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Title:	<i>Smoothness of functions versus smoothness of approximation processes</i>
Journal Information:	<i>Bulletin of Mathematical Sciences,</i>
Author(s):	Kolomoitsev Y.S., Tikhonov S.Y..
Volume, pages:	10 1, DOI:[10.1142/S1664360720300029]

## Smoothness of functions versus smoothness of approximation processes

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Received 15 June 2020

Revised 26 August 2020

Accepted 12 September 2020

Published 28 October 2020

Communicated by Ari Laptev

We provide a comprehensive study of interrelations between different measures of smoothness of functions on various domains and smoothness properties of approximation processes. Two general approaches to this problem have been developed: The first based on geometric properties of Banach spaces and the second on Littlewood–Paley and Hörmander-type multiplier theorems. In particular, we obtain new sharp inequalities for measures of smoothness given by the  $K$ -functionals or moduli of smoothness. As examples of approximation processes we consider best polynomial and spline approximations, Fourier multiplier operators on  $\mathbb{T}^d$ ,  $\mathbb{R}^d$ ,  $[-1, 1]$ , nonlinear wavelet approximation, etc.

**Keywords:** Measures of smoothness;  $K$ -functionals; best approximation; Jackson and Bernstein inequalities; Littlewood–Paley decomposition; Fourier multipliers.

Mathematics Subject Classification: Primary: 41A65, 41A63, 41A50, 41A17, 42B25; Secondary: 41A15, 42A45, 41A35, 41A25

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## 1. Introduction

The fundamental problem in approximation theory is to find for a complicated function  $f$  in a quasinormed space  $X$  a close-by, simple approximant  $P_n$  from a subset of  $X$  such that the error of approximation  $\|f - P_n\|_X$  can be controlled by a specific majorant. In many cases, this problem is solved completely and necessary and sufficient conditions are given in terms of smoothness properties of either the function  $f$  or approximants  $P_n$  of  $f$ .

We illustrate this by considering the well-known case of approximation of periodic functions by trigonometric polynomials on  $\mathbb{T} = [0, 2\pi]$ . If  $f \in L_p(\mathbb{T})$ ,  $1 \leq p \leq \infty$ , and  $0 < \alpha < r$ , for the best approximant  $T_n^*$  and the modulus of smoothness  $\omega_r(f, t)_p$ , the following conditions are equivalent:

$$(i_1) \quad \|f - T_n^*\|_p = \mathcal{O}(n^{-\alpha}),$$

$$(i_2) \quad \omega_r(f, t)_p = \mathcal{O}(t^\alpha),$$

$$(i_3) \quad \|(T_n^*)^{(r)}\|_p = \mathcal{O}(n^{r-\alpha}),$$

see [53, 5; 16, Chap. 7]; for functions on  $\mathbb{T}^d$  see [31]. Let us also mention earlier results by Salem and Zygmund [50], Zamansky [69], and Civin [6]. Similar results in the case of approximation by algebraic polynomials of functions on  $[-1, 1]$  can be found in [20, Chap. 8; 4].

Equivalence  $(i_1) \Leftrightarrow (i_2)$  easily follows from the classical Jackson and Bernstein approximation theorems, see, e.g. [16, Chap. 7], given by

$$E_n(f)_p \lesssim \omega_r(f, 1/n)_p \lesssim \frac{1}{n^r} \sum_{k=0}^n (k+1)^{r-1} E_k(f)_p, \quad 1 \leq p \leq \infty,$$

or their sharper versions for  $1 < p < \infty$ , see, e.g. [11]

$$\frac{1}{n^r} \left( \sum_{k=0}^n (k+1)^{r\tau-1} E_k(f)_p^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_r(f, 1/n)_p \lesssim \frac{1}{n^r} \left( \sum_{k=0}^n (k+1)^{r\theta-1} E_k(f)_p^\theta \right)^{\frac{1}{\theta}},$$

where  $E_n(f)_p$  is the error of the best approximation,  $\tau = \max(p, 2)$  and  $\theta = \min(p, 2)$ .

The equivalence  $(i_2) \Leftrightarrow (i_3)$  follows from the inequalities

$$n^{-r} \|(T_n^*)^{(r)}\|_p \lesssim \omega_r(f, 1/n)_p \lesssim \sum_{k=n}^{\infty} k^{-r-1} \|(T_k^*)^{(r)}\|_p, \quad 1 \leq p \leq \infty. \quad (1.1)$$

The left-hand side estimate is a corollary of the well-known Nikolskii–Stechkin inequality  $\|(T_n^*)^{(r)}\|_p \lesssim n^r \omega_r(T_n, 1/n)_p$ . The right-hand side estimate was proved in [70].

Jackson and Bernstein approximation theorems as well as the corresponding equivalence  $(i_1) \Leftrightarrow (i_2)$  are known to be true in various settings. Surprisingly enough the results involving the smoothness of approximation processes given in the strong form, i.e. similar to inequalities (1.1), or, even in the weak form, i.e. similar to

equivalence  $(i_2) \Leftrightarrow (i_3)$ , are much less known in the literature. It is clear that such results provide additional information on smoothness properties of approximants and, therefore, they are useful for applications. As an example, we mention that the smooth function spaces (Lipschitz, Sobolev, Besov) can be characterize in terms of smoothness of approximation processes.

The main goal of this paper is to present a thoughtful study of interrelations between smoothness properties of functions on various domains and smoothness properties of approximation processes. In particular, we extend inequalities (1.1) as follows: For  $f \in L_p(\mathbb{T})$ ,  $1 < p < \infty$

$$\left( \sum_{k=n+1}^{\infty} 2^{-kr\tau} \|(T_{2^k}^*)^{(r)}\|_p^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_r(f, 2^{-n})_p \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-kr\theta} \|(T_{2^k}^*)^{(r)}\|_p^\theta \right)^{\frac{1}{\theta}},$$

where  $T_{2^k}^*$  stands for the best approximants, partial sums of the Fourier series, de la Vallée Poussin means, Fejér means, etc.

In the general form, our main results state that for  $f \in X$

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f)\|_Y^\tau \right)^{\frac{1}{\tau}} \lesssim \Omega(f, 2^{-n\alpha}, X, Y) \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y^\theta \right)^{\frac{1}{\theta}}, \quad (1.2)$$

where the parameters  $\tau$  and  $\theta$  are related to geometry of the space  $X$ , and, in particular, for  $X = L_p$ ,  $0 < p \leq \infty$ , are given by

$$\tau = \begin{cases} \max(p, 2), & 1 < p < \infty, \\ \infty, & \text{otherwise,} \end{cases} \quad \theta = \begin{cases} \min(p, 2), & p < \infty, \\ 1, & p = \infty. \end{cases}$$

Here,  $Y$  is a smooth function space (Sobolev or Besov spaces),  $P_n(f)$  is a suitable (linear or nonlinear) approximation method, and  $\Omega(f, 2^{-n\alpha}, L_p, Y)$  is some measure of smoothness related to the spaces  $L_p$  and  $Y$ . It is worth mentioning that the classical modulus of smoothness is equivalent to the  $K$ -functional for a couple  $(L_p, W_p^r)$ , namely,  $K(f, t; L_p(\mathbb{T}), W_p^r(\mathbb{T}))_p \asymp \omega_r(f, t)$ , see, e.g. [16, p. 177]. Therefore, as a measure of smoothness it is natural to consider the  $K$ -functional  $K(f, 2^{-n\alpha}, L_p, Y)$  in the case  $1 \leq p \leq \infty$  and either an appropriate modulus of smoothness or a realization of the  $K$ -functional for any  $0 < p \leq \infty$ .

The rest of the paper is organized as follows. In Sec. 2, we consider general (Banach) spaces and investigate smoothness properties of the best approximants. Using geometric properties of  $X$ , we obtain sharp inequalities (1.2) for appropriate  $\theta$  and  $\tau$ . In more detail, if the space  $X$  is  $\theta$ -uniformly smooth and  $\tau$ -uniformly convex, then (1.2) holds.

Section 3 studies the smoothness properties of Fourier means of functions from  $L_{p,w}(\mathcal{D})$ . Our approach is based on Littlewood–Paley-type inequalities and Hörmander’s-type multiplier theorems. In particular, inequalities (1.2) are obtained for a wide class of Fourier multiplier operators, which includes partial sums of

Fourier series, de la Valée Poussin means, Fejér means, Riesz means, etc. Sharpness of the parameters in (1.2) will be discussed in Sec. 9.

In Sec. 4, we deal with general approximation processes  $\{P_{2^n}(f)\}$  and abstract measures of smoothness  $\Omega(f, t)_X$  in the metric space  $X$ . In particular, we treat the case of  $X = L_p$  for  $0 < p < 1$ . We prove that

$$\left\| \{ \Omega(P_{2^k}(f), 2^{-k})_X \}_{k \geq n} \right\|_{\ell_\infty} \lesssim \Omega(f, 2^{-n})_X \lesssim \left\| \{ \Omega(P_{2^k}(f), 2^{-k})_X \}_{k \geq n} \right\|_{\ell_\lambda},$$

where  $\lambda$  is a parameter related to the geometry of  $X$ . Let us emphasize that this result holds under very mild conditions on the approximants  $P_{2^n}(f)$ . Moreover, these inequalities easily imply the results similar to those given in the equivalence  $(i_2) \Leftrightarrow (i_3)$ .

In Secs. 5–8, we illustrate our main results obtained in Secs. 4–3 by several important examples. In particular, in Sec. 5, we investigate relationship between smoothness of periodic functions on  $\mathbb{T}^d$  and smoothness of the best trigonometric approximants, various Fourier means, and smoothness of interpolation operators. Moreover, we consider approximations in Hardy spaces  $H_p(\mathbb{D})$ ,  $0 < p \leq 1$ , and smooth (Lipschitz, Sobolev) spaces. Section 6 is devoted to approximation processes on  $\mathbb{R}^d$ . In this case, we study smoothness properties of band-limited functions that approximate functions from  $L_p(\mathbb{R}^d)$ .

In Sec. 7, we deal with functions on  $L_{p,w}[-1, 1]$ , where  $w$  is the Jacobi weight. In particular, we study smoothness properties of algebraic polynomials and splines of the best approximation and consider some Fourier means related to Fourier–Jacobi series.

In Sec. 8, we show that the results of Secs. 4 and 2 can be applied to study smoothness properties of nonlinear approximation processes. As examples, we treat nonlinear wavelet approximation and splines with free knots.

Finally, in Sec. 9, we study the optimality of inequalities (1.2), showing that the parameters  $\tau$  and  $\theta$  cannot be improved in general. Moreover, we define function classes such that the right-hand side and the left-hand side sums in (1.2) (with appropriate values of  $\tau$  and  $\theta$ ) are equivalent to the corresponding modulus of smoothness.

Throughout the paper, we use the notation  $F \lesssim G$ , with  $F, G \geq 0$ , for the estimate  $F \leq CG$ , where  $C$  is a positive constant independent of the essential variables in  $F$  and  $G$  (usually,  $f$ ,  $\delta$ , and  $n$ ). If  $F \lesssim G$  and  $G \lesssim F$  simultaneously, we write  $F \asymp G$  and say that  $F$  is equivalent to  $G$ .

## 2. $K$ -Functionals and Smoothness of Best Approximants

Let  $(X, Y)$  be a couple of normed function spaces with (semi-)norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , respectively, and  $Y \subset X$ . The Peetre  $K$ -functional for this couple is given by

$$K(f, t; X, Y) = \inf \{ \|f - g\|_X + t\|g\|_Y : g \in Y \} \quad (2.1)$$

for any  $f \in X$  and  $t > 0$ .

Let  $\{G_n\}_{n=1}^\infty$  be a family of subsets of  $Y$  such that

- (i)  $0 \in G_1$ ,
- (ii)  $G_n \subset G_{n+1}$ ,
- (iii)  $G_n = -G_n$ ,
- (iv) the closure of  $\{G_n\}_{n=1}^\infty$  in  $X$  is  $X$ .

The best approximation of  $f \in X$  by elements from  $G_n$  is given by

$$E_n(f)_X = \inf\{\|f - g\|_X : g \in G_n\}.$$

Moreover, we suppose that the family  $\{G_n\}$  is such that Jackson- and Bernstein-type inequalities are valid. Namely, there are positive constants  $c_1$ ,  $c_2$ , and  $\alpha$  such that for any  $n \in \mathbb{N}$  we have

$$E_n(f)_X \leq c_1 n^{-\alpha} \|f\|_Y, \quad f \in Y, \quad (2.2)$$

$$\|g_1 - g_2\|_Y \leq c_2 n^\alpha \|g_1 - g_2\|_X, \quad g_1, g_2 \in G_n. \quad (2.3)$$

The latter condition implies that, for every  $g \in G_n$ ,

$$\|g\|_Y \leq c_2 n^\alpha \|g\|_X, \quad g \in G_n. \quad (2.4)$$

Clearly, if  $G_n$  is a linear space, then (2.3) and (2.4) are equivalent.

It is also plain to see that the Jackson-type inequality (2.2) implies the direct approximation theorem given by

$$E_n(f)_X \lesssim K(f, n^{-\alpha}; X, Y), \quad f \in X, \quad n \in \mathbb{N}, \quad (2.5)$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

Our main goal in this section is to obtain inequalities for  $K(f, t; X, Y)$  in terms of the best approximation of  $f$  by elements from  $G_n$ .

In what follows, we denote by  $P_n(f)$  an element of the best approximation of  $f \in X$  by functions from  $G_n$  (assuming it exists), i.e.

$$E_n(f)_X = \|f - P_n(f)\|_X \leq \|f - g\|_X \quad \text{for any } g \in G_n.$$

An element of the near best approximation of  $f \in X$  by functions from  $G_n$  is denoted by  $Q_n(f)$ , i.e., there exists a constant  $c > 0$  independent of  $f$  and  $n$  such that

$$\|f - Q_n(f)\|_X \leq c E_n(f)_X.$$

One of our main tools is the realization of  $K$ -functional given by

$$R(f, n^{-\alpha}; X, G_n) = \inf\{\|f - g\|_X + n^{-\alpha} \|g\|_Y : g \in G_n\}. \quad (2.6)$$

Clearly,

$$K(f, n^{-\alpha}; X, Y) \leq R(f, n^{-\alpha}; X, G_n), \quad f \in X, \quad n \in \mathbb{N},$$

but for applications it is important to know when

$$K(f, n^{-\alpha}; X, Y) \asymp R(f, n^{-\alpha}; X, G_n).$$

The next proposition describes such cases.

**Proposition 2.1.** *Let inequalities (2.2) and (2.3) hold. Then the following conditions are equivalent:*

(i) *for every  $f \in X$  and  $n \in \mathbb{N}$ ,*

$$R(f, n^{-\alpha}; X, G_n) \lesssim K(f, n^{-\alpha}; X, Y), \quad (2.7)$$

(ii) *for every  $f \in X$  and  $n \in \mathbb{N}$ ,*

$$\|f - Q_n(f)\|_X + n^{-\alpha} \|Q_n(f)\|_Y \lesssim K(f, n^{-\alpha}; X, Y),$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

Even though Proposition 2.1 in this form was not mentioned in [26], its proof easily follows from [26, Theorem 2.2] taking into account that by (2.5), for the near best approximation  $Q_n(f)$ , we have

$$\|f - Q_n(f)\|_X \lesssim E_n(f)_X \lesssim K(f, n^{-\alpha}; X, Y)$$

for any  $f \in X$  and  $n \in \mathbb{N}$ .

**Remark 2.1.** It follows from [26, Theorem 2.2] that under conditions of Proposition 2.1, assertions (i) and (ii) are equivalent to the following conditions:

(iii) *for every  $f \in X$  and  $n \in \mathbb{N}$ ,*

$$\|P_n(f)\|_Y \lesssim n^\alpha K(f, n^{-\alpha}; X, Y),$$

(iv) *for every  $g \in G_n$  and  $n \in \mathbb{N}$ ,*

$$\|g\|_Y \lesssim n^\alpha K(g, n^{-\alpha}; X, Y).$$

The next lemma is a crucial result of this section.

**Lemma 2.1.** *Let  $f \in X$  and inequalities (2.2), (2.3), and (2.7) hold.*

(A) *Suppose that there exist positive constants  $A$  and  $\tau$  such that*

$$\|f - P_n(f)\|_X^\tau \leq \|f - g\|_X^\tau - A \|g - P_n(f)\|_X^\tau, \quad (2.8)$$

*for any  $g \in G_n$ . Then, for any  $n \in \mathbb{N}$ , we have*

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f)\|_Y^\tau \right)^{\frac{1}{\tau}} \lesssim K(f, 2^{-n\alpha}; X, Y), \quad (2.9)$$

*where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .*

(B) *Suppose that there exist positive constants  $B$  and  $\theta$  such that*

$$\|f - g\|_X^\theta \leq \|f - P_n(f)\|_X^\theta + B \|g - P_n(f)\|_X^\theta \quad (2.10)$$

*for all  $g \in G_n$ . Then, for any  $n \in \mathbb{N}$ , we have*

$$K(f, 2^{-n\alpha}; X, Y) \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y^\theta \right)^{\frac{1}{\theta}}, \quad (2.11)$$

*where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .*

**Proof.** (A) Using the representation

$$P_{2^k}(f) = \sum_{l=n+1}^k (P_{2^l}(f) - P_{2^{l-1}}(f)) + P_{2^n}(f),$$

we derive

$$\begin{aligned} & \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f)\|_Y^\tau \\ & \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\| \sum_{l=n+1}^k P_{2^l}(f) - P_{2^{l-1}}(f) \right\|_Y^\tau + 2^{-n\alpha\tau} \|P_{2^n}(f)\|_Y^\tau \\ & \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left( \sum_{l=n+1}^k \|P_{2^l}(f) - P_{2^{l-1}}(f)\|_Y \right)^\tau + 2^{-n\alpha\tau} \|P_{2^n}(f)\|_Y^\tau \\ & =: L + 2^{-n\alpha\tau} \|P_{2^n}(f)\|_Y^\tau. \end{aligned} \quad (2.12)$$

Next, by Hardy's inequality

$$\sum_{k=n}^{\infty} 2^{-k\alpha} \left( \sum_{s=n}^k A_s \right)^q \asymp \sum_{k=n}^{\infty} 2^{-\alpha k} A_k^q, \quad A_k \geq 0, \quad q > 0, \quad (2.13)$$

and Bernstein's inequality (2.3), we obtain

$$L \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f) - P_{2^{k-1}}(f)\|_Y^\tau \lesssim \sum_{k=n+1}^{\infty} \|P_{2^k}(f) - P_{2^{k-1}}(f)\|_X^\tau. \quad (2.14)$$

Using (2.8) with  $g = P_{2^{k-1}}(f)$  and  $n = 2^k$ , we derive

$$\|P_{2^k}(f) - P_{2^{k-1}}(f)\|_X^\tau \leq \frac{1}{A} (\|f - P_{2^{k-1}}(f)\|_X^\tau - \|f - P_{2^k}(f)\|_X^\tau). \quad (2.15)$$

Thus, combining (2.14) and (2.15) and taking into account that  $E_{2^k}(f)_X = \|f - P_{2^k}(f)\|_X \rightarrow 0$  as  $k \rightarrow \infty$ , we have

$$L \lesssim \|f - P_{2^n}(f)\|_X^\tau. \quad (2.16)$$

Finally, combining (2.12) and (2.16) and using Proposition 2.1, we obtain (2.9).

(B) By the definition of the  $K$ -functional, we have

$$K(f, 2^{-n\alpha}; X, Y) \leq \|f - P_{2^{n+1}}(f)\|_X + 2^{-n\alpha} \|P_{2^{n+1}}(f)\|_Y.$$

Thus, to prove (2.11) it is enough to show that

$$\|f - P_{2^{n+1}}(f)\|_X^\theta \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y^\theta. \quad (2.17)$$



Since  $E_{2^k}(f)_X \rightarrow 0$  as  $k \rightarrow \infty$ , we derive

$$\|f - P_{2^{n+1}}(f)\|_X^\theta = \sum_{k=n+2}^{\infty} (\|f - P_{2^{k-1}}(f)\|_X^\theta - \|f - P_{2^k}(f)\|_X^\theta). \quad (2.18)$$

Next, by the definition of the best approximation

$$\|f - P_{2^{k-1}}(f)\|_X \leq \|f - P_{2^{k-1}}(P_{2^k}(f))\|_X.$$

Then, inequality (2.10) with  $n = 2^k$  and  $g = P_{2^{k-1}}(P_{2^k}(f))$  and the Jackson inequality (2.2) imply

$$\begin{aligned} \|f - P_{2^{k-1}}(f)\|_X^\theta - \|f - P_{2^k}(f)\|_X^\theta &\leq \|f - P_{2^{k-1}}(P_{2^k}(f))\|_X^\theta - \|f - P_{2^k}(f)\|_X^\theta \\ &\leq B\|P_{2^k}(f) - P_{2^{k-1}}(P_{2^k}(f))\|_X^\theta \\ &\lesssim 2^{-(k-1)\alpha\theta} \|P_{2^k}(f)\|_Y^\theta. \end{aligned} \quad (2.19)$$

Thus, (2.18) and (2.19) yield (2.17), completing the proof.  $\square$

**Remark 2.2.** (i) It follows from the proof of Lemma 2.1 that conditions (2.8) and (2.10) can be replaced by the following weaker conditions:

$$\|f - P_{2n}(f)\|_X^\tau \leq \|f - P_n(f)\|_X^\tau - A\|P_n(f) - P_{2n}(f)\|_X^\tau$$

and

$$\|f - P_n(P_{2n}(f))\|_X^\theta \leq \|f - P_{2n}(f)\|_X^\theta + B\|P_n(P_{2n}(f)) - P_{2n}(f)\|_X^\theta,$$

respectively.

