_1 k} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |\xi_m|^p \right)^{1/p} \right) \lesssim 2^{(\alpha-\alpha_0)k} \left(\|f_0\|_{WB_{p,\infty}^{\alpha_0}} + 2^{(\alpha_0-\alpha_1)k} \|f_1\|_{WB_{p,\infty}^{\alpha_1}} \right). \end{aligned}$$

Thus we derive that

$$2^{\alpha k} \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |c_m|^p \right)^{1/p} \lesssim 2^{(\alpha-\alpha_0)k} K(f, 2^{(\alpha_0-\alpha_1)k}; WB_{p,\infty}^{\alpha_0}, WB_{p,\infty}^{\alpha_1}),$$

which yields

$$\begin{aligned} \|f\|_{WB_{p,q}^\alpha} &\lesssim \left(\sum_{k=0}^{\infty} \left(2^{(\alpha-\alpha_0)k} K(f, 2^{(\alpha_0-\alpha_1)k}; WB_{p,\infty}^{\alpha_0}, WB_{p,\infty}^{\alpha_1}) \right)^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}} = \|f\|_{(WB_{p,\infty}^{\alpha_0}, WB_{p,\infty}^{\alpha_1})_{\theta,q}}. \end{aligned}$$

Using now the embedding properties of Beurling–Wiener spaces, we obtain

$$(WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q} \hookrightarrow (WB_{p,\infty}^{\alpha_0}, WB_{p,\infty}^{\alpha_1})_{\theta,q} \hookrightarrow WB_{p,q}^{\alpha}.$$

To show the reverse embedding, let $f \in WB_{p,q}^{\alpha}$ and $\tau = \min(q_0, q_1, q), r \in \mathbb{Z}$. Define functions f_0 and f_1 as follows

$$f_0 = \begin{cases} \sum_{|m| < 2^r} c_m(f) e^{imx} & \text{for } r \in \mathbb{Z}_+ = \{z \in \mathbb{Z}, z \geq 0\}, \\ 0 & \text{for } r \notin \mathbb{Z}_+. \end{cases}$$

and $f_1 = f - f_0$. Then, letting $\Delta_k := \Delta_k(p) := \left(\sum_{[2^{k-1}] \leq |m| < 2^k} |c_m|^p \right)^{1/p}$, we obtain

$$\begin{aligned} K(f, 2^{(\alpha_0 - \alpha_1)r}; WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1}) &\lesssim K(f, 2^{(\alpha_0 - \alpha_1)r}; WB_{p,\tau}^{\alpha_0}, WB_{p,\tau}^{\alpha_1}) \\ &\lesssim \|f_0\|_{WB_{p,\tau}^{\alpha_0}} + 2^{(\alpha_0 - \alpha_1)r} \|f_1\|_{WB_{p,\tau}^{\alpha_1}} \\ (5.7) \quad &= \left(\sum_{k=0}^r 2^{\alpha_0 k \tau} \Delta_k^\tau \right)^{1/\tau} + 2^{(\alpha_0 - \alpha_1)r} \left(\sum_{k=r+1}^{\infty} 2^{\alpha_1 k \tau} \Delta_k^\tau \right)^{1/\tau}. \end{aligned}$$

It follows that

$$\begin{aligned} \|f\|_{(WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q}} &\lesssim \|f\|_{(WB_{p,\tau}^{\alpha_0}, WB_{p,\tau}^{\alpha_1})_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty \left(t^{-\theta(\alpha_0 - \alpha_1)} K(f, t^{(\alpha_0 - \alpha_1)}) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^1 \left(t^{-\theta(\alpha_0 - \alpha_1)} t^{(\alpha_0 - \alpha_1)} \|f\|_{WB_{p,\tau}^{\alpha_1}} \right)^q \frac{dt}{t} + \int_1^\infty \left(t^{-\theta(\alpha_0 - \alpha_1)} K(f, t^{(\alpha_0 - \alpha_1)}) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_{WB_{p,\tau}^{\alpha_1}} + \left(\sum_{k=0}^\infty \left(2^{(\alpha_1 - \alpha_0)k} K(f, 2^{(\alpha_0 - \alpha_1)k}) \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Using $\|f\|_{WB_{p,\tau}^{\alpha_1}} \lesssim \|f\|_{WB_{p,q}^{\alpha}}$ for $\alpha_1 < \alpha$ and estimate (5.7), we arrive at

$$\begin{aligned} \|f\|_{(WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q}} &\lesssim \|f\|_{WB_{p,q}^{\alpha}} \\ &\quad + \left(\sum_{r=0}^\infty 2^{(\alpha - \alpha_0)r q} \left(\left(\sum_{k=0}^r 2^{\alpha_0 k \tau} \Delta_k^\tau \right)^{1/\tau} + 2^{(\alpha_0 - \alpha_1)r} \left(\sum_{k=r+1}^\infty 2^{\alpha_1 k \tau} \Delta_k^\tau \right)^{1/\tau} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Finally, taking into account $\alpha_1 < \alpha < \alpha_0$ and using Hardy's inequality, we derive

$$\|f\|_{(WB_{p,q_0}^{\alpha_0}, WB_{p,q_1}^{\alpha_1})_{\theta,q}} \lesssim \|f\|_{WB_{p,q}^{\alpha}}.$$

