INTERSECTION PAIRING AND INTERSECTION MOTIVE OF SURFACES

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Abstract. We construct the Chow motive modelling intersection cohomology of a proper surface. We then study its functoriality properties. Using Murre's decompositions of the motive of a desingularization into Künneth components [Mr1], we show that such decompositions exist also for the intersection motive.

Contents

0. Introduction

Let Y be a variety over a field k. Depending on k, different cohomology theories may be available, which associate to Y objects $H^n(Y)$ of linear algebra. When Y is proper and smooth, then these objects should be considered as being *pure*. Grothendieck's theory of motives provides a way to study those properties of these objects which are intrinsically geometrical, in particular, which do not depend on the choice of cohomology theory.

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Intersection cohomology allows to associate pure objects to proper, but possibly singular varieties Y. It seems a natural question to ask whether it is possible to construct a motive modelling intersection cohomology. In different contexts, affirmative answers are known. Scholl [S1, Sect. 1.1, 1.2] constructed the Chow motive modelling intersection cohomology of a modular curve with coefficients. Gordon, Hanamura and Murre [GHaMr] gave a construction of a Chow motive over $\mathbb{C}$ modelling intersection cohomology of the Baily–Borel compactification of a Hilbert–Blumenthal variety with coefficients. So far, nobody seems to have given a name to this motive. Practical rather than other considerations (aesthetical ones, for example) led me to baptize it the *intersection motive*. In the category of Grothendieck motives, Nair [N] constructed the intersection motive of the Baily–Borel compactification of any (pure) Shimura variety, essentially characterizing it by its stability under the action of the Hecke algebra.

The modest aim of the present work is to construct the intersection (Chow) motive of a proper surface $\overline{X}$ over an arbitrary base field k.

In order to deduce the recipe for its construction, we compute in Section 1 the intersection cohomology of $\overline{X}$ in terms of the cohomology of a desingularization X' (Theorem 1.1). The result, predicted by the decomposition theorem of [BBD], is that the former is a direct factor of the latter. In fact, we get more: it is a direct factor with a *canonical* complement, which is given by the second cohomology of the exceptional divisor D of X'. This is a consequence of the very special geometric situation we are considering. More concretely, it boils down to the well-known non-degeneracy of the intersection pairing on the components of D.

This latter observation allows to directly translate the construction into the motivic world. This is done in Section 2. We get a canonical decomposition

$$h(X') = h_{!*}(\overline{X}) \oplus h^2(D)$$

in the category of Chow motives over k. This category is pseudo-Abelian. The above decomposition should be considered as remarkable: to construct a sub-motive of $h(X')$ does not a priori necessitate the *construction*, but only the *existence* of a complement. In our situation, the complement *is* canonical, thanks to the very special geometrical situation. This point is reflected by the rather subtle functoriality properties of $h_{!*}(\overline{X})$ (Proposition 2.4): viewed as a sub-motive of $h(X')$, it is respected by pull-backs, viewed as a quotient, it is respected by push-forwards under dominant morphisms of surfaces.

The final Section 3 is devoted to the existence of Künneth decompositions of $h_{!*}(\overline{X})$. The main ingredient is of course Murre's construction of projectors for the motive $h(X')$ [Mr1]. Theorem 3.5 shows how to adapt these to our construction.

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Notations and convention: k denotes a fixed base field, and CH stands for the tensor product with $\mathbb{Q}$ of the Chow group. Our standard reference for Chow motives is Scholl's survey article [S2].

1. Intersection cohomology of surfaces

In order to motivate the construction of the intersection motive, to be given in next section, we shall compute the *intersection cohomology* of a complex surface.

Thus, throughout this section, our base field k will be equal to $\mathbb{C}$. We consider the following situation:

$$X \overset{j}{\hookrightarrow} X^* \overset{i}{\hookleftarrow} Z$$

The morphism i is a closed immersion of a sub-scheme Z, with complement j. The scheme X^* is a surface over $\mathbb{C}$, all of whose singularities are contained in Z. Thus, the surface X is smooth.

Our aim is to compute the intersection cohomology groups $H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q})$. Note that since X is smooth, the complex $\mathbb{Q}_X[2]$ consisting of the constant local system $\mathbb{Q}$, placed in degree -2, can be viewed as a *perverse sheaf* on $X(\mathbb{C})$ (for the middle perversity) [BBD, Sect. 2.2.1]. Hence its *intermediate* *extension* $j_{!*}\mathbb{Q}_X[2]$ [BBD, (2.2.3.1)] is defined as a perverse sheaf on $X^*(\mathbb{C})$. By definition,

$$H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q}) = H^{n-2}(X^*(\mathbb{C}), j_{!*}\mathbb{Q}_X[2]) \ , \ \forall\, n \in \mathbb{Z} \ .$$

In order to identify $H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q})$, note first that the normalization of X^* is finite over X^*, and the direct image under finite morphisms is exact for the perverse t-structure [BBD, Cor. 2.2.6 (i)]. Therefore, intersection cohomology is invariant under passage to the normalization. In the sequel, we therefore assume that X^* is normal. In particular, its singularities are isolated.