- (ii) Note that by triangle inequality, estimate (2.10) is always valid with  $\theta = B = 1$ .
- (iii) Lemma 2.1 remains valid without assumption (2.7) with the realization  $R(f, 2^{-n\alpha}; X, Y)$  in place of the  $K$ -functional  $K(f, 2^{-n\alpha}; X, Y)$  in (2.9) and (2.11).

In what follows, we need some terminology from the theory of Banach spaces (see, e.g. [13, Chap. IV]). Let  $X$  be a Banach space with the norm  $\|\cdot\| = \|\cdot\|_X$ . The moduli of convexity and smoothness of  $X$  are defined, respectively, by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1 \text{ and } \|x-y\| = \varepsilon \right\}, \quad 0 \leq \varepsilon \leq 2,$$

and

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = t \right\}, \quad t > 0.$$

Let  $\tau, \theta > 1$  be real numbers. Then  $X$  is said to be  $\tau$ -uniformly convex (respectively,  $\theta$ -uniformly smooth) if there exists a constant  $c > 0$  such that  $\delta_X(\varepsilon) \geq c\varepsilon^\tau$  (respectively,  $\rho_X(t) \leq ct^\theta$ ). Note that by the Day–Nordlander theorem we always have  $\theta \leq 2 \leq \tau$ , see, e.g. [30] or [44].

**Theorem 2.1.** Let  $G_n$  be convex,  $f \in X$ , and inequalities (2.2), (2.3), and (2.7) hold.

(A) Suppose  $X$  is  $\tau$ -uniformly convex for some  $\tau > 1$ . Then, for any  $n \in \mathbb{N}$ , we have

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f)\|_Y^\tau \right)^{\frac{1}{\tau}} \lesssim K(f, 2^{-n\alpha}; X, Y), \quad (2.20)$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

(B) Suppose  $X$  is  $\theta$ -uniformly smooth for some  $\theta > 1$ . Then, for any  $n \in \mathbb{N}$ , we have

$$K(f, 2^{-n\alpha}; X, Y) \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y^\theta \right)^{\frac{1}{\theta}}, \quad (2.21)$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

**Proof.** (A) Since  $X$  is  $\tau$ -uniformly convex, then there exists a constant  $c > 0$  such that for all  $x, y \in X$  and  $t \in [0, 1]$

$$\|tx + (1-t)y\|^\tau \leq t\|x\|^\tau + (1-t)\|y\|^\tau W_\tau(t)c\|x-y\|^\tau, \quad (2.22)$$

where  $\|\cdot\| = \|\cdot\|_X$  and  $W_\tau(t) = t(1-t)^\tau + t^\tau(1-t)$  (see the proof of Theorem 1 in [68], see also [49, 52]). Consider the following Gateaux derivative at  $y$  in the direction  $x - y$ :

$$g_\tau(y, x - y) = \lim_{t \rightarrow +0} \frac{\|y - t(x - y)\|^\tau - \|y\|^\tau}{t}.$$

Dividing both sides of (2.22) by  $t \in (0, 1)$  and taking limit as  $t \rightarrow +0$ , we get

$$g_\tau(y, x - y) \leq \|x\|^\tau - \|y\|^\tau - c\|x - y\|^\tau.$$

Now, let  $g \in G_n$ . Replacing  $x$  by  $f - g$  and  $y$  by  $f - P_n(f)$ , we have that

$$g_\tau(f - P_n(f), P_n(f) - g) \leq \|f - g\|^\tau - \|f - P_n(f)\|^\tau - c\|P_n(f) - g\|^\tau.$$

By the Kolmogorov criterion, see, e.g. [51, p. 90], we have  $g_\tau(f - P_n(f), P_n(f) - g) = 0$ , which implies (2.8). Thus, using Lemma 2.1, we get (2.20).

(B) The proof of (2.21) is similar. We only note that by [68, Theorem 1'],  $X$  is  $\theta$ -uniformly smooth if and only if there exists a constant  $d > 0$  such that

$$\|tx + (1-t)y\|^\theta \geq t\|x\|^\theta + (1-t)\|y\|^\theta W_\theta(t)d\|x - y\|^\theta. \quad (2.23)$$

Then, as above, we derive

$$g_\tau(f - P_n(f), P_n(f) - g) \geq \|f - g\|^\theta - \|f - P_n(f)\|^\theta - c\|P_n(f) - g\|^\theta$$

and apply the Kolmogorov criterion. Lemma 2.1 completes the proof.  $\square$

Let us give two important examples of Banach space  $X$  to illustrate Theorem 2.1, namely, Lebesgue and Orlicz spaces.

**Proposition 2.2** (see [39, p. 63]). *Let  $X$  be an abstract  $L_p$  space with  $1 < p < \infty$ , i.e. let  $X$  be a Banach lattice for which*

$$\|x + y\|^p = \|x\|^p + \|y\|^p,$$

*whenever  $x, y \in X$  and  $\min(x, y) = 0$ . Then there exists a constant  $c > 0$  such that  $\delta_X(\varepsilon) \geq c\varepsilon^{\max(2,p)}$  for all  $0 \leq \varepsilon \leq 2$  and  $\rho_X(t) \leq ct^{\min(2,p)}$  for all  $t > 0$ .*

Making use of Theorem 2.1 and Proposition 2.2, we obtain the following result.

**Theorem 2.2.** *Let inequalities (2.2), (2.3), and (2.7) be valid for  $X = L_p$ ,  $1 < p < \infty$ , and let  $G_n$  be convex. Then, for any  $f \in L_p$  and  $n \in \mathbb{N}$ , we have*

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f)\|_Y^\tau \right)^{\frac{1}{\tau}} \lesssim K(f, 2^{-n\alpha}; L_p, Y), \quad \tau = \max(2, p),$$

and

$$K(f, 2^{-n\alpha}; L_p, Y) \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y^\theta \right)^{\frac{1}{\theta}}, \quad \theta = \min(2, p),$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

In Sec. 9, we will see that the parameters  $\tau$  and  $\theta$  in Theorem 2.2 are optimal.

**Corollary 2.1.** *Let inequalities (2.2), (2.3), and (2.7) be valid for  $X = L_p$ ,  $1 < p < \infty$ , and let  $G_n$  be convex. Then, for any  $f \in L_p$ , the following assertions are equivalent:*

(i) *for any  $n \in \mathbb{N}$*

$$K(f, 2^{-n\alpha}; L_p, Y) \asymp 2^{-n\alpha\theta} \|P_{2^n}(f)\|_Y,$$

*where the constants in  $\asymp$  are independent of  $f$  and  $n$ ,*

(ii) *for any  $n \in \mathbb{N}$*

$$\sum_{k=n}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y \lesssim 2^{-n\alpha\theta} \|P_{2^n}(f)\|_Y,$$

*where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .*

**Proof.** The proof easily follows from Theorem 2.2 and (4.14). □

Finally, we consider Orlicz spaces. Recall that the Orlicz function  $M(t)$  on  $[0, \infty)$  is an increasing convex function satisfying  $M(0) = 0$ . We assume that  $M$  satisfies  $\Delta_2$ -condition, that is,  $M(2t) \leq cM(t)$  for all  $t > 0$ . The Orlicz class of functions

$X = X_M$  on some domain  $\mathcal{D}$  with a positive measure  $d\mu(x)$  is the class of functions  $f$ , for which

$$\int_{\mathcal{D}} M(|f(x)|) d\mu(x) < \infty, \quad (2.24)$$

and the (Luxemburg) norm is

$$\|f\|_{X_M} = \inf \left\{ \sigma > 0 : \int_{\mathcal{D}} M(|f(x)|/\sigma) d\mu(x) \leq 1 \right\}. \quad (2.25)$$

**Proposition 2.3.** (A) Suppose that  $M(u)$  is an Orlicz function such that  $M(u^{1/\tau})$  is concave for some  $\tau$ ,  $2 \leq \tau < \infty$ , and  $M(lt) \leq \frac{1}{2}M(t)$  for some  $l < 1$ . Then there exists an Orlicz function  $N(u)$  such that  $C^{-1}N(u) \leq M(u) \leq CN(u)$  and  $\delta_{X_N}(\varepsilon) \geq c\varepsilon^\tau$  with the norm of the space  $X_N$  given by

$$\|f\|_{X_N} = \inf \left\{ \sigma > 0 : \int_{\mathcal{D}} N(|f(x)|/\sigma) d\mu(x) \leq 1 \right\}. \quad (2.26)$$

(B) Suppose that  $M(u)$  is an Orlicz function such that  $M(u^{1/\theta})$  is convex for some  $\theta$ ,  $1 < \theta \leq 2$ . Then there exists an Orlicz function  $N(u)$  such that  $C^{-1}N(u) \leq M(u) \leq CN(u)$  and  $\rho_{X_N}(t) \leq ct^\theta$  with the norm of the space  $X_N$  given by (2.26).

**Proof.** The proof of (B) can be found in [19, Lemma 2.2]. Assertion (A) can be proved similarly employing [43, Theorem 1].  $\square$

Using Theorem 2.1 and Proposition 2.3, we obtain the following result.

**Theorem 2.3.** Let inequalities (2.2), (2.3), and (2.7) be valid for the Orlicz space  $X = X_M$  defined by (2.24) and (2.25), and let  $G_n$  be convex.

(A) Suppose that the function  $M$  and the parameter  $\tau$  are the same as in Proposition 2.3(A). Then, for any  $f \in X$  and  $n \in \mathbb{N}$ , we have

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f)\|_Y^\tau \right)^{\frac{1}{\tau}} \lesssim K(f, 2^{-n\alpha}; X, Y),$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

(B) Suppose that the function  $M$  and the parameter  $\theta$  are the same as in Proposition 2.3(B). Then, for any  $f \in X$  and  $n \in \mathbb{N}$ , we have

$$K(f, 2^{-n\alpha}; X, Y) \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y^\theta \right)^{\frac{1}{\theta}},$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

### 3. Smoothness of Fourier Multiplier Operators

#### 3.1. Realization and Littlewood–Paley-type inequality

First, we introduce basic notations and collect auxiliary results. We follow the discussion in the paper [11].

Let  $L_{p,w}(\mathcal{D})$  be a weighted  $L_p$  space with the norm

$$\|f\|_{L_{p,w}(\mathcal{D})} = \|f\|_{p,w} = \left( \int_{\mathcal{D}} |f|^p w \right)^{\frac{1}{p}}.$$

We assume that  $Q(D)$  is a self-adjoint operator in  $L_{2,w}(\mathcal{D})$ , that is,  $\langle Q(D)f, g \rangle = \langle f, Q(D)g \rangle$  whenever  $Q(D)f, Q(D)g \in L_{2,w}(\mathcal{D})$ , where, as usual,  $\langle f, g \rangle = \int_{\mathcal{D}} fgw$ . We further assume that the eigenvalues  $(-1)^j \lambda_k$ ,  $j$  fixed, of  $Q(D)$  satisfy  $0 \leq \lambda_0 < \lambda_k < \lambda_{k+1}$ ,  $G_k = \{\varphi: Q(D)\varphi = \lambda_k \varphi\}$  is finite-dimensional,  $G_k \subset L_{p,w}(\mathcal{D})$  for  $1 \leq p \leq \infty$ , and  $\text{span } \cup_k G_k$  is dense in  $L_{p,w}(\mathcal{D})$  for  $1 \leq p < \infty$ . Examples of such operators and matching spaces are:  $-(\frac{d}{dx})^2$  for  $L_p(\mathbb{T})$ ;  $-\frac{d}{dx}(1-x^2)\frac{d}{dx}$  for  $L_p[-1, 1]$ ;  $-\Delta + |x|^2$ , where  $\Delta$  is the Laplacian for  $L_p(\mathbb{R}^d)$ ; and  $-w_{\alpha,\beta}^{-1}\frac{d}{dx}w_{\alpha,\beta}(1-x^2)\frac{d}{dx}$  for  $L_{p,w_{\alpha,\beta}}[-1, 1]$ , where  $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$  with  $\alpha, \beta > -1$ .

We define

$$A_k f = \sum_{\ell=1}^{d_k} \langle f, \psi_{k,\ell} \rangle \psi_{k,\ell},$$

where  $d_k$  is the dimension of  $G_k$  and  $\{\psi_{k,\ell}\}$  an orthonormal basis of  $G_k$  in  $L_{2,w}(\mathcal{D})$ .

For  $f \in L_{p,w}(\mathcal{D})$ ,  $f \sim \sum_{k=0}^{\infty} A_k f$ , we define  $Q(D)^\gamma$  by

$$Q(D)^\gamma f \sim \sum_k \lambda_k^\gamma A_k f$$

and we say that  $Q(D)^\gamma f \in L_{p,w}(\mathcal{D})$  if there exists  $g \in L_{p,w}(\mathcal{D})$  such that  $\lambda_k^\gamma A_k f = A_k g$ .

In what follows, we suppose that  $\lambda_k \asymp k^\sigma$  for some positive  $\sigma > 0$ . Note that in the example above  $\sigma = 2$  except for the eigenvalues of  $-\Delta + |x|^2$  where  $\sigma = 1$  (see [17]).

As usual, we define the  $K$ -functional  $K_\gamma(f, Q(D), t^{\sigma\gamma})_{p,w}$  by

$$K_\gamma(f, Q(D), t^{\sigma\gamma})_{p,w} := \inf_{Q(D)^\gamma g \in L_{p,w}(\mathcal{D})} \{\|f - g\|_{L_{p,w}(\mathcal{D})} + t^{\sigma\gamma} \|Q(D)^\gamma g\|_{L_{p,w}(\mathcal{D})}\}. \quad (3.1)$$

In this section, we consider approximation processes, which are defined by means of the Fourier multiplier operator  $T_\mu$  given by

$$T_\mu f \sim \sum_{k=0}^{\infty} \mu_k A_k f \quad \text{for } f \sim \sum_{k=0}^{\infty} A_k f.$$

We will use the following assumption related to a Hörmander–Mikhlin-type theorem.

**Assumption 3.1.** *For some  $\ell_0 \geq 0$ , the condition*

$$|\Delta^\ell \mu_k| \leq A(k+1)^{-\ell} \quad \text{for } 0 \leq \ell \leq \ell_0, \quad (3.2)$$

where

$$\Delta^0 \mu_k = \mu_k, \quad \Delta \mu_k = \mu_{k+1} - \mu_k \quad \text{and} \quad \Delta^\ell \mu_k = \Delta(\Delta^{\ell-1} \mu_k),$$

implies

$$\|T_\mu f\|_{L_{p,w}(\mathcal{D})} \leq C(A, p, w, \{G_k\}) \|f\|_{L_{p,w}(\mathcal{D})}, \quad 1 < p < \infty.$$

It is clear that under Assumption 3.1, the de la Vallée Poussin-type operator

$$\eta_N f := \sum_{k=0}^{\infty} \eta\left(\frac{k}{N}\right) A_k, \quad f \sim \sum_{k=0}^{\infty} A_k f,$$

satisfies

$$\|\eta_N f\|_{p,w} \leq A \|f\|_{p,w}. \quad (3.3)$$

Here and in what follows, we assume that

$$\eta(\xi) \in C^\infty[0, \infty), \quad \eta(\xi) = \begin{cases} 1, & \xi \leq 1/2, \\ 0, & \xi \geq 1. \end{cases}$$

Moreover, the following realization result (see [17, Theorem 7.1]) holds:

$$K_\gamma(f, Q(D), \lambda_N^{-\gamma})_{p,w} \asymp \|f - \eta_N f\|_{p,w} + \lambda_N^{-\gamma} \|Q(D)^\gamma \eta_N f\|_{p,w}, \quad (3.4)$$

where the constants in  $\asymp$  are independent of  $f$  and  $N$ .

Denote

$$\theta_0(f) := \eta_1 f \quad \text{and} \quad \theta_j(f) := \eta_{2^j} f - \eta_{2^{j-1}} f \quad \text{for } j > 0. \quad (3.5)$$

The following Littlewood–Paley-type theorem plays a crucial role in our further study.

**Theorem 3.1** (see [10, Theorem 2.1; 11, Theorem 3.1]). *Let  $f \in L_{p,w}(\mathcal{D})$ ,  $1 < p < \infty$ , and Assumption 3.1 be satisfied, then*

$$\left\| \left\{ \sum_{j=0}^{\infty} (\theta_j(f))^2 \right\}^{1/2} \right\|_{L_{p,w}(\mathcal{D})} \asymp \|f\|_{L_{p,w}(\mathcal{D})}.$$

If in addition,  $\gamma > 0$ , then

$$\left\| \left\{ \sum_{j=1}^{\infty} (2^{j\gamma} \theta_j(f))^2 \right\}^{1/2} \right\|_{L_{p,w}(\mathcal{D})} \asymp \|Q(D)^\gamma f\|_{L_{p,w}(\mathcal{D})}. \quad (3.6)$$

In the above relations, the constants in  $\asymp$  are independent of  $f$  and  $n$ .

### 3.2. Smoothness of the de la Vallée Poussin means in $L_{p,w}$

**Theorem 3.2.** Let  $f \in L_{p,w}(\mathcal{D})$ ,  $1 < p < \infty$ ,  $\gamma > 0$ ,  $\tau = \max(2, p)$ ,  $\theta = \min(2, p)$ ,  $n \in \mathbb{N}$ , and Assumption 3.1 hold. Then

$$\left( \sum_{k=n+1}^{\infty} 2^{-\sigma\gamma\tau k} \|Q(D)^{\gamma} \eta_{2^k} f\|_{p,w}^{\tau} \right)^{\frac{1}{\tau}} \lesssim K_{\gamma}(f, Q(D), 2^{-n\gamma\sigma})_{p,w} \quad (3.7)$$

and

$$K_{\gamma}(f, Q(D), 2^{-n\gamma\sigma})_{p,w} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-\sigma\gamma\theta k} \|Q(D)^{\gamma} \eta_{2^k} f\|_{p,w}^{\theta} \right)^{\frac{1}{\theta}}, \quad (3.8)$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

**Proof.** Denote  $\alpha = \sigma\gamma$  and

$$I^{\tau} = \sum_{k=n+1}^{\infty} 2^{-\alpha\tau k} \|Q(D)^{\gamma} \eta_{2^k} f\|_{p,w}^{\tau}.$$

Then

$$\begin{aligned} I^{\tau} &\lesssim \sum_{k=n+1}^{\infty} 2^{-\alpha\tau k} \|Q(D)^{\gamma} (\eta_{2^k} f - \eta_{2^n} f)\|_{p,w}^{\tau} + 2^{-n\alpha\tau} \|Q(D)^{\gamma} \eta_{2^n} f\|_{p,w}^{\tau} \\ &= J + 2^{-n\alpha\tau} \|Q(D)^{\gamma} \eta_{2^n} f\|_{p,w}^{\tau}. \end{aligned} \quad (3.9)$$

By (3.6), we have

$$\begin{aligned} J &\lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ \int_{\mathcal{D}} \left( \sum_{j=1}^{\infty} 2^{2\alpha j} (\theta_j (\eta_{2^k} f - \eta_{2^n} f))^2 \right)^{\frac{p}{2}} w \right\}^{\frac{\tau}{p}} \\ &= \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ \int_{\mathcal{D}} \left( \sum_{j=n}^{k+1} 2^{2\alpha j} (\theta_j (\eta_{2^k} f - \eta_{2^n} f))^2 \right)^{\frac{p}{2}} w \right\}^{\frac{\tau}{p}} \\ &\lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ 2^{\alpha n} \|\theta_n (\eta_{2^k} f - \eta_{2^n} f)\|_{p,w} + 2^{\alpha(n+1)} \|\theta_{n+1} (\eta_{2^k} f - \eta_{2^n} f)\|_{p,w} \right\}^{\tau} \\ &\quad + \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ \int_{\mathcal{D}} \left( \sum_{j=n+2}^{k-1} 2^{2j\alpha} \theta_j(f)^2 \right)^{\frac{p}{2}} w \right\}^{\frac{\tau}{p}} \\ &\quad + \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ 2^{k\alpha} \|\theta_k (\eta_{2^k} f - \eta_{2^n} f)\|_{p,w} + 2^{(k+1)\alpha} \|\theta_{k+1} (\eta_{2^k} f - \eta_{2^n} f)\|_{p,w} \right\}^{\tau} \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Let us estimate the first sum  $J_1$ . By (3.3), we have

$$\begin{aligned}\|\theta_j(\eta_{2^k} f - \eta_{2^n} f)\|_{p,w} &\leq 2A\|\eta_{2^k} f - \eta_{2^n} f\|_{p,w} \\ &\leq 2A(\|f - \eta_{2^n} f\|_{p,w} + \|f - \eta_{2^k} f\|_{p,w}).\end{aligned}$$

In light of the fact that

$$\eta_{2^k}(\eta_{2^n} f) = \eta_{2^n} f \quad \text{for } k \geq n+1,$$

we derive

$$\begin{aligned}\|f - \eta_{2^k} f\|_{p,w} &= \|f - \eta_{2^n} f + \eta_{2^n}(\eta_{2^n} f - f)\|_{p,w} \\ &\leq (1+A)\|f - \eta_{2^n} f\|_{p,w}.\end{aligned}\tag{3.10}$$

Therefore,

$$\|\theta_j(\eta_k f - \eta_n f)\|_{p,w} \lesssim \|f - \eta_{2^n} f\|_{p,w}$$

and we get

$$J_1 \lesssim \|f - \eta_{2^n} f\|_{p,w}.$$

Regarding  $J_2$ , we note that

$$\theta_j(f) = \theta_j(f - \eta_{2^n} f) \quad \text{for } j \geq n+2,$$

and, therefore,

$$J_2 = \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ \int_{\mathcal{D}} \left( \sum_{j=n+2}^{k-1} 2^{2j\alpha} \theta_j(f - \eta_{2^n} f)^2 \right)^{\frac{p}{2}} w \right\}^{\frac{\tau}{p}}.$$

Dealing with  $J_3$ , we observe that  $\theta_k(\eta_{2^n} f) = \eta_{2^n}(\theta_k(f))$ . Then

$$\begin{aligned}\|\theta_k(\eta_{2^k} f - \eta_{2^n} f)\|_{p,w} &\leq \|\theta_k(\eta_{2^k} f - f)\|_{p,w} + \|\theta_k(\eta_{2^n} f - f)\|_{p,w} \\ &= \|\eta_{2^k}(\theta_k(f)) - \theta_k(f)\|_{p,w} + \|\theta_k(\eta_{2^n} f - f)\|_{p,w} \\ &\lesssim \|\theta_k(\eta_{2^n} f - f)\|_{p,w},\end{aligned}$$

where in the last estimate we used (3.10) with  $\theta_k(f)$  in place of  $f$ .