To obtain (5.6), we first apply the power theorem for interpolation spaces (see [BL, Th. 3.11.6]), we obtain

$$(5.8) \quad ((WB_{p_0,q_0}^{\alpha_0}, WB_{p_1,q_1}^{\alpha_1})_{\theta,q})^q = ((WB_{p_0,q_0}^{\alpha_0})^{q_0}, (WB_{p_1,q_1}^{\alpha_1})^{q_1})_{\eta,1},$$

where

$$\eta = \frac{q\theta}{q_1}, \quad q = (1 - \eta)q_0 + \eta q_1, \quad 0 < \eta < 1.$$

Observe that for $f(x) = \sum_{m \in \mathbb{Z}} c_m e^{imx}$ we have

$$\begin{aligned} K(t, f; (WB_{p_0, q_0}^{\alpha_0})^{q_0}, (WB_{p_1, q_1}^{\alpha_1})^{q_1}) &= \inf_{f=f_0+f_1} \left(\|f_0\|_{WB_{p_0, q_0}^{\alpha_0}}^{q_0} + t \|f_1\|_{WB_{p_1, q_1}^{\alpha_1}}^{q_1} \right) \\ &= \sum_{k=0}^{\infty} \inf_{b_k=b_{0,k}+b_{1,k}} \left(2^{\alpha_0 q_0 k} \|b_{0,k}\|_{l_{p_0}}^{q_0} + t 2^{\alpha_1 q_1 k} \|b_{1,k}\|_{l_{p_1}}^{q_1} \right), \end{aligned}$$

where $b_k = \{c_m\}_{m=2^k}^{2^{k+1}-1}$, $b_{0,k} = \{c_m^0\}_{m=2^k}^{2^{k+1}-1}$, $b_{1,k} = \{c_m^1\}_{m=2^k}^{2^{k+1}-1}$, $c_m = c_m^0 + c_m^1$, $m \in \mathbb{Z}$.

This together with (5.8) implies

$$\begin{aligned} \|f\|_{(WB_{p_0, q_0}^{\alpha_0}, WB_{p_1, q_1}^{\alpha_1})_{\theta, q}}^q &\asymp \|f\|_{((WB_{p_0, q_0}^{\alpha_0})^{q_0}, (WB_{p_1, q_1}^{\alpha_1})^{q_1})_{\eta, 1}} \\ &= \int_0^\infty t^{-\eta} K(t, f; (WB_{p_0, q_0}^{\alpha_0})^{q_0}, (WB_{p_1, q_1}^{\alpha_1})^{q_1}) \frac{dt}{t} \\ &= \sum_{k=0}^{\infty} 2^{\alpha_0 q_0 k} \int_0^\infty t^{-\eta} K(t 2^{(\alpha_1 q_1 - \alpha_0 q_0)k}, b_k; (l_{p_0})^{q_0}, (l_{p_1})^{q_1}) \frac{dt}{t}. \end{aligned}$$

Changing variables (note that $\alpha_0 q_0 + (\alpha_1 q_1 - \alpha_0 q_0)\eta = \alpha q$) and applying again the power theorem, we obtain

$$\begin{aligned} \|f\|_{(WB_{p_0, q_0}^{\alpha_0}, WB_{p_1, q_1}^{\alpha_1})_{\theta, q}}^q &= \sum_{k=0}^{\infty} 2^{\alpha q k} \|b_k\|_{((l_{p_0})^{q_0}, (l_{p_1})^{q_1})_{\eta, 1}} \asymp \sum_{k=0}^{\infty} 2^{\alpha q k} \|b_k\|_{(l_{p_0}, l_{p_1})_{\theta, q}}^q \\ &\asymp \sum_{k=0}^{\infty} 2^{\alpha q k} \|b_k\|_{l_q}^q, \quad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \end{aligned}$$