Combining the above inequalities, we obtain that

$$J \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ \int_{\mathcal{D}} \left( \sum_{j=n}^{k+1} 2^{2j\alpha} (\theta_j(f - \eta_{2^n} f))^2 \right)^{\frac{p}{2}} w \right\}^{\frac{\tau}{p}} + \|f - \eta_{2^n} f\|_{p,w}^{\tau}.$$



Next, using Minkowski's inequality with  $\frac{\tau}{p} \geq 1$ , Hardy's inequality (2.13), the inequality  $\|\{a_k\}\|_{\ell_\tau} \leq \|\{a_k\}\|_{\ell_2}$ , and Theorem 3.1, we get

$$\begin{aligned} & \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left\{ \int_{\mathcal{D}} \left( \sum_{j=n}^{k+1} 2^{2j\alpha} (\theta_j(f - \eta_{2^n} f))^2 \right)^{\frac{p}{2}} w \right\}^{\frac{\tau}{p}} \\ & \lesssim \left\{ \int_{\mathcal{D}} \left[ \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \left( \sum_{j=n}^{k+1} 2^{2j\alpha} (\theta_j(f - \eta_{2^n} f))^2 \right)^{\frac{p}{2}} \right]^{\frac{\tau}{p}} w \right\}^{\frac{\tau}{p}} \\ & \lesssim \left\{ \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} |\theta_j(f - \eta_{2^n} f)|^\tau \right]^{\frac{p}{\tau}} w \right\}^{\frac{\tau}{p}} \\ & \lesssim \left\{ \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} |\theta_j(f - \eta_{2^n} f)|^2 \right]^{\frac{p}{2}} w \right\}^{\frac{\tau}{p}} \\ & \lesssim \|f - \eta_{2^n} f\|_{p,w}^\tau. \end{aligned}$$

Therefore,

$$J \lesssim \|f - \eta_{2^n} f\|_{p,w}^\tau. \quad (3.11)$$

In light of (3.4), estimates (3.9) and (3.11) imply

$$\begin{aligned} I^\tau & \lesssim \|f - \eta_{2^n} f\|_{p,w}^\tau + 2^{-n\alpha\tau} \|Q(D)^\gamma \eta_{2^n} f\|_{p,w}^\tau \\ & \lesssim K_\gamma(f, Q(D), 2^{-n\gamma\sigma})_{p,w}^\tau, \end{aligned}$$

which proves (3.7).

Let us prove (3.8). By (3.4), we have

$$K_\gamma(f, Q(D), 2^{-n\gamma\sigma})_{p,w}^\theta \lesssim \|f - \eta_{2^n} f\|_{p,w}^\theta + 2^{-n\alpha\theta} \|Q(D)^\gamma \eta_{2^n} f\|_{p,w}^\theta. \quad (3.12)$$

By Theorem 3.1, taking into account that

$$\begin{aligned} (\theta_j(f - \eta_{2^n} f))^2 & \leq 4(\theta_j(f))^2 + 4(\theta_j(\eta_{2^n} f))^2, \\ \theta_j(\eta_{2^n} f) & = 0 \quad \text{for } j \geq n+2, \\ \|\theta_j(\eta_{2^n} f)\|_{p,w} & \leq A\|\theta_j(f)\|_{p,w}, \end{aligned}$$

and  $\|\{a_k\}\|_{\ell_2} \leq \|\{a_k\}\|_{\ell_\theta}$ , we derive

$$\begin{aligned}
 \|f - \eta_{2^n} f\|_{p,w}^\theta &\lesssim \left( \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} \theta_j (f - \eta_{2^n} f)^2 \right]^{\frac{p}{2}} w \right)^{\frac{\theta}{p}} \\
 &\lesssim \left( \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} \theta_j (f)^2 \right]^{\frac{p}{2}} w \right)^{\frac{\theta}{p}} \lesssim \left( \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} |\theta_j(f)|^\theta \right]^{\frac{p}{\theta}} w \right)^{\frac{\theta}{p}} \\
 &= \left( \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} 2^{-j\alpha\theta} (\theta_j(f)^2 2^{2j\alpha})^{\frac{\theta}{2}} \right]^{\frac{p}{2}} w \right)^{\frac{\theta}{p}} \\
 &\lesssim \left( \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} 2^{-j\alpha\theta} \left( \sum_{k=n}^j \theta_k(f)^2 2^{2k\alpha} \right. \right. \right. \\
 &\quad \left. \left. \left. + \theta_{j+1}(\eta_{2^{j+1}} f)^2 2^{2(j+1)\alpha} + \theta_{j+2}(\eta_{2^{j+1}} f)^2 2^{2(j+2)\alpha} \right)^{\frac{\theta}{2}} \right]^{\frac{p}{\theta}} w \right)^{\frac{\theta}{p}} \\
 &= \left( \int_{\mathcal{D}} \left[ \sum_{j=n}^{\infty} 2^{-j\alpha\theta} \left( \sum_{k=n}^{j+2} \theta_k(\eta_{2^{j+1}} f)^2 2^{2k\alpha} \right)^{\frac{\theta}{2}} \right]^{\frac{p}{\theta}} w \right)^{\frac{\theta}{p}}.
 \end{aligned}$$

Next, Minkowski's inequality with  $\frac{p}{\theta} \geq 1$  and Theorem 3.1 (see (3.6)), yield

$$\begin{aligned}
 \|f - \eta_{2^n} f\|_{p,w}^\theta &\lesssim \sum_{j=n}^{\infty} 2^{-j\alpha\theta} \left( \int_{\mathcal{D}} \left[ \sum_{k=n}^{j+2} \theta_k(\eta_{2^{j+1}} f)^2 2^{2k\alpha} \right]^{\frac{p}{2}} w \right)^{\frac{\theta}{p}} \\
 &\lesssim \sum_{j=n}^{\infty} 2^{-j\alpha\theta} \|Q(D)^\gamma \eta_{2^{j+1}} f\|_{p,w}^\theta \lesssim \sum_{j=n}^{\infty} 2^{-j\alpha\theta} \|Q(D)^\gamma \eta_{2^j} f\|_{p,w}^\theta.
 \end{aligned} \tag{3.13}$$

Finally, combining (3.12) and (3.13), we derive (3.8).  $\square$

**Corollary 3.1.** *Under the conditions of Theorem 3.2, we have*

$$\left( \sum_{k=n+1}^{\infty} k^{-\sigma\gamma\tau-1} \|Q(D)^\gamma \eta_k f\|_{p,w}^\tau \right)^{\frac{1}{\tau}} \lesssim K_\gamma(f, Q(D), n^{-\gamma\sigma})_{p,w}$$

and

$$K_\gamma(f, Q(D), n^{-\gamma\sigma})_{p,w} \lesssim \left( \sum_{k=n+1}^{\infty} k^{-\sigma\gamma\theta-1} \|Q(D)^\gamma \eta_k f\|_{p,w}^\theta \right)^{\frac{1}{\theta}},$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

**Proof.** The proof easily follows from inequalities (3.7) and (3.8) and the fact that

$$\|Q(D)^\gamma \eta_\mu f\|_{p,w} \lesssim \|Q(D)^\gamma \eta_\nu f\|_{p,w}, \quad \nu \geq 2\mu.$$

The latter holds in light of boundedness of the de la Vallée Poussin-type operator in  $L_{p,w}$  given by (3.3) and the fact that  $\eta_\mu(\eta_\nu f) = \eta_\mu f$  for  $\nu \geq 2\mu$ . We also take into account that  $K_\gamma(f, Q(D), 2t)_{p,w} \asymp K_\gamma(f, Q(D), t)_{p,w}$  for any  $t > 0$ .  $\square$

### 3.3. General Fourier multiplier operators

In this subsection, we extend Theorem 3.2 considering general Fourier multiplier operators given by

$$\Psi_n f \sim \sum_{k=0}^{\infty} \psi\left(\frac{k}{n}\right) A_k f,$$

where a function  $\psi: [0, \infty) \rightarrow \mathbb{R}$  is such that  $\text{supp } \psi \subset [0, 1)$ . Together with the operator  $\Psi_n$ , additionally assuming that  $\psi(x) \neq 0$  for all  $x \in [0, 2^{-m}]$  for some  $m \in \mathbb{Z}_+$ , we will also use the operator

$$\tilde{\Psi}_n \sim \sum_{k=0}^{\infty} \tilde{\psi}\left(\frac{k}{n}\right) A_k f, \quad \tilde{\psi}(\xi) = \frac{\eta(\xi)}{\psi(2^{-m}\xi)},$$

which plays a role of the inverse operator to  $\Psi_n$ .

**Theorem 3.3.** *Suppose that the conditions of Theorem 3.2 are satisfied.*

(A) *Let the operators  $\Psi_{2^n}$  be such that, for any  $f \in L_{p,w}(\mathcal{D})$  and  $n \in \mathbb{N}$ ,*

$$\|\Psi_{2^n} f\|_{p,w} \leq C(\psi, p, w) \|f\|_{p,w}. \quad (3.14)$$

*Then*

$$\left( \sum_{k=n+1}^{\infty} 2^{-\sigma\gamma\tau k} \|Q(D)^\gamma \Psi_{2^k} f\|_{p,w}^\tau \right)^{\frac{1}{\tau}} \lesssim K_\gamma(f, Q(D), 2^{-n\gamma\sigma})_{p,w}, \quad (3.15)$$

*where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .*

(B) *Suppose that there exists  $m \in \mathbb{N}$  such that  $\psi(x) \neq 0$  for all  $x \in [0, 2^{-m}]$  and the operators  $\tilde{\Psi}_{2^n}$  are such that, for any  $f \in L_{p,w}(\mathcal{D})$  and  $n \in \mathbb{N}$ ,*

$$\|\tilde{\Psi}_{2^n} f\|_{p,w} \leq C(\psi, p, w) \|f\|_{p,w}. \quad (3.16)$$

Then

$$K_\gamma(f, Q(D), 2^{-n\gamma\sigma})_{p,w} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-\sigma\gamma\theta k} \|Q(D)^\gamma \Psi_{2^k} f\|_{p,w}^\theta \right)^{\frac{1}{\theta}}, \quad (3.17)$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

**Proof.** To prove inequality (3.15), it is enough to note that by (3.14) one has

$$\|Q(D)^\gamma \Psi_{2^n} f\|_{p,w} = \|Q(D)^\gamma \Psi_{2^n} (\eta_{2^{n+1}} f)\|_{p,w} \leq C \|Q(D)^\gamma \eta_{2^{n+1}} f\|_{p,w}.$$

Thus, (3.7) clearly implies (3.15).

To show (3.17), we note that by (3.16), we have

$$\begin{aligned} \left\| \sum_{k=0}^n \eta(2^{-n}k) A_k(f) \right\|_{p,w} &= \left\| \sum_{k=0}^n \eta(2^{-n}k) \psi(2^{-n-m}k) (\psi(2^{-n-m}k))^{-1} A_k(f) \right\|_{p,w} \\ &\leq C \left\| \sum_{k=0}^{n+m} \psi(2^{-n-m}k) A_k(f) \right\|_{p,w}, \end{aligned}$$

which gives

$$\|Q(D)^\gamma \eta_{2^n} f\|_{p,w} \lesssim \|Q(D)^\gamma \Psi_{2^{n+m}} f\|_{p,w}. \quad (3.18)$$

This and (3.8) imply

$$\begin{aligned} K_\gamma(f, Q(D), 2^{-n\gamma\sigma})_{p,w} &\lesssim \left( \sum_{k=n+1}^{\infty} 2^{-\sigma\gamma\theta k} \|Q(D)^\gamma \Psi_{2^{k+m}} f\|_{p,w}^\theta \right)^{\frac{1}{\theta}} \\ &\lesssim \left( \sum_{k=n+1}^{\infty} 2^{-\sigma\gamma\theta k} \|Q(D)^\gamma \Psi_{2^k} f\|_{p,w}^\theta \right)^{\frac{1}{\theta}}, \end{aligned}$$

completing the proof.  $\square$

**Remark 3.1.** (i) By Assumption 3.1, condition (3.14) can be replaced by the condition that the sequence  $\{\psi(k2^{-n})\}_{k \in \mathbb{Z}_+}$  satisfies (3.2). Similarly, condition (3.16) can be replaced by the condition that the sequence  $\{\tilde{\psi}(k2^{-n})\}_{k \in \mathbb{Z}_+}$  satisfies (3.2).

(ii) If  $\psi \in C^r([0, \infty))$ , then both sequences  $\{\psi(k2^{-n})\}_{k \in \mathbb{Z}_+}$  and  $\{\tilde{\psi}(k2^{-n})\}_{k \in \mathbb{Z}_+}$  satisfy (3.2) with  $\ell_0 = r$ .

(iii) Inequalities (3.15) and (3.17) can be written similarly to those in Corollary 3.1.

**Example.** Many classical Fourier means are covered by Theorem 3.3. In particular, these cases include the following operators  $\Psi_n f \sim \sum_{k=0}^n \psi\left(\frac{k}{n}\right) A_k f$ :

- (1) Partial sums of Fourier series, the case  $\psi(x) = \chi_{[0,1]}(x)$ ;
- (2) Fejér means that are generated by the function  $\psi(x) = (1-x)_+$

- (3) More generally, Riesz means for which  $\psi(x) = (1 - x^\alpha)_+^\delta$ ,  $\alpha, \delta > 0$ ;  
 (4) Rogosinskii means that are generated by

$$\psi(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right), & 0 \leq x \leq 1, \\ 0, & x > 1. \end{cases}$$

- (5) Jackson means, the case  $\psi(x) = \frac{3}{2}(1 - |x|)_+ * (1 - |x|)_+$ .

The precise formulation of the corresponding results in the periodic case will be given in Corollary 5.1.

#### 4. General Approximation Processes and Measures of Smoothness

For a fixed positive  $\lambda$ , we consider a metric space  $(X, \rho)$  with the metric  $\rho : X \times X \mapsto \mathbb{R}_+$  defined by

$$\rho(f, g) = \|f - g\|_X^\lambda,$$

where the functional  $\|\cdot\|_X : X \mapsto \mathbb{R}_+$  is such that for all  $f, g \in X$  the following properties hold:

- (i)  $\|f\|_X = 0$  if and only if  $f = 0$ ,
- (ii)  $\|-f\|_X = \|f\|_X$ ,
- (iii)  $\|f + g\|_X^\lambda \leq \|f\|_X^\lambda + \|g\|_X^\lambda$ .

Note that the metric  $\|\cdot\|_X = \rho(f, 0)$  is not a norm in general since the homogeneity property is not assumed. A typical example of  $\|\cdot\|_X$  with  $\lambda = 1$  is given by  $\|f\|_X = \int_A \varphi(|f(t)|) d\mu$ , where  $\varphi$  is a positive continuous function such that  $\varphi(0) = 0$  and  $\varphi(x + y) \leq \varphi(x) + \varphi(y)$  for all  $x, y \in \mathbb{R}_+$ . Other examples of  $\|\cdot\|_X$  concerns the standard (quasi-)norm defined in the Lorentz space  $L_{p,q}$ , the Orlicz spaces  $X_M$  given in Sec. 2, the Wiener-type spaces  $A_p$ , and related spaces.

Let us consider the following functional, which to some extend, plays a role of a measure of smoothness (abstract modulus of smoothness)

$$\Omega(f, \delta)_X : X \times (0, \infty) \mapsto \mathbb{R}_+,$$

which satisfies the following conditions: For any  $f, g \in X$  and  $\delta > 0$ ,

$$\Omega(f, \delta)_X \rightarrow 0 \quad \text{as } \delta \rightarrow +0, \tag{4.1}$$

$$\Omega(f, \delta)_X \leq C_1 \|f\|_X, \tag{4.2}$$

$$\Omega(f + g, \delta)_X \leq C_2 (\Omega(f, \delta)_X + \Omega(g, \delta)_X), \tag{4.3}$$

$$\Omega(f, 2\delta)_X \leq C_3 \Omega(f, \delta)_X, \tag{4.4}$$

where  $C_j = C_j(X, \lambda)$ ,  $j = 1, 2, 3$ . A typical example is the modulus of smoothness defined by

$$\Omega(f, \delta)_X = \sup_{|h| \leq \delta} \|\Delta_h^r f\|_X,$$

where  $\Delta_h^1 f(x) = f(x+h) - f(x)$ ,  $\Delta_h^r = \Delta_h^1 \Delta_h^{r-1}$  (here, we suppose that  $\Delta_h^r f \in X$ ). Other examples of  $\Omega(f, \delta)_X$  are given by the  $K$ -functionals or realizations mentioned in the previous sections as well as by any modulus of smoothness, which will be introduced in Secs. 5–7.

As an approximation tool, we consider the family of operators  $P_n : X \mapsto X$ ,  $n \in \mathbb{N}$ , such that the following two properties hold: For any  $f \in X$  and  $n \in \mathbb{N}$ ,

$$\|f - P_n(f)\|_X \leq \|f - P_n(P_{2n}(f))\|_X, \quad (4.5)$$

$$\|f - P_n(f)\|_X \leq C_4 \Omega(f, n^{-1})_X, \quad (4.6)$$

where  $C_4 = C_4(X, \lambda)$ .

Inequality (4.5) trivially holds when  $P_n(f)$  is a best approximant to  $f$  in  $X$  or  $P_n(f)$  is such that  $P_n(P_{2n}(f)) = P_n(f)$ , for example, take a de la Vallée Poussin-type operator or a projection operator. The second inequality is the Jackson-type theorem.

**Theorem 4.1.** *Let  $f \in X$  and  $n \in \mathbb{N}$ . Then*

$$\Omega(P_{2^n}(f), 2^{-n})_X \lesssim \Omega(f, 2^{-n})_X \lesssim \left( \sum_{k=n+1}^{\infty} \Omega(P_{2^k}(f), 2^{-k})_X^\lambda \right)^{\frac{1}{\lambda}}, \quad (4.7)$$

where the left-hand side inequality holds if we assume only (4.2), (4.3), and (4.6). Here, the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

Note that in the case of the Banach space  $X$ , a similar result for  $K$ -functionals and holomorphic semi-groups was obtained in [3, Lemmas 3.5.4 and 3.5.5].

**Proof of Theorem 4.1.** By (4.3),

$$\Omega(P_{2^n}(f), 2^{-n})_X \lesssim \Omega(P_{2^n}(f) - f, 2^{-n})_X + \Omega(f, 2^{-n})_X,$$

and the left-hand side estimate in (4.11) follows from (4.2) and (4.6).

Let us prove the right-hand side inequality. Denote

$$I_{2^n} := \|P_{2^{n+1}}(f) - P_{2^n}(P_{2^{n+1}}(f))\|_X.$$

Then by (4.6) and (4.4), we have

$$I_{2^n} \lesssim \Omega(P_{2^{n+1}}(f), 2^{-n})_X \lesssim \Omega(P_{2^{n+1}}(f), 2^{-n-1})_X. \quad (4.8)$$

At the same time, by (4.5) we get

$$\begin{aligned} I_{2^n}^\lambda &= \|P_{2^{n+1}}(f) - f + f - P_{2^n}(P_{2^{n+1}}(f))\|_X^\lambda \\ &\geq \|f - P_{2^n}(P_{2^{n+1}}(f))\|_X^\lambda - \|f - P_{2^{n+1}}(f)\|_X^\lambda \\ &\geq \|f - P_{2^n}(f)\|_X^\lambda - \|f - P_{2^{n+1}}(f)\|_X^\lambda \\ &=: E_{2^n}^\lambda - E_{2^{n+1}}^\lambda. \end{aligned} \quad (4.9)$$

By (4.6) and (4.1),  $E_{2^k} \rightarrow 0$  as  $k \rightarrow \infty$ . Thus, (4.8) and (4.9) imply

$$E_{2^n}^\lambda = \sum_{k=n}^{\infty} (E_{2^k}^\lambda - E_{2^{k+1}}^\lambda) \leq \sum_{k=n}^{\infty} I_{2^k}^\lambda \lesssim \sum_{k=n}^{\infty} \Omega(P_{2^{k+1}}(f), 2^{-k-1})_X^\lambda. \quad (4.10)$$

Then, using properties of the modulus of smoothness, namely (4.4), (4.3), and (4.2), we obtain

$$\begin{aligned} \Omega(f, 2^{-n})_X^\lambda &\lesssim \Omega(f, 2^{-n-1})_X^\lambda \\ &\lesssim (\Omega(f - P_{2^{n+1}}(f), 2^{-n-1})_X^\lambda + \Omega(P_{2^{n+1}}(f), 2^{-n-1})_X^\lambda) \\ &\lesssim \|f - P_{2^{n+1}}(f)\|_X^\lambda + \Omega(P_{2^{n+1}}(f), 2^{-n-1})_X^\lambda \\ &= E_{2^{n+1}}^\lambda + \Omega(P_{2^{n+1}}(f), 2^{-n-1})_X^\lambda. \end{aligned}$$

Finally, taking into account (4.10),

$$\begin{aligned} \Omega(f, 2^{-n})_X^\lambda &\lesssim \sum_{k=n}^{\infty} \Omega(P_{2^{k+1}}(f), 2^{-k-1})_X^\lambda + \Omega(P_{2^{n+1}}(f), 2^{-n-1})_X^\lambda \\ &\lesssim \sum_{k=n}^{\infty} \Omega(P_{2^{k+1}}(f), 2^{-k-1})_X^\lambda, \end{aligned}$$

which is the right-hand side inequality of (4.11).  $\square$

**Remark 4.1.** Under the conditions of Theorem 4.1, we have

$$\Omega(f, n^{-1})_X \lesssim \left( \sum_{k=1}^{\infty} \Omega(P_{2^{k_n}}(f), 2^{-k} n^{-1})_X^\lambda \right)^{\frac{1}{\lambda}}, \quad (4.11)$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

This inequality can be obtained by using a slight modification of the proof of Theorem 4.1. See also the proof in [28, Lemma 8]. Similar assertions are also valid for Theorems 2.1–2.3, 3.2, 3.3 as well as for the corresponding examples in Secs. 5–8.

As a simple corollary of Theorem 4.1 and Remark 4.1, we have the following version of Jackson's inequality written in terms of measure of smoothness of  $P_{2^{k_n}}(f)$ .

**Corollary 4.1.** *Let  $f \in X$  and  $n \in \mathbb{N}$ . Then*

$$\|f - P_n(f)\|_X \lesssim \left( \sum_{k=1}^{\infty} \Omega(P_{2^{k_n}}(f), 2^{-k} n^{-1})_X^\lambda \right)^{\frac{1}{\lambda}},$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

**Remark 4.2.** If we assume the more general condition than (4.6), namely,

$$\|f - P_n(f)\|_X \leq C_4 \xi(n) \Omega(f, n^{-1})_X,$$

where  $C_4 = C_4(X, \lambda)$  and  $\xi$  is a positive non-decreasing function on  $[1, \infty)$ , then repeating the proof of Theorem 4.1 gives the following estimates:

$$\xi^{-1}(2^n) \Omega(P_{2^n}(f), 2^{-n})_X \lesssim \Omega(f, 2^{-n})_X \lesssim \left( \sum_{k=n+1}^{\infty} \xi^\lambda(2^k) \Omega(P_{2^k}(f), 2^{-k})_X^\lambda \right)^{\frac{1}{\lambda}} \quad (4.12)$$

and

$$\|f - P_{2^n}(f)\|_X \lesssim \left( \sum_{k=n+1}^{\infty} \xi^\lambda(2^k) \Omega(P_{2^k}(f), 2^{-k})_X^\lambda \right)^{\frac{1}{\lambda}}.$$

A typical example when Remark 4.2 can be applied is considering the partial sums of Fourier series  $P_n(f) = S_n(f)$  in the case  $X = L_p(\mathbb{T})$ ,  $p = 1, \infty$ , and  $\xi(t) = \log(t+1)$ ; for details see Corollary 5.2.

In what follows, we say that  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is the modulus of continuity if  $\omega$  is a positive non-decreasing function,  $\omega(0) = 0$ , and  $\omega(x+y) \leq \omega(x) + \omega(y)$  for any  $x, y \in \mathbb{R}_+$ .

**Corollary 4.2.** *For any modulus of continuity  $\omega$  such that*

$$\sum_{k=n}^{\infty} \omega(2^{-k}) \lesssim \omega(2^{-n}), \quad (4.13)$$

*the following assertions are equivalent:*

- (1)  $\Omega(P_{2^n}(f), 2^{-n})_X \lesssim \omega(2^{-n})$ ,
- (2)  $\Omega(f, 2^{-n})_X \lesssim \omega(2^{-n})$ .

**Proof.** The proof follows from (4.11) and the simple fact that (4.13) is equivalent to

$$\left( \sum_{k=n}^{\infty} \omega(2^{-k})^\lambda \right)^{\frac{1}{\lambda}} \lesssim \omega(2^{-n}) \quad \text{for any } \lambda > 0, \quad (4.14)$$

see, e.g. [59]. □

For a given modulus of continuity  $\omega$ , we define the function class

$$\Xi_\omega = \{f \in X : \Omega(f, \delta)_X \asymp \omega(\delta), \delta \rightarrow 0\}.$$

The next corollary provides sharpness of Theorem 4.1.



**Corollary 4.3.** Let  $f \in \Xi_\omega$  and  $\omega$  satisfy (4.13). Then, for large enough  $n \in \mathbb{N}$ ,

$$\Omega(f, 2^{-n})_X \asymp \Omega(P_{2^n}(f), 2^{-n})_X \asymp \left( \sum_{k=n+1}^{\infty} \Omega(P_{2^k}(f), 2^{-k})_X^\lambda \right)^{\frac{1}{\lambda}}, \quad (4.15)$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$ .

**Proof.** First, we prove that

$$\Omega(f, 2^{-n})_X \asymp \Omega(P_{2^n}(f), 2^{-n})_X. \quad (4.16)$$

The part  $\gtrsim$  in (4.16) is given by (4.11). To show the part  $\lesssim$ , we note that by (4.13) and monotonicity of  $\omega$ , for any  $m < n$ , we have

$$\omega(2^{-n+m}) \gtrsim \sum_{k=n-m}^{\infty} \omega(2^{-k}) \gtrsim \sum_{k=n-m}^n \omega(2^{-k}) \gtrsim (m+1)\omega(2^{-n}).$$

Then, taking into account (4.2)–(4.4), and (4.6) and choosing large enough  $m \in \mathbb{N}$ , we derive

$$\begin{aligned} \Omega(P_{2^n}(f), 2^{-n})_X^\lambda &\geq C_3^{-m\lambda} \Omega(P_{2^n}(f), 2^{-n+m})_X^\lambda \\ &\geq C_3^{-m\lambda} (C_2^{-\lambda} \Omega(f, 2^{-n+m})_X^\lambda - \Omega(f - P_{2^n}(f), 2^{-n-m})_X^\lambda) \\ &\geq C_3^{-m\lambda} (C_2^{-\lambda} \Omega(f, 2^{-n+m})_X^\lambda - C_1^\lambda \|f - P_{2^n}(f)\|_X^\lambda) \\ &\geq C_3^{-m\lambda} (C_2^{-\lambda} \Omega(f, 2^{-n+m})_X^\lambda - (C_1 C_4)^\lambda \Omega(f, 2^{-n})_X^\lambda) \\ &\geq C_3^{-m\lambda} (c' \omega(2^{-n+m})^\lambda - c'' \omega(2^{-n})^\lambda) \\ &\geq C_3^{-m\lambda} (c'(m+1)^\lambda - c'') \omega(2^{-n})^\lambda \\ &\gtrsim \omega(2^{-n})^\lambda \gtrsim \Omega(f, 2^{-n})_X^\lambda. \end{aligned}$$

To prove the second equivalence in (4.15), we note the part  $\lesssim$  follows from the right-hand side inequality of (4.11) and (4.16) while the part  $\gtrsim$  follows from (4.14), the left-hand side inequality in (4.11), and (4.16),

$$\left( \sum_{k=n+1}^{\infty} \Omega(P_{2^k}(f), 2^{-k})_X^\lambda \right)^{\frac{1}{\lambda}} \lesssim \left( \sum_{k=n}^{\infty} \omega(2^{-k})^\lambda \right)^{\frac{1}{\lambda}} \lesssim \omega(2^{-n}) \lesssim \Omega(P_{2^n}(f), 2^{-n})_X. \quad \square$$

**Remark 4.3.** Corollaries 4.2 and 4.3 imply that if  $\omega(\delta) = \delta^\alpha$ ,  $\alpha > 0$ , then, for any  $f \in X$  and  $n \in \mathbb{N}$ , we have

$$\Omega(f, 2^{-n})_X \lesssim \omega(2^{-n}) \quad \text{if and only if} \quad \Omega(P_{2^n}(f), 2^{-n})_X \lesssim \omega(2^{-n}).$$

If, in addition,  $f \in \Xi_\omega$ , then

$$\Omega(f, 2^{-n})_X \asymp \Omega(P_{2^n}(f), 2^{-n})_X \asymp \omega(2^{-n}).$$

The results of Remark 4.3 can be extended to Besov-type spaces.

For a given modulus of smoothness  $\Omega$ ,  $s > 0$ , and  $0 < q \leq \infty$ , we define the Besov-type space as follows:

$$B_{X,q}^s = \left\{ f \in X : |f|_{B_{X,q}^s} = \left( \int_0^1 (t^{-s} \Omega(f, t)_X)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\} \quad (4.17)$$

with the usual modification in the case  $q = \infty$ .

We have the following characterization of  $B_{X,q}^s$ .

**Corollary 4.4.** *Let  $s > 0$  and  $0 < q \leq \infty$ , we have*

$$|f|_{B_{X,q}^s} \asymp \left( \sum_{k=1}^{\infty} 2^{sqk} \Omega(P_{2^k}(f), 2^{-k})_X^q \right)^{\frac{1}{q}},$$

where the constants in  $\asymp$  are independent of  $f$ .

**Proof.** The proof easily follows from Theorem 4.1 and the Hardy-type inequality

$$\sum_{\nu=n}^{\infty} 2^{\nu s} \left( \sum_{k=\nu}^{\infty} A_k \right)^q \asymp \sum_{\nu=n}^{\infty} 2^{\nu s} A_{\nu}^q,$$

where  $A_{\nu} \geq 0$  and  $s, q > 0$ . □

## 5. Smoothness of Approximation Processes on $\mathbb{T}^d$

### 5.1. Smoothness of best approximants

In this subsection, we give analogues of Theorems 4.1 and 2.2 for best trigonometric approximants in  $L_p(\mathbb{T}^d)$  spaces. We recall some basic notations. Denote the set of all trigonometric polynomials of degree at most  $n$  by

$$\mathcal{T}_n = \text{span}\{e^{i(k,x)} : |k| \leq n\},$$

where  $|k| = (k_1^2 + \dots + k_d^2)^{1/2}$ . The best approximation by trigonometric polynomials is given by

$$E_n(f)_{L_p(\mathbb{T}^d)} = \inf\{\|f - \varphi\|_{L_p(\mathbb{T}^d)} : \varphi \in \mathcal{T}_n\}.$$

As above, by  $P_n(f)$  we denote the best approximant of a function  $f$  in  $L_p(\mathbb{T}^d)$ , that is,

$$\|f - P_n(f)\|_{L_p(\mathbb{T}^d)} = E_n(f)_{L_p(\mathbb{T}^d)},$$

where  $P_n(f) \in \mathcal{T}_n$ .

In what follows, we will use the well-known Jackson-type inequality, see, e.g. [62, 56]:

$$E_n(f)_{L_p(\mathbb{T}^d)} \leq C \omega_r \left( f, \frac{1}{n} \right)_{L_p(\mathbb{T}^d)}, \quad f \in L_p(\mathbb{T}^d), \quad 0 < p \leq \infty, \quad r \in \mathbb{N}, \quad (5.1)$$

where  $\omega_r(f, h)_p$  is the classical modulus of smoothness,

$$\omega_r(f, \delta)_p = \sup_{|h| < \delta} \|\Delta_h^r f\|_{L_p(\mathbb{T}^d)},$$

$$\Delta_h f(x) = f(x+h) - f(x), \quad \Delta_h^r = \Delta_h \Delta_h^{r-1}, \quad h \in \mathbb{R}^d, \quad d \geq 1,$$

and  $C = C(r, p, d)$ .

We will also need the following Stechkin–Nikolskii-type inequality (see [36, Theorem 3.2]), which states that, for any  $n \in \mathbb{N}$  and  $0 < \delta \leq \pi/n$ ,

$$\|T_n\|_{\dot{W}_p^r(\mathbb{T}^d)} \asymp \delta^{-r} \omega_r(T_n, \delta)_{L_p(\mathbb{T}^d)}, \quad T_n \in \mathcal{T}_n, \quad 0 < p \leq \infty, \quad r \in \mathbb{N}, \quad (5.2)$$

where the constants in this equivalence are independent of  $T_n$  and  $\delta$ . Here, the homogeneous Sobolev norm is given by

$$\|f\|_{\dot{W}_p^r(\mathbb{T}^d)} = \sum_{|\nu|_1=r} \|D^\nu f\|_{L_p(\mathbb{T}^d)}.$$

Using Theorem 4.1 with  $X = L_p(\mathbb{T}^d)$ ,  $0 < p \leq \infty$ , and  $\Omega(f, \delta)_X = \omega_r(f, \delta)_{L_p(\mathbb{T}^d)}$  for some  $r \in \mathbb{N}$ , one can easily verify that properties (4.1)–(4.6) are valid. Therefore, applying Stechkin–Nikolskii-type inequality (5.2), we obtain the following result.

**Theorem 5.1.** *Let  $f \in L_p(\mathbb{T}^d)$ ,  $0 < p \leq \infty$ , and  $r \in \mathbb{N}$ . Then*

$$2^{-nr} \|P_{2^n}(f)\|_{\dot{W}_p^r(\mathbb{T}^d)} \lesssim \omega_r(f, 2^{-n})_{L_p(\mathbb{T}^d)} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-kr\lambda} \|P_{2^k}(f)\|_{\dot{W}_p^r(\mathbb{T}^d)}^\lambda \right)^{\frac{1}{\lambda}}, \quad (5.3)$$

where  $\lambda = \min(p, 1)$  and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

The above theorem can be also formulated in terms of the fractional smoothness. For this, we recall the following assertion from [36, Corollary 3.1]: *Let  $0 < p \leq \infty$ ,  $\alpha > 0$ ,  $n \in \mathbb{N}$ , and  $0 < \delta \leq \pi/n$ . Then, for any  $T_n \in \mathcal{T}_n$ , we have*

$$\sup_{\xi \in \mathbb{R}^d, |\xi|=1} \left\| \left( \frac{\partial}{\partial \xi} \right)^\alpha T_n \right\|_{L_p(\mathbb{T}^d)} \asymp \delta^{-\alpha} \omega_\alpha(T_n, \delta)_{L_p(\mathbb{T}^d)}, \quad (5.4)$$

where the constants in  $\asymp$  are independent of  $T_n$  and the fractional modulus of smoothness  $\omega_\alpha(f, \delta)_{L_p(\mathbb{T}^d)}$  is given by

$$\omega_\alpha(f, \delta)_{L_p(\mathbb{T}^d)} = \sup_{|h| \leq \delta} \left\| \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\alpha}{\nu} f(\cdot + (\alpha - \nu)h) \right\|_{L_p(\mathbb{T}^d)}$$

and  $\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}$ ,  $\binom{\alpha}{0} = 1$ , see [48].

Our next goal is to obtain a sharp version of (5.3) in the case  $1 < p < \infty$ . For this, we use Theorem 2.2 with  $G_n = \mathcal{T}_n$ ,  $X = L_p(\mathbb{T}^d)$ , and  $Y = H_p^\alpha(\mathbb{T}^d)$ , where

$$H_p^\alpha(\mathbb{T}^d) = \{g \in L_p(\mathbb{T}^d) : \|g\|_{\dot{H}_p^\alpha(\mathbb{T}^d)} = \|(-\Delta)^{\alpha/2} g\|_{L_p(\mathbb{T}^d)} < \infty\}$$

is the fractional Sobolev space. Recall that

$$K(f, t^\alpha, L_p(\mathbb{T}^d); H_p^\alpha(\mathbb{T}^d)) = \inf \{ \|f - g\|_{L_p(\mathbb{T}^d)} + t^\alpha \|g\|_{\dot{H}_p^\alpha(\mathbb{T}^d)} : g \in H_p^\alpha(\mathbb{T}^d) \} \quad (5.5)$$

and

$$R(f, t^\alpha; L_p(\mathbb{T}^d), \mathcal{T}_{[1/t]}) = \inf \{ \|f - T\|_{L_p(\mathbb{T}^d)} + t^\alpha \|T\|_{\dot{H}_p^\alpha(\mathbb{T}^d)} : T \in \mathcal{T}_{[1/t]} \} \quad (5.6)$$

(cf. (2.1) and (2.6)). For any  $f \in L_p(\mathbb{T}^d)$ ,  $1 < p < \infty$ , and  $\alpha > 0$  we have (see, e.g. [36])

$$K(f, t^\alpha; L_p(\mathbb{T}^d), H_p^\alpha(\mathbb{T}^d)) \asymp R(f, t^\alpha; L_p(\mathbb{T}^d), \mathcal{T}_{[1/t]}) \asymp \omega_\alpha(f, t)_{L_p(\mathbb{T}^d)},$$

which, in particular, implies (2.7). Here, the constants in  $\asymp$  are independent of  $f$  and  $t$ .

Jackson and Bernstein inequalities (2.2) and (2.3) are given by (5.1) and the following inequality, see, e.g. [67]:

$$\|(-\Delta)^{\alpha/2} T_n\|_{L_p(\mathbb{T}^d)} \lesssim n^\alpha \|T_n\|_{L_p(\mathbb{T}^d)}, \quad T_n \in \mathcal{T}_n, \quad 1 < p < \infty, \quad \alpha > 0.$$

Thus, Theorem 2.2 implies the following result.

**Theorem 5.2.** *Let  $f \in L_p(\mathbb{T}^d)$ ,  $1 < p < \infty$ , and  $\alpha > 0$ . Then*

$$\begin{aligned} \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|(-\Delta)^{\alpha/2} P_{2^k}(f)\|_{L_p(\mathbb{T}^d)}^\tau \right)^{\frac{1}{\tau}} &\lesssim \omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T}^d)} \\ &\lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|(-\Delta)^{\alpha/2} P_{2^k}(f)\|_{L_p(\mathbb{T}^d)}^\theta \right)^{\frac{1}{\theta}}, \end{aligned}$$

where  $\tau = \max(2, p)$ ,  $\theta = \min(2, p)$ , and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

## 5.2. The case of Fourier multiplier operators

In this subsection, we give an analogue of Theorem 3.3 in the case  $\mathcal{D} = \mathbb{T}^d$ . We start by recalling the multiplier theorem (Assumption 3.1) and the Littlewood–Paley-type theorem in  $L_p(\mathbb{T}^d)$  for  $1 < p < \infty$ .

Concerning Assumption 3.1, the well-known Mikhlin–Hörmander multiplier theorem (see [24, p. 224]) states that the condition

$$|\Delta_{e_1}^{\beta_1} \dots \Delta_{e_d}^{\beta_d} m(k_1, \dots, k_d)| \leq A|k|^{-|\beta|}, \quad |\beta| \equiv \beta_1 + \dots + \beta_d < [d/2] + 1, \quad (5.7)$$

where  $\Delta_{e_i} m(k_1, \dots, k_i, k_d) = m(k_1, \dots, k_i + 1, \dots, k_d) - m(k_1, \dots, k_i, \dots, k_d)$ , implies

$$\|T_m f\|_{L_p(\mathbb{T}^d)} \leq C(A, p, d) \|f\|_{L_p(\mathbb{T}^d)},$$

where

$$(T_m f)^\wedge(k) = m(k) \hat{f}(k)$$

$$\text{and } \hat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(y) e^{-i(k,y)} dy.$$

We define the de la Vallée Poussin-type multiplier operator by

$$(\eta_m f)^\wedge(k) = \eta\left(\frac{|k|}{n}\right) \hat{f}(k)$$

and similarly to (3.5), we set

$$\theta_0(f) = \eta_1 f \quad \text{and} \quad \theta_j(f) = \eta_{2^j} f - \eta_{2^{j-1}} f \quad \text{for } j \geq 1.$$

An analogue of the Littlewood–Paley theorem in the case  $\mathcal{D} = \mathbb{T}^d$  is given by the following two inequalities, see, e.g. [11, Theorem 4.1] or [25, Chap. 6]: For  $f \in L_p(\mathbb{T}^d)$ ,  $1 < p < \infty$ , and  $\alpha > 0$ , we have

$$\left\| \left\{ \sum_{j=0}^{\infty} (\theta_j(f))^2 \right\}^{1/2} \right\|_{L_p(\mathbb{T}^d)} \asymp \|f\|_{L_p(\mathbb{T}^d)}$$

and

$$\left\| \left\{ \sum_{j=1}^{\infty} 2^{2j\alpha} (\theta_j(f))^2 \right\}^{1/2} \right\|_{L_p(\mathbb{T}^d)} \asymp \|(-\Delta)^{\alpha/2} f\|_{L_p(\mathbb{T}^d)},$$

where the constants in  $\asymp$  are independent of  $f$ .

Let us consider the Fourier means given by

$$\begin{aligned} \Psi_n f(x) &= \sum_{k \in \mathbb{Z}^d} \psi\left(\frac{k}{n}\right) \hat{f}(k) e^{i(k,x)}, \\ \tilde{\Psi}_n f(x) &= \sum_{k \in \mathbb{Z}^d} \tilde{\psi}\left(\frac{k}{n}\right) \hat{f}(k) e^{i(k,x)}, \quad \tilde{\psi}(\xi) = \frac{\eta(|\xi|)}{\psi(2^{-m}\xi)}, \end{aligned}$$

where the function  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$  is such that  $\text{supp } \psi \subset [-1, 1]^d$  and for some  $m \in \mathbb{Z}_+$ ,  $\psi(x) \neq 0$  for all  $x \in [-2^{-m}, 2^{-m}]^d$ .

We derive the following analogue of Theorem 3.3 in the case  $\mathcal{D} = \mathbb{T}^d$ .