□

Theorem 5.1 immediately implies the following

COROLLARY 5.1. *Let $0 < q \leq \infty$, $\beta > 0$. We have*

- (i) $(WB_{\beta, q_1}^{\alpha_1}, A^\beta)_{\theta, q} = WB_{\beta, q}^\alpha$, $\alpha = (1-\theta)\alpha_1$, $0 < \theta < 1$, $\alpha_1 \neq 0$.
- (ii) $(A^{*\beta}, WB_{\infty, q}^{\alpha_1})_{\theta, q} = WB_{\infty, q}^\alpha$, $\alpha = \frac{1-\theta}{\beta} + \theta\alpha_1$, $0 < \theta < 1$, $\alpha_1 \neq 1/\beta$.
- (iii) $(A^{*\beta_0}, A^{*\beta_1})_{\theta, q} = WB_{\infty, q}^{\frac{1}{\beta}}$, $\frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$, $0 < \theta < 1$, $0 < \beta_0 < \beta_1 < \infty$.

In particular,

$$(A^{*\beta_0}, A^{*\beta_1})_{\theta, \beta} = A^{*\beta}, \quad \frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}, \quad 0 < \theta < 1.$$

- (iv) *Let $0 < q_0, q_1 < \infty$, $q_0 \neq q_1$, and $0 < r \leq \infty$. Then $f(x) = \sum c_n e^{inx} \in WB_{1, r}^{\frac{1}{q}-1}$ if and only if*

$$(5.9) \quad \{c_n\} \in (cop(q_0), cop(q_1))_{\theta, r}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

In particular, taking $r = q$, we arrive at (5.2).

- (v) *Similarly, for $1 < q_0, q_1 < \infty$, $q_0 \neq q_1$, and $0 < r \leq \infty$, (5.9) can be equivalently replaced by*

$$\{c_n\} \in (ces(q_0), ces(q_1))_{\theta, r}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

In particular, taking $r = q$, we arrive at (5.1).

6. Final remarks

1. Note that the classical Bernstein inequality $\|T_N^{(\alpha)}\|_p \leq N^\alpha \|T_N\|_p$ for trigonometric polynomials $T_N(x) = \sum_{|k| \leq N} c_k e^{ikx}$ has the usual form in the Wiener-Beurling spaces, namely,

$$\|T_N^{(\alpha)}\|_{WB_{p,q}^\alpha} \leq N^\alpha \|T_N\|_{WB_{p,q}^\alpha}.$$

2. Let us also discuss the Nikolskii inequality for trigonometric polynomials. Its analogue for the Wiener-Beurling spaces reads as follows: *If $0 < r \leq p \leq \infty$ and $\alpha + 1/r - \beta - 1/p \geq 0$, then*

$$(6.1) \quad \|T_N\|_{WB_{r,q}^\alpha} \leq N^{(\alpha+1/r)-(\beta+1/p)} \|T_N\|_{WB_{p,q}^\beta}.$$

Indeed, let $2^{m-1} \leq N < 2^m$ and $\gamma = \alpha + 1/r - 1/p$. Taking into account (2.1), we derive that

$$\begin{aligned} \|T_N\|_{WB_{r,q}^\alpha} &\lesssim \|T_N\|_{WB_{p,q}^\gamma} \\ &\lesssim \left(\sum_{k=0}^{m-1} 2^{\gamma q k} \left(\sum_{[2^{k-1}] \leq |n| < 2^k} |c_n|^p \right)^{q/p} + 2^{\gamma q m} \left(\sum_{[2^{m-1}] \leq |n| \leq N} |c_n|^p \right)^{q/p} \right)^{1/q} \\ &\lesssim N^{(\alpha+1/r)-(\beta+1/p)} \|T_N\|_{WB_{p,q}^\beta}. \end{aligned}$$

Note that if the first integrability parameters are the same but the second ones are different, then the Nikolskii inequality has a different form. Namely, *if $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q_1 < q \leq \infty$, then*

$$(6.2) \quad \|T_N\|_{WB_{p,q_1}^\alpha} \leq ([\ln N] + 1)^{\frac{1}{q} - \frac{1}{q_1}} \|T_N\|_{WB_{p,q}^\alpha}.$$

This follows from Hölder's inequality.

Thus, in view of (6.1) and (6.2), the Nikolskii inequalities in the Wiener-Beurling spaces are similar to those in the Lorentz spaces, cf. [NT].