**Theorem 5.3.** Let  $f \in L_p(\mathbb{T}^d)$ ,  $1 < p < \infty$ ,  $n \in \mathbb{N}$ ,  $\alpha > 0$ ,  $\tau = \max(2, p)$ , and  $\theta = \min(2, p)$ .

(A) If  $\{\Psi_{2^k}\}$  are uniformly bounded operators in  $L_p(\mathbb{T}^d)$ , then

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|(-\Delta)^{\alpha/2} \Psi_{2^k} f\|_{L_p(\mathbb{T}^d)}^{\tau} \right)^{\frac{1}{\tau}} \lesssim \omega_{\alpha}(f, 2^{-n})_{L_p(\mathbb{T}^d)},$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

(B) If  $\{\tilde{\Psi}_{2^k}\}$  are uniformly bounded operators in  $L_p(\mathbb{T}^d)$ , then

$$\omega_{\alpha}(f, 2^{-n})_{L_p(\mathbb{T}^d)} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|(-\Delta)^{\alpha/2} \tilde{\Psi}_{2^k} f\|_{L_p(\mathbb{T}^d)}^{\theta} \right)^{\frac{1}{\theta}},$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

**Remark 5.1.** (i) Note that if  $\psi \in A(\mathbb{R}^d) = \{f: f = \widehat{g}, g \in L_1(\mathbb{R}^d)\}$  (the Wiener class of absolutely convergent Fourier integrals), then the operators  $\{\Psi_n\}$  are uniformly bounded in  $L_p(\mathbb{T}^d)$  for all  $1 \leq p \leq \infty$ , see, e.g. [54, Chap. VII]. Various useful conditions to insure that  $\psi \in A(\mathbb{R}^d)$  can be found in the survey [41], see also [64, Chaps. 4 and 6].

(ii) Concerning the uniform boundedness of  $\{\tilde{\Psi}_n\}$ , one can use following version of  $\frac{1}{f}$ -Wiener theorem (see [42, p. 102]): Let  $f \in A(\mathbb{R}^d)$ . If  $f(x) \neq 0$  on a closed bounded set  $V \subset \mathbb{R}^d$ , then  $\frac{1}{f(x)}$  is extendable to a function in  $A(\mathbb{R}^d)$ , i.e. there exists a function  $g \in A(\mathbb{R}^d)$  such that  $f(x) \equiv g(x)$  on  $V$ .

(iii) To verify the uniform boundedness of  $\{\Psi_n\}$  and  $\{\tilde{\Psi}_n\}$  in  $L_p(\mathbb{T}^d)$  for  $1 < p < \infty$ , one can use the Mikhlin–Hörmander multiplier condition (5.7), which is less restrictive than the conditions given in parts (i) and (ii) of this remark.

(iv) Under conditions of Theorem 5.3, we have that for any  $f \in H_p^{\beta}(\mathbb{T}^d)$ ,  $\beta > 0$ ,

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|(-\Delta)^{(\alpha+\beta)/2} \Psi_{2^k} f\|_{L_p(\mathbb{T}^d)}^{\tau} \right)^{\frac{1}{\tau}} \lesssim \omega_{\alpha}((-\Delta)^{\beta/2} f, 2^{-n})_{L_p(\mathbb{T}^d)}$$

and

$$\omega_{\alpha}((-\Delta)^{\beta/2} f, 2^{-n})_{L_p(\mathbb{T}^d)} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|(-\Delta)^{(\alpha+\beta)/2} \tilde{\Psi}_{2^k} f\|_{L_p(\mathbb{T}^d)}^{\theta} \right)^{\frac{1}{\theta}}.$$

As examples, let us consider the following approximation processes:

(1) the  $\ell_q$ -partial Fourier sums

$$S_{n;q}f(x) = \sum_{\|k\|_{\ell_q} \leq n} \widehat{f}(k) e^{i(k,x)}, \quad 1 \leq q \leq \infty;$$

(2) the de la Vallée Poussin-type means

$$\eta_n f(x) = \sum_{k \in \mathbb{Z}^d} \eta \left( \frac{|k|}{n} \right) \widehat{f}(k) e^{i(k,x)};$$

(3) the Riesz spherical means

$$R_n^{\beta, \delta} f(x) = \sum_{|k| \leq n} \left( 1 - \left( \frac{|k|}{n} \right)^\beta \right)_+^\delta \widehat{f}(k) e^{i(k,x)}, \quad \beta, \delta > 0.$$

**Corollary 5.1.** *Let  $f \in L_p(\mathbb{T}^d)$ ,  $1 < p < \infty$ ,  $\alpha > 0$ ,  $\tau = \max(2, p)$ , and  $\theta = \min(2, p)$ . Then*

$$\begin{aligned} \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|(-\Delta)^{\alpha/2} T_{2^k} f\|_p^\tau \right)^{\frac{1}{\tau}} &\lesssim \omega_\alpha \left( f, \frac{1}{2^n} \right)_p \\ &\lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|(-\Delta)^{\alpha/2} T_{2^k} f\|_p^\theta \right)^{\frac{1}{\theta}}, \end{aligned} \quad (5.8)$$

where  $T_{2^k} f = S_{2^k; q} f$  with  $q = 1, \infty$ ,  $\eta_{2^k} f$ , or  $R_{2^k}^{\beta, \delta} f$  with  $\delta > (d-1)/2$ , and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

**Proof.** It is enough to note that these means are uniformly bounded in  $L_p(\mathbb{T}^d)$ ,  $1 < p < \infty$ , see, e.g. [54, Chap. VII; 66], and to apply the Mikhlin–Hörmander multiplier condition to show that the corresponding inverse operators  $\{\widetilde{\Psi}_n\}$  are also uniformly bounded in  $L_p(\mathbb{T}^d)$ .  $\square$

**Remark 5.2.** In the univariate case of the Fejér means  $T_{2^k} f = R_{2^k}^{1,1} f$ , the right-hand side of inequality (5.8) was obtained earlier by Zhuk and Natanson in [70].

Note that for  $\alpha \in \mathbb{N}$  and  $1 < p < \infty$  inequality (5.8) can be equivalently written as follows:

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|T_{2^k} f\|_{\dot{W}_p^\alpha(\mathbb{T}^d)}^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_\alpha \left( f, \frac{1}{2^n} \right)_p \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|T_{2^k} f\|_{\dot{W}_p^\alpha(\mathbb{T}^d)}^\theta \right)^{\frac{1}{\theta}}.$$

We give its analogue for the cases  $p = 1, \infty$ .

**Corollary 5.2.** *Let  $f \in L_p(\mathbb{T}^d)$ ,  $p = 1, \infty$ , and  $\alpha \in \mathbb{N}$ . Then*

$$2^{-n\alpha} \xi_q^{-1}(2^n) \|S_{2^n; q} f\|_{\dot{W}_p^\alpha(\mathbb{T}^d)} \lesssim \omega_\alpha \left( f, \frac{1}{2^n} \right)_p \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha} \xi_q(2^k) \|S_{2^k; q} f\|_{\dot{W}_p^\alpha(\mathbb{T}^d)}, \quad (5.9)$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ ,

$$\xi_q(t) = \begin{cases} \log^d(t+1), & q = 1, \infty, \\ t^{\frac{d-1}{2}}, & 1 < q < \infty, q \neq 1, \end{cases}$$

and

$$2^{-n\alpha} \|T_{2^n} f\|_{\dot{W}_p^\alpha(\mathbb{T}^d)} \lesssim \omega_\alpha \left( f, \frac{1}{2^n} \right)_p \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha} \|T_{2^k} f\|_{\dot{W}_p^\alpha(\mathbb{T}^d)}, \quad (5.10)$$

where  $T_{2^k} f = \eta_{2^k} f$  or  $R_{2^k}^{\beta, \delta} f$  with  $\delta > (d-1)/2$ .

**Proof.** Estimates (5.9) follow from Remark 4.2 with  $\xi(t) = \xi_q(t)$  since

$$\|f - S_{n;q} f\|_{L_p(\mathbb{T}^d)} \lesssim \|S_{n;q}\|_{L_1 \rightarrow L_1} E_{cn}(f)_{L_p(\mathbb{T}^d)} \lesssim \xi_q(n) \omega_\alpha(f, n^{-1})_{L_p(\mathbb{T}^d)}.$$

For calculation of  $\xi_q(t)$  see, e.g. [40, 21] for the case  $1 < q < \infty$  and [64, Sec. 9.2; 34] for the case  $q = 1, \infty$ .

The proof of (5.10) for  $T_{2^k} f = \eta_{2^k} f$  follows from Theorem 4.1 and the uniform boundedness of the de la Vallée Poussin means in  $L_1(\mathbb{T}^d)$ , see also Remark 5.1. The case  $T_{2^k} f = R_{2^k}^{\beta, \delta} f$  can be proved similarly using the uniform boundedness of  $R_{2^k}^{\beta, \delta}$ , see, e.g. [54, Chap. VII], the inequality  $\|f - R_{2^k}^{\beta, \delta} f\|_{L_p(\mathbb{T}^d)} \lesssim \omega_\alpha(f, 2^{-n})_p$ , see [67], and applying the same arguments as in the proof of (3.17).  $\square$

### 5.3. Inequalities in the Hardy spaces $H_p(\mathbb{D})$ , $0 < p \leq 1$

For simplicity, we only consider the analytic Hardy spaces on the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . By definition, an analytic function  $f$  on  $\mathbb{D}$  belongs to the space  $H_p = H_p(\mathbb{D})$  if

$$\|f\|_{H_p} = \sup_{0 < \rho < 1} \left( \int_0^{2\pi} |f(\rho e^{it})|^p dt \right)^{\frac{1}{p}} < \infty.$$

Set

$$\eta_n f(x) = \sum_{k=0}^n \eta \left( \frac{k}{n} \right) c_k e^{ikx},$$

where  $c_k = c_k(f)$  are the Taylor coefficients of  $f$ . Then, the realization result is given as follows (see [36, Sec. 11]):

$$\|f - \eta_{2^n} f\|_{H_p} + 2^{-\alpha n} \|(\eta_{2^n} f)^{(\alpha)}\|_{H_p} \asymp \omega_\alpha(f, 2^{-n})_{H_p},$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$ .

Using the scheme of the proof of Theorem 3.2 and the Littlewood–Paley theorem in the Hardy spaces  $H_p(\mathbb{D})$ ,  $0 < p \leq 1$ , see, e.g. [25, Chap. 6], we obtain the following result.



**Theorem 5.4.** Let  $f \in H_p(\mathbb{D})$ ,  $0 < p \leq 1$ ,  $\alpha \in \mathbb{N} \cup (1/p - 1, \infty)$ ,  $n \in \mathbb{N}$ . Then

$$\left( \sum_{k=n+1}^{\infty} 2^{-2\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{H_p}^2 \right)^{\frac{1}{2}} \lesssim \omega_{\alpha}(f, 2^{-n})_{H_p} \quad (5.11)$$

and

$$\omega_{\alpha}(f, 2^{-n})_{H_p} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-\alpha p k} \|(\eta_{2^k} f)^{(\alpha)}\|_{H_p}^p \right)^{\frac{1}{p}}, \quad (5.12)$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

**Remark 5.3.** (i) Note that the restriction  $\alpha > 1/p - 1$  is needed to correctly define the modulus of smoothness  $\omega_{\alpha}(f, \delta)_{H_p}$ .

(ii) Inequalities (5.11) and (5.12) are also valid if we replace the de la Vallée Poussin means  $\eta_{2^k} f$  by the corresponding means  $\Psi_{2^k} f$  with the properties similar to those indicated in Theorem 3.3.

(iii) Inequality (5.12) also follows from Theorem 4.1 and the Stechkin–Nikolskii inequality (5.4).

#### 5.4. Approximation in smooth function spaces

We will say that  $f \in \text{Lip}(\alpha, p)(\mathbb{T})$ ,  $0 < p \leq \infty$ ,  $\alpha > 0$ , if  $f \in L_p(\mathbb{T})$  and

$$\|f\|_{\text{Lip}(\alpha, p)} = \|f\|_{L_p(\mathbb{T})} + |f|_{\text{Lip}(\alpha, p)} < \infty,$$

where

$$|f|_{\text{Lip}(\alpha, p)} = \sup_{h>0} \frac{\|\Delta_h^r f\|_{L_p(\mathbb{T})}}{h^{\alpha}} = \sup_{h>0} \frac{\omega_r(f, h)_p}{h^{\alpha}}, \quad r = [\alpha] + 1.$$

Let  $0 < p \leq \infty$ ,  $0 < \alpha < \ell$ , and  $\ell, n \in \mathbb{N}$ . The best approximation in  $\text{Lip}(\alpha, p)(\mathbb{T})$  and the modulus of smoothness are given by

$$E_n(f)_{\text{Lip}(\alpha, p)} = \inf_{T \in \mathcal{T}_n} \|f - T\|_{\text{Lip}(\alpha, p)}$$

and

$$\vartheta_{\ell, \alpha}(f, \delta)_p = \sup_{0 < h \leq \delta} \frac{\omega_{\ell}(f, h)_p}{h^{\alpha}}.$$

In light of the Jackson inequality (see [35])

$$E_n(f)_{\text{Lip}(\alpha, p)} \lesssim \vartheta_{\ell, \alpha} \left( f, \frac{1}{n} \right)_p, \quad n \in \mathbb{N},$$

by (5.2), the realization result can be written as follows:

$$\vartheta_{\ell, \alpha}(f, \delta)_p \asymp \|f - T_n\|_{\text{Lip}(\alpha, p)} + \delta^{\ell - \alpha} \|T_n^{(\ell)}\|_{L_p(\mathbb{T})}, \quad n = [1/\delta], \quad (5.13)$$

where  $T_n \in \mathcal{T}_n$  is such that  $E_n(f)_{\text{Lip}(\alpha, p)} = \|f - T_n\|_{\text{Lip}(\alpha, p)}$ .

Therefore, making use of Theorem 4.1 with  $X = \text{Lip}(\alpha, p)$  and  $\Omega(f, \delta)_X = \vartheta_{\ell, \alpha}(f, \delta)_p$ ,  $\alpha < \ell$ ,  $\ell \in \mathbb{N}$ , and (5.13), we obtain the following result.

**Theorem 5.5.** *Let  $f \in \text{Lip}(\alpha, p)$ ,  $0 < p \leq \infty$ ,  $\ell \in \mathbb{N}$ ,  $0 < \alpha < \ell$ , and  $\lambda = \min(p, 1)$ . Then*

$$2^{-n(\ell-\alpha)} \|T_{2^n}^{(\ell)}\|_{L_p(\mathbb{T})} \lesssim \vartheta_{\ell, \alpha}(f, 2^{-n})_p \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k(\ell-\alpha)\lambda} \|T_{2^k}^{(\ell)}\|_{L_p(\mathbb{T})}^{\lambda} \right)^{\frac{1}{\lambda}}, \quad (5.14)$$

where  $T_{2^k} \in \mathcal{T}_{2^k}$  is the best approximant of  $f$  in  $\text{Lip}(\alpha, p)$  and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

In view of Theorem 2.2, we sharpen (5.14) for  $1 < p < \infty$  as follows.

**Theorem 5.6.** *Let  $f \in \text{Lip}(\alpha, p)$ ,  $1 < p < \infty$ ,  $\ell \in \mathbb{N}$ ,  $0 < \alpha < \ell$ , and  $\tau = \max(2, p)$ ,  $\theta = \min(2, p)$ . Then*

$$\begin{aligned} \left( \sum_{k=n+1}^{\infty} 2^{-k(\ell-\alpha)\tau} \|T_{2^k}^{(\ell)}\|_{L_p(\mathbb{T})}^{\tau} \right)^{\frac{1}{\tau}} &\lesssim \vartheta_{\ell, \alpha}(f, 2^{-n})_p \\ &\lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k(\ell-\alpha)\theta} \|T_{2^k}^{(\ell)}\|_{L_p(\mathbb{T})}^{\theta} \right)^{\frac{1}{\theta}}, \end{aligned}$$

where  $T_{2^k} \in \mathcal{T}_{2^k}$  is the best approximant of  $f$  in  $\text{Lip}(\alpha, p)$  and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

**Remark 5.4.** Using the well-known facts about simultaneous approximation of functions and their derivatives in  $L_p(\mathbb{T})$ , see, e.g. [9; 16, Chap. 7, Theorem 2.7], it is not difficult to obtain analogues of Theorems 5.5 and 5.6 in the Sobolev spaces  $W_p^r(\mathbb{T})$ ,  $1 \leq p \leq \infty$ , and  $r \in \mathbb{N}$ , cf. Remark 5.1(iv).

### 5.5. Interpolation operators

In the above sections, we deal with polynomials of the best approximation and Fourier means. It turns out that Theorem 4.1 can be also applied for interpolation operators. As an example, let us consider an interpolation analogue of the de la Vallée Poussin means:

$$V_n f(t) = \frac{1}{3n} \sum_{k=0}^{6n-1} f(t_k) K_n(t - t_k), \quad t_k = \frac{\pi k}{3n}, \quad t \in \mathbb{T},$$

where

$$K_n(t) = \frac{1}{2} + \sum_{k=1}^{2n} \cos kt + \sum_{k=2n+1}^{4n-1} \frac{4n-k}{2n} \cos kt.$$

Recall some basic properties of  $V_n f$  (see [57]).

**Proposition 5.1.** *The following assertions hold:*

- (1)  $\deg V_n f \leq 4n - 1$ ;
- (2)  $V_n f(t_k) = f(t_k)$ ,  $k = 0, \dots, 6n - 1$ ;
- (3)  $V_n T(t) = T(t)$  for any  $T \in \mathcal{T}_{2n}$ ;
- (4) for all  $f \in C(\mathbb{T})$  and  $r, n \in \mathbb{N}$ , we have

$$\|f - V_n f\|_{L_\infty(\mathbb{T})} \lesssim \omega_r(f, 1/n)_\infty.$$

Thus, noting that  $V_n(V_{2n}f) = V_n f$  and using Theorem 4.1, Proposition 5.1, and the Nikolskii–Stechkin-type inequality (5.2), we derive the following result.

**Theorem 5.7.** *Let  $f \in C(\mathbb{T})$  and  $r, n \in \mathbb{N}$ . Then*

$$2^{-nr} \|(V_{2^n} f)^{(r)}\|_{L_\infty(\mathbb{T})} \lesssim \omega_r(f, 2^{-n})_\infty \lesssim \sum_{k=n+1}^{\infty} 2^{-kr} \|(V_{2^k} f)^{(r)}\|_{L_\infty(\mathbb{T})},$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

## 6. Smoothness of Approximation Processes on $\mathbb{R}^d$

### 6.1. Smoothness of best approximants

In what follows, the class of band-limited functions  $\mathcal{B}_p^\sigma$ ,  $1 \leq p \leq \infty$ ,  $\sigma > 0$ , is given by

$$\mathcal{B}_p^\sigma = \{\varphi \in L_p(\mathbb{R}^d) : \text{supp } \widehat{\varphi}(x) \subset \{x : |x| < \sigma\}\},$$

where

$$\widehat{g}(x) = \int_{\mathbb{R}^d} g(y) e^{-i(x,y)} dy.$$

Let

$$E_\sigma(f)_{L_p(\mathbb{R}^d)} = \inf \{\|f - \varphi\|_{L_p(\mathbb{R}^d)} : \varphi \in \mathcal{B}_p^\sigma\}$$

be the best approximation of  $f$  and  $P_\sigma(f) \in \mathcal{B}_p^\sigma$  be a best approximant of  $f$  in  $L_p(\mathbb{R}^d)$ , that is,

$$\|f - P_\sigma(f)\|_{L_p(\mathbb{R}^d)} = E_\sigma(f)_{L_p(\mathbb{R}^d)}.$$

We will use the following Jackson and Nikolskii–Stechkin inequalities, see, e.g. [62, 5.3.2; 67, Theorem 3] for the case  $1 \leq p \leq \infty$  and [37] for the case  $0 < p < 1$ :

$$E_\sigma(f)_p \lesssim \omega_r\left(f, \frac{1}{\sigma}\right)_p, \quad f \in L_p(\mathbb{R}^d), \quad \sigma > 0, \quad r \in \mathbb{N}, \quad (6.1)$$

$$\|P_\sigma\|_{\dot{W}_p^r(\mathbb{R}^d)} \asymp \delta^{-r} \omega_r(P_n, \delta)_{L_p(\mathbb{R}^d)}, \quad P_\sigma \in \mathcal{B}_p^\sigma, \quad \sigma > 0, \quad 0 < \delta \leq \pi/\sigma. \quad (6.2)$$

In the above relations, the constants in  $\lesssim$  and  $\asymp$  are independent of  $f$ ,  $\sigma$ , and  $\delta$ .

Then, Theorem 4.1 together with inequalities (6.1) and (6.2) imply the following result.

**Theorem 6.1.** *Let  $f \in L_p(\mathbb{R}^d)$ ,  $0 < p \leq \infty$ , and  $r \in \mathbb{N}$ . Then*

$$2^{-nr} \|P_{2^n}(f)\|_{\dot{W}_p^r(\mathbb{R}^d)} \lesssim \omega_r(f, 2^{-n})_{L_p(\mathbb{R}^d)} \lesssim \sum_{k=n+1}^{\infty} 2^{-kr} \|P_{2^k}(f)\|_{\dot{W}_p^r(\mathbb{R}^d)},$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

To sharpen this result in the case  $1 < p < \infty$ , we will use Theorem 2.2 with  $G_n = \mathcal{B}_p^n$ ,  $X = L_p(\mathbb{R}^d)$ , and  $Y = H_p^\alpha(\mathbb{R}^d)$ ,  $\alpha > 0$ , where

$$H_p^\alpha(\mathbb{R}^d) = \{g \in L_p(\mathbb{R}^d) : \|g\|_{\dot{H}_p^\alpha(\mathbb{R}^d)} = \|(-\Delta)^{\alpha/2} g\|_{L_p(\mathbb{R}^d)} < \infty\}$$

is the fractional Sobolev spaces. The corresponding  $K$ -functional and its realization are defined similarly to (5.5) and (5.6) and, moreover, for any  $f \in L_p(\mathbb{R}^d)$ ,  $1 < p < \infty$ , and  $\alpha > 0$ ,

$$K(f, t^\alpha; L_p(\mathbb{R}^d), H_p^\alpha(\mathbb{R}^d)) \asymp R(f, t^\alpha; L_p(\mathbb{R}^d), \mathcal{B}_p^{1/t}) \asymp \omega_\alpha(f, t)_{L_p(\mathbb{R}^d)},$$

see [67]. This, in particular, implies

$$\|(-\Delta)^{\alpha/2} P_\sigma\|_{L_p(\mathbb{R}^d)} \lesssim n^\alpha \|P_\sigma\|_{L_p(\mathbb{R}^d)}, \quad P_\sigma \in \mathcal{B}_p^n, \quad 1 < p < \infty.$$

Thus, by Theorem 2.2, we obtain the following theorem.

**Theorem 6.2.** *Let  $f \in L_p(\mathbb{R}^d)$ ,  $1 < p < \infty$ ,  $\alpha > 0$ ,  $\tau = \max(2, p)$ , and  $\theta = \min(2, p)$ . Then*

$$\begin{aligned} & \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|(-\Delta)^{\alpha/2} P_{2^k}(f)\|_{L_p(\mathbb{R}^d)}^\tau \right)^{\frac{1}{\tau}} \\ & \lesssim \omega_\alpha(f, 2^{-n})_{L_p(\mathbb{R}^d)} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|(-\Delta)^{\alpha/2} P_{2^k}(f)\|_{L_p(\mathbb{R}^d)}^\theta \right)^{\frac{1}{\theta}}, \end{aligned}$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

## 6.2. The case of Fourier multipliers operators.

The Mikhlin–Hörmander multiplier theorem (cf. Assumption 3.1) states that the condition

$$\left| \frac{\partial^\beta}{\partial^{\beta_1} x_1 \dots \partial^{\beta_d} x_d} \mu(x) \right| \leq A |x|^{-|\beta|}, \quad |\beta| \equiv \beta_1 + \dots + \beta_d < \left[ \frac{d}{2} \right] + 1$$

(see [24, p. 366]) implies

$$\|T_\mu f\|_{L_p(\mathbb{R}^d)} \leq C(A, p, d) \|f\|_{L_p(\mathbb{R}^d)},$$

where  $(T_\mu f)^\wedge(x) = \mu(x)\hat{f}(x)$ . Setting

$$(\eta_\sigma f)^\wedge(x) = \eta\left(\frac{|x|}{\sigma}\right) \hat{f}(x)$$

and

$$\theta_0(f) = \eta_1 f \quad \text{and} \quad \theta_j(f) = \eta_{2^j} f - \eta_{2^{j-1}} f \quad \text{for } j \geq 1,$$

we have the following analogue of the Littlewood–Paley theorem in the case  $\mathcal{D} = \mathbb{R}^d$  (see [25, p. 20; 11, Theorem 4.1]): for  $f \in L_p(\mathbb{T}^d)$ ,  $1 < p < \infty$ , and  $\gamma > 0$ ,

$$\left\| \left\{ \sum_{j=0}^{\infty} (\theta_j(f))^2 \right\}^{1/2} \right\|_{L_p(\mathbb{R}^d)} \asymp \|f\|_{L_p(\mathbb{R}^d)}$$

and

$$\left\| \left\{ \sum_{j=1}^{\infty} 2^{2j\alpha} (\theta_j(f))^2 \right\}^{1/2} \right\|_{L_p(\mathbb{R}^d)} \asymp \|(-\Delta)^{\alpha/2} f\|_{L_p(\mathbb{R}^d)},$$

where the constants in  $\asymp$  are independent of  $f$ .

We introduce the operators  $\Psi_\sigma$  and  $\tilde{\Psi}_\sigma$  as follows:

$$(\Psi_\sigma f)^\wedge(x) = \psi\left(\frac{|x|}{\sigma}\right) \hat{f}(x),$$

$$(\tilde{\Psi}_\sigma f)^\wedge(x) = \tilde{\psi}\left(\frac{|x|}{\sigma}\right) \hat{f}(x), \quad \tilde{\psi}(\xi) = \frac{\eta(|\xi|)}{\psi(2^{-m}\xi)},$$

where a function  $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$  is such that  $\text{supp } \psi \subset [-1, 1]^d$  and for some  $m \in \mathbb{Z}_+$ ,  $\psi(x) \neq 0$  for all  $x \in [-2^{-m}, 2^{-m}]^d$ .

We are now in a position to give a version of Theorem 3.3 in the case  $\mathcal{D} = \mathbb{R}^d$ .

**Theorem 6.3.** *Let  $f \in L_p(\mathbb{R}^d)$ ,  $1 < p < \infty$ ,  $\alpha > 0$ ,  $\tau = \max(2, p)$ , and  $\theta = \min(2, p)$ .*

(A) *If  $\{\Psi_{2^k}\}$  are uniformly bounded in  $L_p(\mathbb{R}^d)$ , then*

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|(-\Delta)^{\alpha/2} \Psi_{2^k} f\|_{L_p(\mathbb{R}^d)}^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_\alpha(f, 2^{-n})_{L_p(\mathbb{R}^d)}.$$

(B) *If  $\{\tilde{\Psi}_{2^k}\}$  are uniformly bounded in  $L_p(\mathbb{R}^d)$ , then*

$$\omega_\alpha(f, 2^{-n})_{L_p(\mathbb{R}^d)} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|(-\Delta)^{\alpha/2} \tilde{\Psi}_{2^k} f\|_{L_p(\mathbb{R}^d)}^\theta \right)^{\frac{1}{\theta}}.$$

*In the above relations, the constants in  $\lesssim$  are independent of  $f$  and  $n$ .*

An analogue of Corollary 5.1 on  $\mathbb{R}^d$ , namely, inequality (5.8) holds for the following Fourier means:

(1) the  $\ell_q$ -Fourier means given by

$$\widehat{S_{n,q}f}(\xi) = \chi_{\{\xi \in \mathbb{R}^d : \|\xi\|_{\ell_q} \leq n\}}(\xi) \widehat{f}(\xi), \quad q = 1, \infty;$$

(2) the de la Vallée Poussin-type means  $\eta_n f(x)$ ;

(3) the Riesz spherical means  $R_n^{\beta,\delta}$  given by

$$\widehat{R_n^{\beta,\delta}f}(\xi) = \left(1 - \left(\frac{|\xi|}{n}\right)^\beta\right)_+^\delta \widehat{f}(\xi)$$

for  $\beta > 0$  and  $\delta > (d-1)/2$ .

At the same time, an analogue of Corollary 5.2 on  $\mathbb{R}^d$  is valid only for the de la Vallée Poussin-type means and the Riesz spherical means. Namely, for any  $f \in L_p(\mathbb{R}^d)$ ,  $p = 1, \infty$ , and  $\alpha \in \mathbb{N}$ , we have

$$2^{-n\alpha} \|T_{2^n} f\|_{\dot{W}_p(\mathbb{R}^d)} \lesssim \omega_\alpha\left(f, \frac{1}{2^n}\right)_p \lesssim \sum_{k=n+1}^{\infty} 2^{-k\alpha} \|T_{2^k} f\|_{\dot{W}_p(\mathbb{R}^d)},$$

where  $T_{2^k} f = \eta_{2^k} f$  or  $R_{2^k}^{\beta,\delta} f$  with  $\delta > (d-1)/2$ .

Finally, in this section, we give a characterization of the classical Besov spaces  $B_{p,q}^s(\mathbb{R}^d)$  in terms of best approximants and Fourier means. Using Theorems 6.1–6.3 and the same arguments as in Corollary 4.4, we derive the following corollary.

**Corollary 6.1.** *Let  $1 < p < \infty$ ,  $0 < q \leq \infty$ , and  $0 < s < \alpha$ . We have*

$$|f|_{B_{p,q}^s(\mathbb{R}^d)} \asymp \left( \sum_{k=1}^{\infty} 2^{(s-\alpha)qk} \|(-\Delta)^{\alpha/2} P_{2^k}(f)\|_{L_p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}, \quad (6.3)$$

where  $P_{2^k}(f)$  stands for the best approximants or the Fourier means  $\Psi_{2^k} f$  with the properties given in Theorem 6.3.

In the case,  $p = 1$  or  $\infty$  and  $\alpha \in \mathbb{N}$ ,  $s < \alpha$ , we have

$$|f|_{B_{p,q}^s(\mathbb{R}^d)} \asymp \left( \sum_{k=1}^{\infty} 2^{(s-\alpha)qk} \|P_{2^k}(f)\|_{\dot{W}_p^\alpha(\mathbb{R}^d)}^q \right)^{\frac{1}{q}},$$

where  $P_{2^k}(f)$  stands for the best approximants, the de la Vallée Poussin-type means  $\eta_n f(x)$ , or the Riesz spherical means  $R_n^{\beta,\delta}$  with  $\delta > (d-1)/2$ .

In the above relations, the constants in  $\asymp$  are independent of  $f$ .

Note that a similar assertion for the Gauss–Weierstrass semi-group  $W_t f(x) = (4\pi t)^{d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y) dy = (e^{-t|\xi|^2} \widehat{f}(\xi))(x)$ ,  $t > 0$ , was obtained in [3, Theorem 3.4.6, p. 198; 63, Sec. 1.13.2, pp. 76–81].

## 7. Smoothness of Approximation Processes on $[-1, 1]$

### 7.1. Sharp inequalities for algebraic polynomials

Let  $L_{w,p} = L_p([-1, 1]; w)$ ,  $0 < p \leq \infty$ , be the space of all functions  $f$  with the finite (quasi-)norm

$$\|f\|_{w,p} = \|f\|_{L_p([-1,1];w)} = \left( \int_{-1}^1 |f(x)|^p w(x) dx \right)^{\frac{1}{p}},$$

where

$$w(x) = w_{a,b}(x) = (1-x)^a(1+x)^b, \quad a, b > -1,$$

is the Jacobi weight on  $[-1, 1]$ . In the unweighted case,  $w(x) \equiv 1$ , we write  $L_p = L_p[-1, 1]$ ,  $\|f\|_p = \|f\|_{L_p[-1,1]}$ .

Further, let  $\mathcal{P}_n$  be the set of all algebraic polynomials of degree at most  $n$ . As usual, the error of the best approximation of a function  $f \in L_{w,p}$  by algebraic polynomials is defined as follows:

$$E_n(f)_{w,p} = \inf_{P \in \mathcal{P}_n} \|f - P\|_{w,p}.$$

Let  $f \in L_p[-1, 1]$ ,  $0 < p < \infty$ ,  $r \in \mathbb{N}$ ,  $\varphi(x) = \sqrt{1-x^2}$ , and  $\sigma \geq 0$ . Recall that the Ditzian–Totik modulus of smoothness  $\omega_r^\varphi(f, \delta)_p$  is given by

$$\omega_r^\varphi(f, \delta)_p = \sup_{|h| \leq \delta} \|\bar{\Delta}_{h\varphi}^r f\|_{L_p[-1,1]},$$

where

$$\bar{\Delta}_{h\varphi(x)}^r f(x) = \begin{cases} \sum_{k=0}^r (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right), & x \pm \frac{r}{2} h\varphi(x) \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

The Jackson-type theorem for the Ditzian–Totik moduli of smoothness is given by

$$E_n(f)_p \leq C(r, p) \omega_r^\varphi(f, n^{-1})_p, \quad f \in L_p[-1, 1], \quad 0 < p < \infty, \quad n > r,$$

(see [15, Theorem 1.1] for the case  $0 < p < 1$  and [20, p. 79, Theorem 7.2.1] for the case  $p \geq 1$ ). It is also well known, see, e.g. [18], that  $\omega_r^\varphi(f, \delta)_p \leq C(r, p) \|f\|_p$  and  $\omega_r^\varphi(f, 2t)_p \leq C(r, p) \omega_r^\varphi(f, t)_p$ . Thus, taking into account the following Nikolskii–Stechkin-type inequality (see [18, 28])

$$\omega_r^\varphi(P_n, \delta)_p \asymp \delta^r \|\varphi^r P_n^{(r)}\|_p, \quad 0 < p < \infty, \quad P_n \in \mathcal{P}_n, \quad 0 < \delta \leq n^{-1},$$

we see that Theorem 4.1 implies the following result (see also [28]).

**Theorem 7.1.** For any  $f \in L_p[-1, 1]$ ,  $0 < p \leq \infty$ , and  $n > r$ , we have

$$2^{-rn} \|\varphi^r P_{2^n}^{(r)}\|_{L_p[-1, 1]} \lesssim \omega_r^\varphi(f, 2^{-n})_{L_p[-1, 1]} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-r\lambda k} \|\varphi^r P_{2^k}^{(r)}\|_{L_p[-1, 1]}^\lambda \right)^{\frac{1}{\lambda}},$$

where  $\lambda = \min(1, p)$ ,  $P_n$  is a polynomial of the best approximation of  $f$  in  $L_p[-1, 1]$ , and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

Now, we are going to apply Theorems 2.2 and 3.2 in the case of the weighted  $L_p$  spaces for  $1 < p < \infty$ . First, we introduce some notations.

For  $a, b > -1$ , denote by  $P_k^{(a, b)}(x)$ ,  $k \in \mathbb{Z}_+$ , the system of Jacobi polynomials, orthogonal on  $[-1, 1]$ , such that  $P_k^{(a, b)}(1) = \binom{k+a}{k}$ ,  $k \in \mathbb{Z}_+$ . Let also  $R_k^{(a, b)}$  be the normalized Jacobi polynomials,  $R_k^{(a, b)}(x) = P_k^{(a, b)}(x)/P_k^{(a, b)}(1)$ ,  $k \in \mathbb{Z}_+$ .

The Fourier–Jacobi series of  $f \in L_{w, p}$ ,  $1 \leq p \leq \infty$ ,  $a, b > -1$ , is given by

$$f(x) \sim \sum_{k=0}^{\infty} c_k^{(a, b)}(f) \mu_k^{(a, b)} R_k^{(a, b)}(x),$$

with the Fourier coefficients

$$c_k^{(a, b)}(f) = \int_{-1}^1 f(x) R_k^{(a, b)}(x) w(x) dx, \quad k \in \mathbb{Z}_+,$$

and  $\mu_k^{(a, b)} = \|R_k^{(a, b)}\|_{L_{w, 2}}^{-2} \asymp k^{2a+1}$ .

Note that the Jacobi polynomials are the eigenfunctions of the differential operator

$$Q(D) = Q_{\alpha, \beta}(D) = \frac{-1}{w(x)} \frac{d}{dx} w(x) (1 - x^2) \frac{d}{dx},$$

$$Q(D) P_k^{(a, b)} = \lambda_k^{(a, b)} P_k^{(a, b)}, \quad \lambda_k^{(a, b)} = k(k + a + b + 1).$$

Then the corresponding  $K$ -functional is given by (3.1) with  $\sigma = 2$  and  $\mathcal{D} = [-1, 1]$ .

Recall that by (3.4) and [10, Sec. 6], we have

$$K_\gamma(f, Q(D), n^{-2\gamma})_{L_{p, w}[-1, 1]} \asymp \|f - \eta_n f\|_{L_{p, w}[-1, 1]} + n^{-2\gamma} \|Q(D)^\gamma \eta_n f\|_{L_{p, w}[-1, 1]},$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$  and the de la Vallée Poussin means  $\eta_n f$  are given by

$$\eta_n f(x) = \sum_{k=0}^{\infty} \eta \left( \frac{k}{n} \right) c_k^{(a, b)}(f) \mu_k^{(a, b)} R_k^{(a, b)}(x).$$

Thus, employing Theorems 2.2 and 3.2, and the needed facts from [10, Sec. 6], we obtain the following result.



**Theorem 7.2.** Let  $f \in L_{p,w}[-1,1]$ ,  $1 < p < \infty$ ,  $\gamma > 0$ ,  $\tau = \max(2, p)$ , and  $\theta = \min(2, p)$ . Then

$$\left( \sum_{k=n+1}^{\infty} 2^{-2\gamma\tau k} \|Q(D)^{\gamma} \eta_{2^k} f\|_{L_{p,w}[-1,1]}^{\tau} \right)^{\frac{1}{\tau}} \lesssim K_{\gamma}(f, Q(D), 2^{-2n\gamma})_{L_{p,w}[-1,1]}, \quad (7.1)$$

$$K_{\gamma}(f, Q(D), 2^{-2n\gamma})_{L_{p,w}[-1,1]} \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-2\gamma\theta k} \|Q(D)^{\gamma} \eta_{2^k} f\|_{L_{p,w}[-1,1]}^{\theta} \right)^{\frac{1}{\theta}}, \quad (7.2)$$

where the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

Inequalities (7.1) and (7.2) are also valid if we replace the de la Vallée Poussin means  $\eta_{2^k} f$  by the best approximants  $P_{2^k}(f)$ , or by the Fourier–Jacobi means  $\Psi_{2^k} f$  with the properties similar to those indicated in Theorem 3.3.

**Remark 7.1.** Note that the results given in Theorems 7.1 and 7.2 essentially improve the corresponding results for the best approximants in  $L_{p,w}[-1,1]$ ,  $1 \leq p < \infty$ , obtained early in [20, Theorem 8.3.1; 4, 28, 38, 65].

## 7.2. Sharp inequalities for splines

In this subsection, we consider approximation of functions by splines in the space  $L_p[0,1]$  with the (quasi-)norm  $\|\cdot\|_p = \|\cdot\|_{L_p[0,1]}$ .

Denote by  $\mathcal{S}_{m,n}$  the set of all spline functions of degree  $m-1$  with the knots  $t_j = t_{j,n} = j/n$ ,  $j = 0, \dots, n$ , i.e.  $S \in \mathcal{S}_{m,n}$  if  $S \in C^{m-2}[0,1]$  and  $S$  is some algebraic polynomial of degree  $m-1$  in each interval  $(t_{j-1}, t_j)$ ,  $j = 1, \dots, n$ .

Let

$$\mathcal{E}_{m,n}(f)_p = \inf_{S \in \mathcal{S}_{m,n}} \|f - S\|_{L_p[0,1]}$$

be the best approximation of a function  $f$  by splines  $S \in \mathcal{S}_{m,n}$  in  $L_p[0,1]$ .

The Jackson-type inequality is given by ([46, Theorem 1], see also [16, Chap. 12, p. 379])

$$\mathcal{E}_{r,n}(f)_p \leq C(r, p) \omega_r(f, n^{-1})_p, \quad (7.3)$$

where  $f \in L_p[0,1]$ ,  $0 < p \leq \infty$ ,  $n \in \mathbb{N}$ , and

$$\omega_r(f, \delta)_p = \sup_{0 < h \leq \delta} \|\Delta_h^r f\|_{L_p[0,1-rh]}$$

is the modulus of smoothness of order  $r \in \mathbb{N}$ .

Note that any spline  $S_n \in \mathcal{S}_{r,n}$  can be represented (see [46]) as follows:

$$S_n(x) = P(x) + \sum_{j=1}^{n-1} a_j (x - t_j)_+^{r-1},$$

where  $P \in \mathcal{P}_{r-1}$ ,  $x_+ = x$  if  $x \geq 0$  and  $x_+ = 0$  if  $x < 0$ . Moreover, one has

$$C(r, p)^{-1} n^{-(1+(r-1)p)} \sum_{j=1}^{n-1} |a_j|^p \leq \omega_r(S_n, n^{-1})_p^p \leq C(r, p) n^{-(1+(r-1)p)} \sum_{j=1}^{n-1} |a_j|^p, \quad (7.4)$$

Inequalities (7.4) were proved in [29, Lemma 2.1] (see also [27]) in the case  $1 \leq p < \infty$ . It is easy to see that the same also holds in the case  $0 < p < 1$ .

It is important to mention that (7.4) implies that for any  $S_n \in \mathcal{S}_{r,n}$ ,  $n, r \in \mathbb{N}$ , one has

$$\omega_r(S_n, n^{-1})_p \asymp n^{-(r-1)-\frac{1}{p}} V(S_n^{(r-1)})_p, \quad 0 < p < \infty, \quad (7.5)$$

where  $V(f)_p$  denotes the  $p$ -variation of the function  $f$ , that is,

$$V(f)_p = \sup_{0=x_0 < x_1 < \dots < x_n=1} \left( \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^p \right)^{\frac{1}{p}}.$$

In its turn, (7.5) implies the following analogue of the Bernstein inequality:

$$n^{-(r-1)-\frac{1}{p}} V(S_n^{(r-1)})_p \leq C(r, p) \|S_n\|_p, \quad (7.6)$$

Moreover, by (7.3) and (7.5), for any  $S_n \in \mathcal{S}_{r,n}$ ,  $n, r \in \mathbb{N}$ , such that  $\|f - S_n\|_{L_p[0,1]} = \mathcal{E}_{r,n}(f)_p$ , we have

$$\|f - S_n\|_p + n^{-(r-1+\frac{1}{p})} V(S_n^{(r-1)})_p \asymp \omega_r(f, n^{-1})_p, \quad (7.7)$$

where the constants in  $\asymp$  do not depend on  $f$ ,  $S_n$ , and  $n$ .

The above results allow us to apply Theorem 4.1 to obtain the following result.