3. Now we study the Fourier series with averages of Fourier coefficients. Given a series $f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}$, define

$$Hf(x) = \sum_{k \in \mathbb{Z}} c_k^h e^{ikx}, \quad c_k^h = \frac{1}{2|k|+1} \sum_{m=-|k|}^{|k|} |c_m|$$

and

$$Bf(x) = \sum_{k \in \mathbb{Z}} c_k^b e^{ikx}, \quad c_k^b = \sum_{n=|k|}^{\infty} \frac{|c_n| + |c_{-n}|}{2n} < \infty.$$

This topic was originated by Hardy [Ha], who proved that if $f \in L_p$, $1 < p < \infty$, then $Hf \in L_p$. See also [An, Be]; for the recent results see [DNT].

We will see that Wiener-Beurling spaces have the remarkable property that series with Hardy-Cesàro and Copson averages belong to Wiener-Beurling spaces if and only if the original series belong to similar spaces with different values of parameters.

THEOREM 6.1. *If $0 < p, q \leq \infty$, then*

$$(6.3) \quad Hf \in WB_{p,q}^\alpha \iff f \in WB_{1,q}^{\alpha + \frac{1}{p} - 1} \quad \text{if and only if} \quad \alpha < 1 - \frac{1}{p}$$

and

$$(6.4) \quad Bf \in WB_{p,q}^\alpha \iff f \in WB_{1,q}^{\alpha+\frac{1}{p}-1} \quad \text{if and only if} \quad \alpha + \frac{1}{p} > 0.$$

REMARK 6.1. In a particular case $p = 1$ and $\alpha = \frac{1}{q} - 1 < 0$ we arrive at the known results for Cesáro and Copson sequence spaces by Bennett [Ben, Theorem 20.31]:

$$\{c_m\} \in ces(q) \iff \{c_m^h\} \in ces(q) \iff \{c_m^b\} \in ces(q), \quad 1 < q < \infty$$

and

$$\{c_m\} \in cop(q) \iff \{c_m^h\} \in cop(q) \iff \{c_m^b\} \in cop(q), \quad 1 < q < \infty.$$

PROOF. To show (6.3), we observe that

$$\begin{aligned} \|Hf\|_{WB_{p,q}^\alpha} &\asymp \left(\sum_{k=0}^{\infty} 2^{k\alpha q} \left(\sum_{m=2^k}^{2^{k+1}} \left(\frac{1}{m} \sum_{|s|=0}^m |c_s| \right)^p \right)^{q/p} \right)^{1/q} \\ &\asymp \left(\sum_{k=0}^{\infty} 2^{kq(\alpha+\frac{1}{p}-1)} \left(\sum_{|s|=0}^{2^k} |c_s| \right)^q \right)^{1/q}. \end{aligned}$$

The latter is equivalent to $\|f\|_{WB_{1,q}^{\alpha+\frac{1}{p}-1}}$ for $\alpha + \frac{1}{p} - 1 < 0$, cf. (1.2). To verify the sharpness of this condition, assume that $\beta = \alpha + \frac{1}{p} - 1 \geq 0$ and define

$$c_k = \frac{1}{(|k| + 1)^{1+\beta+\varepsilon}}, \quad \varepsilon > 0.$$

It is routine to show that $\|f\|_{WB_{p,q}^\beta} < \infty$ and $\|Hf\|_{WB_{p,q}^\alpha} = \infty$ for $f = \sum c_k e^{ikx}$.

The proof of (6.4) is similar. The sharpness can be seen considering $f = \sum c_k e^{ikx}$ with

$$c_k = \frac{1}{(\ln(|k| + 1))^{\frac{1+\varepsilon}{q}} (|k| + 1)^{\alpha+\frac{1}{p}}}, \quad \alpha + \frac{1}{p} \leq 0, \quad 0 < \varepsilon \leq \frac{1}{q},$$

for which one has $\|f\|_{WB_{1,q}^{\alpha+\frac{1}{p}-1}} < \infty$ but $\|Bf\|_{WB_{p,q}^\alpha} = \infty$.

□

References

- [An] K. F. Andersen, On the representation of Fourier coefficients of certain classes of functions, Pacific J. Math. 100(2) (1982), 243–248.
- [ATT] M. Ash, S. Tikhonov, J. Tung, Wiener's positive Fourier coefficients theorem in variants of L^p spaces, Michigan Math. J., Vol. 59, 1 (2010), 143–152.
- [AM1] S.V. Astashkin, L. Maligranda, Interpolation of Cesáro sequence and function spaces, Studia Math. 215 (1) (2013) 39–69.
- [AM2] S.V. Astashkin, L. Maligranda, Structure of Cesáro function spaces: a survey. Function spaces X, Banach Center Publ., 102, Polish Acad. Sci. Inst. Math., Warsaw, 13–40, 2014.
- [BLT] E. S. Belinskii, E. R. Liflyand, R. M. Trigub, The Banach algebra A^* and its properties, J. Fourier Anal. Appl. 3 (1997), no. 2, 103–129.
- [Be] R. Bellman, A note on a theorem of Hardy on Fourier constants, Bull. Amer. Math. Soc. 50(10) (1944), 741–744.
- [BS] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press, 1988.
- [Ben] G. Bennett, Factorizing the Classical Inequalities, Memoirs of Amer. Math. Soc., 120, no 576 (1996), 1–130.
- [BL] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction. Springer, Berlin, 1976.