**Theorem 7.3.** *Let  $f \in L_p[0, 1]$ ,  $0 < p < \infty$ ,  $r, n \in \mathbb{N}$ , and  $\lambda = \min(1, p)$ . Then*

$$2^{-n(r-1+\frac{1}{p})} V(S_{2^k}^{(r-1)})_p \lesssim \omega_r(f, 2^{-n})_p \lesssim \left( \sum_{k=n+1}^{\infty} \left( 2^{-k(r-1+\frac{1}{p})} V(S_{2^k}^{(r-1)})_p \right)^\lambda \right)^{\frac{1}{\lambda}},$$

where  $S_{2^k} \in \mathcal{S}_{r,2^k}$  is such that  $\|f - S_{2^k}\|_{L_p[0,1]} = \mathcal{E}_{r,2^k}(f)_p$  and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

In the case  $1 < p < \infty$ , using (7.5)–(7.7) and Theorem 2.2, we arrive at the next statement.

**Theorem 7.4.** Let  $f \in L_p[0, 1]$ ,  $1 < p < \infty$ ,  $r, n \in \mathbb{N}$ , and  $\tau = \max(2, p)$ ,  $\theta = \min(2, p)$ . Then

$$\left( \sum_{k=n+1}^{\infty} 2^{-k(r-1+\frac{1}{p})\tau} V(S_{2^k}^{(r-1)})_p^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_r(f, 2^{-n})_p$$

$$\lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k(r-1+\frac{1}{p})\theta} V(S_{2^k}^{(r-1)})_p^\theta \right)^{\frac{1}{\theta}},$$

where  $S_{2^k} \in \mathcal{S}_{r, 2^k}$  is such that  $\|f - S_{2^k}\|_{L_p[0,1]} = \mathcal{E}_{r, 2^k}(f)_p$  and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

## 8. Nonlinear Methods of Approximation

### 8.1. Nonlinear wavelet approximation

We restrict ourselves to the case of compactly supported biorthogonal wavelets and follow the discussion in [14, Sec. 7]. Let  $\varphi$  and  $\tilde{\varphi}$  be two refinable compactly supported functions in  $L_2(\mathbb{R})$ . Suppose that  $\varphi$  and  $\tilde{\varphi}$  generate two multiresolution analysis (see, e.g. [45]) and are in duality as follows:

$$\int_{\mathbb{R}} \varphi(x-j) \tilde{\varphi}(x-k) dx = \delta_{jk},$$

where  $\delta_{jk}$  is the Kronecker delta. For such functions  $\varphi$  and  $\tilde{\varphi}$ , we have

$$\varphi(x) = \sum_{k \in \mathbb{Z}} c_k \varphi(2x - k), \quad \tilde{\varphi}(x) = \sum_{k \in \mathbb{Z}} \tilde{c}_k \tilde{\varphi}(2x - k).$$

Then the corresponding wavelet functions  $\psi$  and  $\tilde{\psi}$  are given by

$$\psi(x) = \sum_{k \in \mathbb{Z}} (-1)^k \tilde{c}_{1-k} \varphi(2x - k), \quad \tilde{\psi}(x) = \sum_{k \in \mathbb{Z}} (-1)^k c_{1-k} \tilde{\varphi}(2x - k).$$

The classical example of wavelet functions is the Haar system. Set  $\varphi = \tilde{\varphi} = \chi_{[0,1]}$ , then (see, e.g. [45, p. 23])

$$\psi(x) = \tilde{\psi}(x) = \varphi(2x+1) - \varphi(2x+2) = \begin{cases} -1, & -1 \leq x < -1/2; \\ 1, & -1/2 < x \leq 0; \\ 0, & x = -1/2, \quad x > 0, \quad x < -1. \end{cases}$$

It is well known that each function  $f \in L_p(\mathbb{R})$  has the following wavelet decomposition:

$$f = \sum_{I \in \mathfrak{D}} c_{I,p}(f) \psi_{I,p}, \quad c_{I,p}(f) = \langle f, \tilde{\psi}_{I,p/(p-1)} \rangle,$$

see, e.g. [7, 12]. In the above formula,  $\mathfrak{D}$  is the set of all dyadic intervals in  $\mathbb{R}$ ,  $I$  denotes the dyadic cube  $I = 2^{-k}(j + [0, 1])$  associated with  $j, k \in \mathbb{Z}$  and

$$\psi_{I,p}(x) = |I|^{-1/p} \psi(2^k x - j).$$

Let  $\Sigma_n^w$  denote the set of all functions

$$S = \sum_{I \in \Lambda} a_I \psi_I,$$

where  $\Lambda \subset \mathfrak{D}$  is a set of dyadic intervals of cardinality  $\#\Lambda \leq n$ . Thus  $\Sigma_n^w$  is the set of all functions which are a linear combination of  $n$  wavelet functions. We define

$$\sigma_n^w(f)_p = \inf_{S \in \Sigma_n^w} \|f - S\|_{L_p(\mathbb{R})}.$$

Let  $B_{p,q}^r(\mathbb{R})$ ,  $r > 0$ ,  $0 < p, q \leq \infty$ , be the classical Besov spaces. The Jackson- and Bernstein-type inequalities are given in the following two propositions (see [8, Corollary 4.1 and Theorem 4.3]).

**Proposition 8.1.** *Let  $1 < p < \infty$ ,  $r > 0$ , and  $f \in L_p(\mathbb{R})$ ,  $1/\gamma = r + 1/p$ . If  $\psi$  has  $m$  vanishing moments with  $m > r$  and  $\psi$  is in  $B_{\gamma,q}^\rho(\mathbb{R})$  for some  $q > 0$  and some  $\rho > r$ , then*

$$\sigma_n^w(f)_p \lesssim K(f, n^{-r}; L_p(\mathbb{R}), B_{\gamma,\gamma}^r(\mathbb{R})), \quad n \in \mathbb{N},$$

where the constant in  $\lesssim$  is independent of  $f$  and  $n$ .

**Proposition 8.2.** *Let  $1 < p < \infty$ ,  $r > 0$ ,  $1/\gamma = r + 1/p$ . If  $S = \sum_{I \in \Lambda} c_{I,p}(f) \psi_I$ , with  $\#\Lambda \leq n$ , then*

$$|S|_{B_{\gamma,\gamma}^r(\mathbb{R})} \lesssim n^r \|S\|_{L_p(\mathbb{R})},$$

where the constant in  $\lesssim$  is independent of  $S$  and  $n$ .

We will also use the fact that there exists  $Q_n f \in \Sigma_n^w$  such that  $\|f - Q_n f\|_{L_p(\mathbb{R})} \lesssim \sigma_n^w(f)_p$  and

$$K(f, n^{-r}; L_p(\mathbb{R}), B_{\gamma,\gamma}^r(\mathbb{R})) \asymp \|f - Q_n f\|_{L_p(\mathbb{R})} + n^{-r} |Q_n f|_{B_{\gamma,\gamma}^r(\mathbb{R})},$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$  (see for details [8]). This realization result in particular implies the Nikolskii–Stechkin-type inequality

$$K(S, n^{-r}; L_p(\mathbb{R}), B_{\gamma,\gamma}^r(\mathbb{R})) \asymp n^{-r} |S|_{B_{\gamma,\gamma}^r(\mathbb{R})}, \quad S \in \Sigma_n^w.$$

Thus, in light of Theorem 2.2, Propositions 8.1 and 8.2, we obtain the following result.

**Theorem 8.1.** *Under conditions of Proposition 8.1, we have*

$$\begin{aligned} \left( \sum_{k=n+1}^{\infty} 2^{-r\tau k} |P_{2^k} f|_{B_{\gamma,\gamma}^r(\mathbb{R})}^\tau \right)^{\frac{1}{\tau}} &\lesssim K(f, 2^{-rn}; L_p(\mathbb{R}), B_{\gamma,\gamma}^r(\mathbb{R})) \\ &\lesssim \left( \sum_{k=n+1}^{\infty} 2^{-r\theta k} |P_{2^k} f|_{B_{\gamma,\gamma}^r(\mathbb{R})}^\theta \right)^{\frac{1}{\theta}}, \end{aligned}$$

where  $P_{2^k} f \in \Sigma_{2^k}^w$  is such that  $\|f - P_{2^k} f\|_{L_p(\mathbb{R})} = \sigma_{2^k}^w(f)_p$ ,  $\tau = \max(2, p)$ ,  $\theta = \min(2, p)$ , and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

As a corollary, we obtain the characterization of the Besov space  $B_{X,q}^r$  (interpolation space) given in (4.17) with  $X = L_p(\mathbb{R})$  and  $\Omega(f, 2^{-k})_X = K(f, 2^{-rk}, L_p(\mathbb{R}), B_{\gamma,\gamma}^r(\mathbb{R}))$ .

**Corollary 8.1.** *Under conditions of Proposition 8.1, if  $0 < \sigma < r$  and  $0 < q \leq \infty$ , then*

$$|f|_{B_{X,q}^\sigma(\mathbb{R})} \asymp \left( \sum_{k=1}^{\infty} 2^{(\sigma-r)qk} |P_{2^k} f|_{B_{\gamma,\gamma}^r(\mathbb{R})}^q \right)^{\frac{1}{q}},$$

where  $P_{2^k} f \in \Sigma_{2^k}^w$  is such that  $\|f - P_{2^k} f\|_{L_p(\mathbb{R})} = \sigma_{2^k}^w(f)_p$  and the constants in  $\asymp$  are independent of  $f$ .

## 8.2. Free knot piecewise polynomial approximation

Let  $r \in \mathbb{N}$  be fixed and for each  $n = 1, 2, \dots$ , let  $\Sigma_{r,n}$  be the set of piecewise polynomials of degree  $r$  with  $n$  pieces on  $[0, 1]$ . That is, for each element  $S \in \Sigma_{r,n}$  there is a partition  $\Lambda$  of  $[0, 1]$  consisting of  $n$  disjoint intervals  $I \subset [0, 1]$  and polynomials  $P_I \in \mathcal{P}_r$  such that

$$S = \sum_{I \in \Lambda} P_I \chi_I.$$

For each  $0 < p < \infty$ , we define the error of the best approximation by

$$\sigma_{r,n}(f)_p = \inf_{S \in \Sigma_{r,n}} \|f - S\|_{L_p[0,1]}.$$

Recall the well-known Jackson-type inequality (see [47, Theorem 2.3]).

**Proposition 8.3.** *Let  $f \in L_p[0, 1]$ ,  $0 < p < \infty$ ,  $r > 0$ ,  $k \in \mathbb{N}$ , and  $1/\gamma = r + 1/p$ . Then*

$$\sigma_{r,n}(f)_p \lesssim K(f, n^{-r}; L_p[0, 1], B_{\gamma,\gamma;k}^r[0, 1]), \quad n \in \mathbb{N}, \quad (8.1)$$

where  $B_{\gamma,\gamma;k}^r[0, 1]$  is the non-periodic Besov space, which consists of  $f \in L_\gamma[0, 1]$  such that

$$|f|_{B_{p,q;k}^r[0,1]} = \left( \int_0^{1/k} (t^{-r} \omega_k(f, t)_{L_\gamma[0,1]})^\gamma \frac{dt}{t} \right)^{1/\gamma} < \infty.$$

The constant in  $\lesssim$  is independent of  $f$  and  $n$ .

Now, using (8.1) and Theorem 4.1, we derive the following result.

**Theorem 8.2.** *Under conditions of Proposition 8.3, we have*

$$\begin{aligned} K\left(S_{2^n}, 2^{-rn}; L_p[0, 1], B_{\gamma, \gamma; k}^r[0, 1]\right) &\lesssim K\left(f, 2^{-rn}; L_p[0, 1], B_{\gamma, \gamma; k}^r[0, 1]\right) \\ &\lesssim \left(\sum_{k=n+1}^{\infty} K\left(S_{2^k}, 2^{-rk}; L_p[0, 1], B_{\gamma, \gamma; k}^r[0, 1]\right)^{\lambda}\right)^{\frac{1}{\lambda}}, \end{aligned}$$

where  $S_{2^k} \in \Sigma_{r, 2^k}$  is such that  $\|f - S_{2^k}\|_{L_p[0, 1]} = \sigma_{r, 2^k}(f)_p$ ,  $\lambda = \min(p, 1)$ , and the constants in  $\lesssim$  are independent of  $f$  and  $n$ .

Finally, we characterize the Besov space  $B_{X, q}^r$  given in (4.17) with  $X = L_p[0, 1]$  and  $\Omega(f, 2^{-k})_X = K\left(f, 2^{-rk}, L_p[0, 1], B_{\gamma, \gamma; k}^r[0, 1]\right)$ .

**Corollary 8.2.** *Let  $0 < \sigma < r$  and  $0 < q \leq \infty$ , we have*

$$|f|_{B_{X, q}^{\sigma}[0, 1]} \asymp \left(\sum_{k=1}^{\infty} 2^{\sigma q k} K\left(S_{2^k}, 2^{-rk}; L_p[0, 1], B_{\gamma, \gamma; k}^r[0, 1]\right)^q\right)^{\frac{1}{q}},$$

where  $S_{2^k} \in \Sigma_{2^k, r}$  is such that  $\|f - S_{2^k}\|_{L_p[0, 1]} = \sigma_{r, 2^k}(f)_p$  and the constants in  $\asymp$  are independent of  $f$ .

## 9. Optimality

In the previous sections, we derived the following inequalities:

$$\left(\sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|P_{2^k}(f)\|_Y^{\tau}\right)^{\frac{1}{\tau}} \lesssim K(f, 2^{-n\alpha}; L_p, Y) \lesssim \left(\sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|P_{2^k}(f)\|_Y^{\theta}\right)^{\frac{1}{\theta}}, \quad (9.1)$$

where  $f \in L_p$ ,  $1 \leq p \leq \infty$ ,

$$\tau = \begin{cases} \max(p, 2), & 1 < p < \infty, \\ \infty, & \text{otherwise,} \end{cases} \quad \theta = \begin{cases} \min(p, 2), & p < \infty, \\ 1, & p = \infty, \end{cases}$$

$Y$  is an appropriate smooth function space, and  $P_n(f)$  is a suitable approximation method. In this section, we show that the parameters  $\theta$  and  $\tau$  are optimal.

For this, we restrict ourselves to the case of  $\mathcal{D} = \mathbb{T}$  and approximation of periodic  $L_p$ -functions by  $S_n(f)$ , the  $n$ th partial sums of the Fourier series of  $f$ , and the de la Vallée Poussin means  $\eta_n f$ .

Recall that if  $f \in L_p(\mathbb{T})$ ,  $1 < p < \infty$ , then inequality (9.1) in particular implies

$$\left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\tau} \|S_{2^k}^{(\alpha)}(f)\|_p^\tau \right)^{\frac{1}{\tau}} \lesssim \omega_\alpha \left( f, \frac{1}{2^n} \right)_p \lesssim \left( \sum_{k=n+1}^{\infty} 2^{-k\alpha\theta} \|S_{2^k}^{(\alpha)}(f)\|_p^\theta \right)^{\frac{1}{\theta}}. \quad (9.2)$$

If  $f \in L_p(\mathbb{T})$ ,  $p = 1, \infty$ , and  $P_n(f) = \eta_n f$ , estimate (9.1) can be written by

$$2^{-\alpha n} \|(\eta_{2^n} f)^{(\alpha)}\|_{L_p(\mathbb{T})} \lesssim \omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T})} \lesssim \sum_{k=n}^{\infty} 2^{-2\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_p(\mathbb{T})}.$$

### 9.1. Optimality of (9.1) in the case $1 < p < \infty$

In this subsection, we deal with not only sharpness of the parameters  $\tau = \max(2, p)$  and  $\theta = \min(2, p)$  but we also show that for the classes of functions with lacunary and general monotone Fourier coefficients, inequality (9.1) becomes an equivalence with  $\tau = \theta = 2$  and  $\tau = \theta = p$ , respectively.

We start with lacunary series and first give a simple proof of Zygmund's theorem in  $L_p$ ,  $1 < p < \infty$ , based on the Littlewood–Paley technique given in Sec. 3.1. We deal with the general case of functions represented by

$$f \sim \sum_{k=0}^{\infty} A_k f, \quad A_k f = \sum_{\ell=1}^{d_k} \langle f, \psi_{k,\ell} \rangle \psi_{k,\ell}.$$

For convenience, we suppose that the dimension  $d_k = 1$  for all  $k \in \mathbb{Z}_+$ .

We will say that the Fourier expansion of  $f \in L_{p,w}(\mathcal{D})$  is lacunary, written  $f \in \Lambda$ , if  $f \sim \sum_{j=0}^{\infty} A_{2^j} f$ , i.e.  $A_k f = 0$  for  $k \neq 2^j$ ,  $j \in \mathbb{Z}_+$ .

Let us first derive an analogue of Zygmund's theorem.

**Lemma 9.1.** *Let  $1 < p < \infty$ ,  $f \in \Lambda$ , and Assumption 3.1 hold. Suppose that  $w \in L_1(\mathcal{D})$  and the functions  $\psi_k = \psi_{k,1}$  are such that*

$$0 < \xi_2 \leq \|\psi_k\|_{p,w} \leq \xi_1 < \infty \quad \text{for any } k \in \mathbb{Z}_+. \quad (9.3)$$

Then

$$\|f\|_{p,w} \asymp \left( \sum_{k=0}^{\infty} c_{2^k}(f)^2 \right)^{\frac{1}{2}}, \quad c_k(f) = \int_{\mathcal{D}} f \psi_k w.$$

In particular,  $\|f\|_{p,w} \asymp \|f\|_{2,w}$ . Here, the constants in  $\asymp$  are independent of  $f$ .

**Proof.** First, let us prove the estimate from above. Let  $1 < p \leq 2$ . Then by Hölder's inequality and Parseval's inequality, we obtain  $\|f\|_{p,w} \lesssim \|f\|_{2,w} \asymp$

$(\sum_{k=0}^{\infty} c_{2^k}(f)^2)^{\frac{1}{2}}$ . If  $p \geq 2$ , noting that

$$\theta_j(A_{2^k}f) = (\eta_{2^j} - \eta_{2^{j-1}})(A_{2^k}f) = \begin{cases} A_{2^{j-1}}f, & j = k+1, \\ 0, & j \neq k+1, \end{cases}$$

and using the Littlewood–Paley decomposition (Theorem 3.1), Minkowski’s inequality, and (9.3), we derive

$$\begin{aligned} \|f\|_{p,w} &\asymp \left\| \left( \sum_{k=0}^{\infty} \theta_k(f)^2 \right)^{\frac{1}{2}} \right\|_{p,w} = \left\| \left( \sum_{k=0}^{\infty} A_{2^k}(f)^2 \right)^{\frac{1}{2}} \right\|_{p,w} \\ &= \left( \int_{\mathcal{D}} \left( \sum_{k=0}^{\infty} (c_{2^k}(f)\psi_{2^k})^2 \right)^{\frac{p}{2}} w \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{k=0}^{\infty} \left( \int_{\mathcal{D}} |c_{2^k}(f)\psi_{2^k}|^p w \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{k=0}^{\infty} |c_{2^k}(f)|^2 \right)^{\frac{1}{2}} \max_k \left( \int_{\mathcal{D}} |\psi_{2^k}|^p w \right)^{\frac{1}{p}} \lesssim \left( \sum_{k=0}^{\infty} |c_{2^k}(f)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

To show the inverse inequality for  $p \leq 2$ , we similarly obtain

$$\begin{aligned} \|f\|_{p,w} &\gtrsim \left( \int_{\mathcal{D}} \left( \sum_{k=0}^{\infty} (c_{2^k}(f)\psi_{2^k})^2 \right)^{\frac{p}{2}} w \right)^{\frac{1}{p}} \\ &\geq \left( \sum_{k=0}^{\infty} \left( \int_{\mathcal{D}} |c_{2^k}(f)\psi_{2^k}|^p w \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \\ &\geq \left( \sum_{k=0}^{\infty} c_{2^k}(f)^2 \right)^{\frac{1}{2}} \min_k \|\psi_{2^k}\|_{p,w} \gtrsim \left( \sum_{k=0}^{\infty} c_{2^k}(f)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

If  $p \geq 2$ , Hölder’s inequality implies  $\|f\|_{2,w} \lesssim \|f\|_{p,w}$ , which proves the lemma.  $\square$

**Remark 9.1.** As an example of the system  $\{\psi_k\}$  in Lemma 9.1, one can take the trigonometric system, the Walsh system, systems of the Chebyshev polynomials and, more generally, the system of normalized Jacobi polynomials for specific range of parameters  $\alpha, \beta > -1$  indicated in [2].

**Theorem 9.1.** Under all assumptions of Lemma 9.1, we have for  $f \in L_{p,w}(\mathcal{D}) \cap \Lambda$

$$\left( \sum_{k=n+1}^{\infty} 2^{-2\gamma\sigma k} \|Q(D)^{\gamma} \eta_{2^k} f\|_{p,w}^2 \right)^{\frac{1}{2}} \asymp K_{\gamma}(f, Q(D), 2^{-n\gamma\sigma})_{p,w}, \quad \gamma > 0,$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$ .