- [Be] A. Beurling, On the spectral synthesis of bounded functions, *Acta Math.* 81, 225–238 (1949).
- [Bo] D. Borwein, Linear functionals connected with strong Cesàro summability, *J. Lond. Math. Soc.* 40 (1965), 628–634.
- [DL] R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, 1993.
- [DNT] M. Dyachenko, E. Nursultanov, S. Tikhonov, Hardy–Littlewood and Pitt’s inequalities for Hausdorff operators, *Bull. Sci. Math.*, 147 (2018), 40–57.
- [Ha] G. H. Hardy, Notes on some points in the integral calculus, *Messenger of Math.* 58 (1928), 50–52.
- [He] J. H. Hedlund, Multipliers of H^p spaces, *J. Math. Mech.* 18 (1969), 1067–1074.
- [Ka] J.-P. Kahane, *Séries de Fourier Absolument Convergentes*, Springer-Verlag, 1970.
- [Ke] C. N. Kellogg, An extension of the Hausdorff–Young Theorem, *Michigan Math. J.*, 18 (1971), 121–127.
- [LL] A. Lerner, E. Liflyand, Interpolation properties of a scale of spaces, *Collect. Math.* 54 (2003), no. 2, 153–161.
- [LM] K. Lesnik, L. Maligranda, Interpolation of abstract Cesàro, Copson and Tandori spaces, *Indag. Math., New Ser.* 27, No. 3, 764–785 (2016).
- [Li] E. Liflyand, *Functions of Bounded Variation and their Fourier transforms*. Birkhäuser, 2019.
- [LST] E. Liflyand, S. Samko, R. Trigub, The Wiener algebra of absolutely convergent Fourier integrals: an overview, *Anal. Math. Phys.* 2 (2012), no. 1, 1–68.
- [Ni] S. M. Nikol’skii, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer, Berlin (1975).
- [NRT] E. Nursultanov, M. Ruzhansky, S. Tikhonov, Nikolskii inequality and Besov, Triebel–Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds, *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, Vol. XVI (2016), 981–1017.
- [NT] E. Nursultanov, S. Tikhonov, A sharp Remez inequality for trigonometric polynomials, *Constructive Approx.*, 38 (1) (2013), 101–132.
- [Pi] H. R. Pitt, Theorems on Fourier series and power series, *Duke Math. J.* 3 (1937), 747–755.
- [Ri] W. Ricker, Convolution operators in discrete Cesàro spaces, *Arch. Math.* 112, No. 1 (2019), 71–82.
- [Sl] W. T. Sledd, Some inequalities related to the Hausdorff–Young theorem, *Proc. Amer. Math. Soc.* 42 (1974), 535–540.
- [St] E. M. Stein, Interpolation of linear operators, *Trans. Amer. Math. Soc.* 83 (1956), 482–492.
- [SW] E. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*. Princeton, N.J., 1971.
- [Su] Q. Sun, Wiener’s lemma for localized integral operators, *Appl. Comput. Harmon. Anal.* 25 (2008), no. 2, 148–167.
- [Sz] O. Szasz, Fourier series and mean moduli of continuity, *Trans. Amer. Math. Soc.* 42(3), 366395 (1937).
- [Ti] S. Tikhonov, Trigonometric series with general monotone coefficients. *J. Math. Anal. Appl.* **326**, 721–735 (2007).
- [TB] R. Trigub, E. Bellinsky, *Fourier Analysis and Approximation of Functions*, Kluwer Academic Publishers, 2004.
- [Zy] A. Zygmund, *Trigonometric Series*, Vols. I, II, third ed., Cambridge Univ. Press, 2002.

E. NURSULTANOV, LOMONOSOV MOSCOW STATE UNIVERSITY (KAZAKH BRANCH) AND GUMILYOV EURASIAN NATIONAL UNIVERSITY, MUNATPASOVA 7, 010010 ASTANA KAZAKHSTAN
E-mail address: `er-nurs@yandex.ru`

S. TIKHONOV, ICREA AND CENTRE DE RECERCA MATEMÀTICA (CRM), E-08193, BELLATERRA, BARCELONA
E-mail address: `stikhonov@crm.cat`