**Proof.** Using the realization result (3.4) and Lemma 9.1, we get

$$\begin{aligned} K_\gamma(f, Q(D), 2^{-n\gamma\sigma})_{p,w} &\asymp \|f - \eta_{2^n} f\|_{p,w} + 2^{-\gamma\sigma n} \|Q(D)^\gamma \eta_{2^n} f\|_{p,w} \\ &\asymp \left( \sum_{k=n-1}^{\infty} c_{2^k}(f)^2 \right)^{\frac{1}{2}} + 2^{-\gamma\sigma n} \left( \sum_{k=1}^{n-1} 2^{2\gamma\sigma k} c_{2^k}(f)^2 \right)^{\frac{1}{2}} \end{aligned} \quad (9.4)$$

and

$$2^{-2\gamma\sigma k} \|Q(D)^\gamma \eta_{2^k} f\|_{p,w}^2 \asymp 2^{-2\gamma\sigma k} \sum_{l=1}^{k-1} 2^{2\gamma\sigma l} c_{2^l}(f)^2.$$

Then

$$\begin{aligned} \sum_{k=n+1}^{\infty} 2^{-2\gamma\sigma k} \|Q(D)^\gamma \eta_{2^k} f\|_{p,w}^2 &\asymp \sum_{k=n+1}^{\infty} 2^{-2\gamma\sigma k} \sum_{l=1}^{k-1} 2^{2\gamma\sigma l} c_{2^l}(f)^2 \\ &= \sum_{k=n+1}^{\infty} 2^{-2\gamma\sigma k} \left( \sum_{l=1}^n + \sum_{l=n+1}^{k-1} 2^{2\gamma\sigma l} c_{2^l}(f)^2 \right) \\ &\asymp 2^{-2\gamma\sigma n} \sum_{l=1}^n 2^{2\gamma\sigma l} c_{2^l}(f)^2 + \sum_{l=n+1}^{\infty} 2^{2\gamma\sigma l} c_{2^l}(f)^2 \\ &\asymp K_\gamma(f, Q(D), 2^{-n\gamma\sigma})_{p,w}^2. \quad \square \end{aligned}$$

In particular, for the classical Fourier series on  $\mathcal{D} = \mathbb{T}$ , we obtain

$$\omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T})} \asymp \left( \sum_{k=n}^{\infty} 2^{-2\alpha k} \|(S_{2^k} f)^{(\alpha)}\|_{L_p(\mathbb{T})}^2 \right)^{\frac{1}{2}}, \quad f \in L_p(\mathbb{T}) \cap \Lambda, \quad (9.5)$$

where  $1 < p < \infty$  and  $\alpha > 0$ ; cf. (9.2).

**Remark 9.2.** It is clear that (9.5) gives the sharpness of the parameter  $\theta$  for  $p \geq 2$  and  $\tau$  for  $p \leq 2$  in inequality (9.2).

**Proof.** Assume that  $p \geq 2$  and there holds

$$\omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T})} \lesssim \left( \sum_{k=n}^{\infty} \left( 2^{-\alpha k} \|(S_{2^k} f)^{(\alpha)}\|_{L_p(\mathbb{T})} \right)^{2+\varepsilon} \right)^{\frac{1}{2+\varepsilon}} \quad (9.6)$$

with some  $\varepsilon > 0$ . Consider  $f(x) = \sum_{n=1}^{\infty} a_{2^n} \cos 2^n x$ , where  $a_{2^n} = 1/n$ . Then  $f \in L_p(\mathbb{T}) \cap \Lambda$  and, by (9.4), one has

$$\begin{aligned} \omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T})} &\asymp \left( \sum_{k=n}^{\infty} a_{2^k}^2 \right)^{\frac{1}{2}} + 2^{-\alpha n} \left( \sum_{k=1}^n 2^{2\alpha k} a_{2^k}^2 \right)^{\frac{1}{2}} \asymp \frac{1}{n^{1/2}}, \\ 2^{-\alpha k} \|(S_{2^k} f)^{(\alpha)}\|_{L_p(\mathbb{T})} &\asymp \frac{1}{k}, \quad \left( \sum_{k=n}^{\infty} 2^{-(2+\varepsilon)\alpha k} \|(S_{2^k} f)^{(\alpha)}\|_{L_p(\mathbb{T})}^{2+\varepsilon} \right)^{\frac{1}{2+\varepsilon}} \asymp n^{-\frac{1+\varepsilon}{2+\varepsilon}}, \end{aligned}$$

which contradicts (9.6). Similarly, if  $p \leq 2$ , then the inequality

$$\omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T})} \gtrsim \left( \sum_{k=n}^{\infty} \left( 2^{-\alpha k} \| (S_{2^k} f)^{(\alpha)} \|_{L_p(\mathbb{T})} \right)^{2-\varepsilon} \right)^{\frac{1}{2-\varepsilon}}$$

with some  $\varepsilon \in (0, 2)$  does not hold for  $f(x) = \sum_{n=1}^{\infty} a_{2^n} \cos 2^n x \in L_p$ , where  $a_{2^n} = n^{-1/(2-\varepsilon)}$ .  $\square$

Now, let us consider the case of the classical Fourier series with general monotone coefficients. In what follows, we say (see [60]) that a (complex) sequence  $\{d_n\}$  is general monotone, written  $\{d_n\} \in GM$ , if

$$\sum_{k=n}^{2n} |d_k - d_{k+1}| \leq C |d_n|,$$

where  $C$  does not depend on  $n$ . Note that any monotone (quasi-monotone) sequences are general monotone. We denote by  $\widehat{GM}$  the class of integrable functions such that  $f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$  with  $\{a_n\}, \{b_n\} \in GM$ .

**Theorem 9.2.** *Let  $f \in L_p(\mathbb{T}) \cap \widehat{GM}$ ,  $1 < p < \infty$ , and  $\alpha > 0$ . Then*

$$\omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T})} \asymp \left( \sum_{k=n}^{\infty} 2^{-p\alpha k} \| (S_{2^k} f)^{(\alpha)} \|_{L_p(\mathbb{T})}^p \right)^{\frac{1}{p}}, \quad (9.7)$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$ .

**Proof.** First, we recall the following Hardy–Littlewood theorem:

$$\|f\|_{L_p(\mathbb{T})} \asymp \left( \sum_{n=1}^{\infty} (|a_n| + |b_n|)^p n^{p-2} \right)^{\frac{1}{p}}, \quad f \in L_p(\mathbb{T}) \cap \widehat{GM}, \quad 1 < p < \infty.$$

This is a well-known fact for functions with monotone coefficients, see [71, Chap. XII]. For the class  $\widehat{GM}$  (in fact for a more general class and for Lorentz spaces) this has been recently proved in [22]. Moreover, it is also shown in [22] that

$$\omega_\alpha(f, n^{-1})_{L_p(\mathbb{T})} \asymp n^{-\alpha} \left( \sum_{k=0}^n (|a_k| + |b_k|)^p k^{p\alpha+p-2} \right)^{\frac{1}{p}} + \left( \sum_{k=n}^{\infty} (|a_k| + |b_k|)^p k^{p-2} \right)^{\frac{1}{p}}.$$

Now, we note that the sequences  $\{d_1, \dots, d_n, 0, 0, \dots\}$  and  $\{n^\alpha d_n\}$  belong to  $GM$  whenever  $\{d_n\} \in GM$ , which implies that the Hardy–Littlewood theorem can be applied for the partial Fourier sums of  $f$ . Moreover, since any general monotone sequence  $\{d_n\}$  satisfies the following property, see [60]:  $|d_k| \leq C |d_n|$  for  $n \leq k \leq 2n$ ,

we have

$$\|(S_{2^n} f)^{(\alpha)}\|_{L_p(\mathbb{T})} \asymp \left( \sum_{k=0}^n (|a_{2^k}| + |b_{2^k}|)^p 2^{k(p\alpha+p-1)} \right)^{\frac{1}{p}}.$$

Thus, we derive

$$\begin{aligned} \omega_\alpha(f, 2^{-n})_{L_p(\mathbb{T})} &\asymp 2^{-\alpha n} \left( \sum_{k=0}^n (|a_{2^k}| + |b_{2^k}|)^p 2^{k(p\alpha+p-1)} \right)^{\frac{1}{p}} \\ &\quad + \left( \sum_{k=n}^{\infty} (|a_{2^k}| + |b_{2^k}|)^p 2^{k(p-1)} \right)^{\frac{1}{p}} \\ &\asymp \left( \sum_{k=n}^{\infty} 2^{-p\alpha k} \sum_{l=0}^k (|a_{2^l}| + |b_{2^l}|)^p 2^{l(p\alpha+p-1)} \right)^{\frac{1}{p}} \\ &\asymp \left( \sum_{k=n}^{\infty} 2^{-p\alpha k} \|(S_{2^k} f)^{(\alpha)}\|_{L_p(\mathbb{T})}^p \right)^{\frac{1}{p}}, \end{aligned}$$

completing the proof.  $\square$

**Remark 9.3.** Similarly to Remark 9.2, equivalence (9.7) provides the sharpness of the parameter  $\theta$  for  $p \leq 2$  and  $\tau$  for  $p \geq 2$  in (9.2).

## 9.2. Optimality of the right-hand inequality in (9.1) for $p = 1$ and $p = \infty$

We start by obtaining two simple results for lacunary Fourier series.

**Theorem 9.3.** *Let  $f \in L_1(\mathbb{T}) \cap \Lambda$  and  $\alpha > 0$ . Then*

$$\omega_\alpha(f, 2^{-n})_{L_1(\mathbb{T})} \asymp \left( \sum_{k=n}^{\infty} 2^{-2\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_1(\mathbb{T})}^2 \right)^{\frac{1}{2}},$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$ .

**Proof.** The proof repeats the one of Theorem 9.1 since by Zygmund's theorem (see [24, Theorem 3.7.4]), we have

$$\omega_\alpha(f, 2^{-n})_{L_1(\mathbb{T})} \asymp \left( \sum_{k=n}^{\infty} |c_{2^k}|^2 \right)^{\frac{1}{2}} + 2^{-\alpha n} \left( \sum_{k=1}^n 2^{2\alpha k} |c_{2^k}|^2 \right)^{\frac{1}{2}},$$

where  $\{c_k\}$  are the Fourier coefficients of  $f$ .  $\square$

**Theorem 9.4.** *Let  $f \in L_\infty(\mathbb{T}) \cap \Lambda$  and  $\alpha > 0$ . Then*

$$\omega_\alpha(f, 2^{-n})_{L_\infty(\mathbb{T})} \asymp \sum_{k=n}^{\infty} 2^{-\alpha k} \|\eta_{2^k}^{(\alpha)} f\|_{L_\infty(\mathbb{T})},$$

where the constants in  $\asymp$  are independent of  $f$  and  $n$ .

**Proof.** By Stechkin's theorem (see [24, Theorem 3.7.6]), we have

$$\begin{aligned} \sum_{k=n}^{\infty} 2^{-\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_\infty(\mathbb{T})} &\asymp \sum_{k=n}^{\infty} 2^{-\alpha k} \sum_{s=1}^{n-1} 2^{\alpha s} |c_{2^s}| + \sum_{k=n}^{\infty} 2^{-\alpha k} \sum_{s=n}^k 2^{\alpha s} |c_{2^s}| \\ &\asymp 2^{-\alpha n} \|(\eta_{2^n} f)^{(\alpha)}\|_{L_\infty(\mathbb{T})} + \sum_{k=n}^{\infty} |c_{2^k}| \\ &\asymp 2^{-\alpha n} \|(\eta_{2^n} f)^{(\alpha)}\|_{L_\infty(\mathbb{T})} + E_{2^n}(f)_\infty \asymp \omega_\alpha(f, 2^{-n})_{L_\infty(\mathbb{T})}. \end{aligned}$$

□

Note that Theorem 9.4 shows that in the case  $p = \infty$ , the right-hand inequality (9.1) is sharp for  $\theta = 1$ , in other words this inequality cannot be improved for some  $\theta > 1$  in the general case. At the same time, we remark that Theorem 9.3 only shows that in the case  $p = 1$ , the right-hand inequality (9.1) is sharp for  $\theta = 2$ , that is, (9.1) cannot be sharpen with any  $\theta > 2$ .

Now, we show that (9.1) is in fact sharp for  $\theta = 1$ .

**Theorem 9.5.** *Let  $\alpha \in \mathbb{N}$ . Then for any  $q > 1$  there exists a function  $f \in L_1(\mathbb{T})$  such that*

$$\omega_\alpha(f, 2^{-n})_{L_1(\mathbb{T})} \leq C \left( \sum_{k=n+1}^{\infty} 2^{-q\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_1(\mathbb{T})}^q \right)^{\frac{1}{q}} \quad (9.8)$$

is not valid with a constant  $C$  independent of  $f$  and  $n$ .

**Proof.** We will use the following well-known Kolmogorov's estimates for the  $L_1$ -norms of trigonometric series:

$$\int_0^\pi \left| \sum_{k=1}^{\infty} a_k \cos kx \right| dx \lesssim \sum_{k=1}^{\infty} k |\Delta^2 a_k|, \quad (9.9)$$

$$\int_0^\pi \left| \sum_{k=1}^{\infty} a_k \sin kx \right| dx \lesssim \sum_{k=1}^{\infty} k |\Delta^2 a_k| + \sum_{k=1}^{\infty} \frac{|a_k|}{k}, \quad (9.10)$$

where  $\Delta^2 a_k = a_{k+2} - 2a_{k+1} + a_k$ . Inequality (9.9) was obtained in [32], see also [58]; for inequality (9.10) see [58].

We will also need the following estimate for the error of the best approximation given by (see [23, Lemma 2]):

$$E_n(g)_{L_1(\mathbb{T})} \gtrsim \left| \sum_{k=n+1}^{\infty} \frac{a_k}{k} \right|, \quad g(x) \sim \sum_{k=1}^{\infty} a_k \sin kx \in L_1(\mathbb{T}). \quad (9.11)$$

Now, consider the function

$$f_N(x) = \sum_{k=1}^N \frac{\sin kx}{\log^{\gamma}(k+1)},$$

where  $N > 2^n$  and  $0 < \gamma < 1/q$ . By the Jackson inequality and (9.11), we obtain

$$\begin{aligned} \omega_{\alpha}(f_N, 2^{-n})_{L_1(\mathbb{T})} &\gtrsim E_{2^n}(f_N)_{L_1(\mathbb{T})} \\ &\gtrsim \sum_{k=2^n+1}^N \frac{1}{k \log^{\gamma}(k+1)} \asymp \log^{1-\gamma} N - \log^{1-\gamma} 2^n. \end{aligned} \quad (9.12)$$

Next, if  $\alpha$  is odd, by (9.9), we derive

$$\begin{aligned} \|(\eta_{2^m} f_N)^{(\alpha)}\|_{L_1(\mathbb{T})} &= \left\| \eta_{2^m} \left( \sum_{k=1}^N \frac{k^{\alpha}}{\log^{\gamma}(k+1)} \cos kx \right) \right\|_{L_1(\mathbb{T})} \\ &\lesssim \sum_{k=1}^{2^m} \frac{k^{\alpha-1}}{\log^{\gamma}(k+1)} \lesssim \frac{2^{\alpha m}}{m^{\gamma}}. \end{aligned}$$

Similarly, if  $\alpha$  is even, (9.10) implies that

$$\|(\eta_{2^m} f_N)^{(\alpha)}\|_{L_1(\mathbb{T})} \lesssim \frac{2^{\alpha m}}{m^{\gamma}}.$$

Thus, for all  $\alpha \in \mathbb{N}$ , we have

$$\begin{aligned} &\sum_{m=n}^{\infty} \left( 2^{-\alpha m} \|(\eta_{2^m} f_N)^{(\alpha)}\|_{L_1(\mathbb{T})} \right)^q \\ &\lesssim \sum_{m=n}^{\lfloor \log N \rfloor} \frac{1}{m^{\gamma q}} + \sum_{m=\lfloor \log N \rfloor+1}^{\infty} 2^{-\alpha q m} \frac{N^{\alpha q}}{(\log N)^{\gamma q}} \\ &\lesssim 7 \frac{\log N}{(\log N)^{\gamma q}}. \end{aligned} \quad (9.13)$$

Combining (9.12) and (9.13), it is easy to see that inequality (9.8) is not valid for  $f = f_N$  with sufficiently large  $N$ .  $\square$

### 9.3. Optimality of the left-hand inequality in (9.1) for $p = 1$ and $p = \infty$

In this subsection, we show that the left-hand inequality in (9.1) cannot be improved in general. In particular, for  $p = 1$  or  $p = \infty$ , the following inequality is not valid for any  $q > 0$ :

$$\left( \sum_{k=n+1}^{\infty} 2^{-q\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_p(\mathbb{T})}^q \right)^{\frac{1}{q}} \leq C \omega_{\alpha}(f, 2^{-n})_{L_p(\mathbb{T})}. \quad (9.14)$$

**Theorem 9.6.** *Let  $p = 1$  or  $\infty$  and  $\alpha \in \mathbb{N}$ . Then for any  $q > 0$  there exists a function  $f \in L_p(\mathbb{T})$  such that inequality (9.14) is not valid with a constant  $C$  independent of  $f$  and  $n$ .*

**Proof.** Let  $p = \infty$ . We take

$$f(x) = \sum_{m=1}^{\infty} a_m \sin mx, \quad a_m = \frac{1}{m \log^{\gamma}(m+1)}, \quad \gamma > 1.$$

Since  $a_m \searrow 0$  and  $ma_m \rightarrow 0$ , we have  $f \in C(\mathbb{T})$ , see, e.g. [71, Chap. V].

By [61], we get

$$E_n(f)_{L_{\infty}(\mathbb{T})} \asymp \max_{\nu \geq 1} \nu a_{\nu+n} \asymp \max_{\nu \geq 1} \frac{\nu}{(\nu+n) \log^{\gamma}(\nu+n+1)} \asymp \frac{1}{\log^{\gamma} n}. \quad (9.15)$$

Next,

$$\|(\eta_{2^k} f)^{(\alpha)}\|_{L_{\infty}(\mathbb{T})} = \left\| \eta_{2^k} \left( \sum_{m=1}^{\infty} \frac{m^{\alpha-1} \cos(mx + \alpha\pi)}{\log^{\gamma}(m+1)} \right) \right\|_{L_{\infty}(\mathbb{T})}.$$

If  $\alpha$  is even, we obviously have

$$\|(\eta_{2^k} f)^{(\alpha)}\|_{L_{\infty}(\mathbb{T})} \asymp \sum_{m=1}^{2^k} \eta \left( \frac{m}{2^k} \right) \frac{m^{\alpha-1}}{\log^{\gamma}(m+1)} \asymp \frac{2^{\alpha k}}{k^{\gamma}}. \quad (9.16)$$

For odd  $\alpha$ , using Bernstein's inequality, we derive

$$\begin{aligned} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_{\infty}(\mathbb{T})} &\geq \frac{1}{2^k} \|(\eta_{2^k} f)^{(\alpha+1)}\|_{L_{\infty}(\mathbb{T})} \\ &= \frac{1}{2^k} \left\| \eta_{2^k} \left( \sum_{m=1}^{\infty} \frac{m^{\alpha} \cos mx}{\log^{\gamma}(m+1)} \right) \right\|_{L_{\infty}(\mathbb{T})} \asymp \frac{2^{\alpha k}}{k^{\gamma}}. \end{aligned} \quad (9.17)$$

Due to (9.15)–(9.17), and using the realization result, we have

$$\omega_{\alpha}(f, 2^{-n})_{L_{\infty}(\mathbb{T})} \asymp \frac{1}{n^{\gamma}}.$$

At the same time, by (9.16) and (9.17), we derive

$$\left( \sum_{k=n+1}^{\infty} 2^{-q\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_{\infty}(\mathbb{T})}^q \right)^{\frac{1}{q}} \asymp \frac{n^{\frac{1}{q}}}{n^{\gamma}}.$$

The last two formula imply that inequality (9.14) is not valid in the case  $p = \infty$ .

Now, let us consider the case  $p = 1$ . We put

$$f(x) = \sum_{m=1}^{\infty} a_m \cos mx, \quad a_m = \frac{1}{\log^{\gamma}(m+1)}, \quad \gamma > 1.$$

Since  $a_m \searrow 0$  and  $\Delta^2 a_m \geq 0$ , we have  $f \in L_1(\mathbb{T})$ , see, e.g. [71, Chap. V].

Recall that if a convex sequence  $\{a_m\}$  is the sequence of cosine Fourier coefficients of an even function  $f \in L_1(\mathbb{T})$ , then applying [1, Theorem 1], we have

$$\omega_{\alpha}(f, 2^{-n})_{L_1(\mathbb{T})} \lesssim \frac{1}{2^{\alpha n}} \sum_{m=1}^{2^n} m^{\alpha-1} a_m \lesssim \frac{1}{n^{\gamma}}. \quad (9.18)$$

Next, since for any  $g \in L_1(\mathbb{T})$  and  $k \in \mathbb{N}$ , one has  $\|g\|_{L_1(\mathbb{T})} \geq 2\pi|\widehat{g}(2^k)|$ , it follows that

$$\|(\eta_{2^k} f)^{(\alpha)}\|_{L_1(\mathbb{T})} = \left\| \eta_{2^k} \left( \sum_{m=1}^{\infty} \frac{m^{\alpha} \cos(mx + \alpha\pi)}{\log^{\gamma}(m+1)} \right) \right\|_{L_1(\mathbb{T})} \gtrsim \frac{2^{\alpha k}}{k^{\gamma}}$$

and, therefore,

$$\left( \sum_{k=n+1}^{\infty} 2^{-q\alpha k} \|(\eta_{2^k} f)^{(\alpha)}\|_{L_1(\mathbb{T})}^q \right)^{\frac{1}{q}} \gtrsim \frac{n^{\frac{1}{q}}}{n^{\gamma}}. \quad (9.19)$$

Finally, combining (9.18) and (9.19), we obtain contradiction to (9.14).  $\square$

## Acknowledgments

The first author was supported by the DFG Project KO 5804/1-1. The second author was partially supported by the MTM 2017-87409-P, 2017 SGR 358, and by the CERCA Programme of the Generalitat de Catalunya. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme “Approximation, sampling and compression in data science” where part of the work on this paper was undertaken. This work was supported by the EPSRC Grant No. EP/K032208/1.

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