Next, note that if X^* is smooth, then the complex $j_{!*}\mathbb{Q}_X[2]$ equals $\mathbb{Q}_{X^*}[2]$. Transitivity of $j_{!*}$ [BBD, (2.1.7.1)] shows that we may enlarge X, and hence assume that the closed sub-scheme Z is finite.

Choose a resolution of singularities. More precisely, consider in addition the following diagram, assumed to be cartesian:

$$\begin{tikzcd} X \arrow[r, hook, "j'"] \arrow[d, equal] & X' \arrow[d, "\pi"'] & D \arrow[l, hook', "i'"'] \arrow[d, "\pi"'] \\ X \arrow[r, hook, "j"] & X^* & Z \arrow[l, hook', "i"'] \end{tikzcd}$$

The morphism π is assumed proper (and birational) and the surface X', smooth. We then have the following result.

Theorem 1.1. *(i) For $n \neq 2$,*

$$H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q}) = H^n(X'(\mathbb{C}),\mathbb{Q}) \ .$$

(ii) The group $H^2_{!*}(X^*(\mathbb{C}),\mathbb{Q})$ is a direct factor of $H^2(X'(\mathbb{C}),\mathbb{Q})$, with a canonical *complement. As a sub-group, this complement is given by the map*

$$i'_* : H^2_{D(\mathbb{C})}(X'(\mathbb{C}),\mathbb{Q}) \longrightarrow H^2(X'(\mathbb{C}),\mathbb{Q})$$

from cohomology with support in $D(\mathbb{C})$; this map is injective. As a quotient, the complement is given by the restriction

$$i'^* : H^2(X'(\mathbb{C}),\mathbb{Q}) \longrightarrow H^2(D(\mathbb{C}),\mathbb{Q}) \ ;$$

this map is surjective.

Note that this result is compatible with further blow-up of X' in points belonging to D.

In order to prove Theorem 1.1, we need to construct the maps between $H^n_{!*}(X^*(\mathbb{C}),\mathbb{Q})$ and $H^n(X'(\mathbb{C}),\mathbb{Q})$ leading to the above identifications.

Consider the total direct image $\pi_*\mathbb{Q}_{X'}$; following the convention used in [BBD], we drop the letter "R" from our notation.

Lemma 1.2. *The complex $\pi_*\mathbb{Q}_{X'}[2]$ is a perverse sheaf on X^*.*

Proof. Let P be a point (of Z) over which π is not an isomorphism, and denote by i_P its inclusion into X^*. By definition [BBD, Déf. 2.1.2], we need to check that (a) the higher inverse images $H^n i_P^* \pi_* \mathbb{Q}_{X'}$ vanish for $n > 2$, (b) the higher exceptional inverse images $H^n i^!_P \pi_* \mathbb{Q}_{X'}$ vanish for $n < 2$.

(a) By proper base change, the group in question equals $H^n(\pi^{-1}(P),\mathbb{Q})$. Since $\pi^{-1}(P)$ is of dimension at most one, there is no cohomology above degree two.

(b) The surface X' is smooth. Duality and proper base change imply that the group in question is abstractly isomorphic to the dual of $H^{4-n}(\pi^{-1}(P),\mathbb{Q})$. This group vanishes if $4-n$ is strictly larger than two. **q.e.d.**

For $a \in \mathbb{Z}$, denote by $\tau_{\le a}$ the functor associating to a complex the a-th step of its canonical filtration (with respect to the classical t-structure). Recall that $j_{!*}\mathbb{Q}_X[2]$ equals $\tau_{\le -1}(j_*\mathbb{Q}_X[2])$ [BBD, Prop. 2.1.11]. We now see how to relate it to $\pi_*\mathbb{Q}_{X'}[2]$: apply π_* to the exact triangle

$$i'_* i'^! \mathbb{Q}_{X'} \longrightarrow \mathbb{Q}_{X'} \longrightarrow j'_* \mathbb{Q}_X \longrightarrow i'_* i'^! \mathbb{Q}_{X'}[1] \ .$$

This gives an exact triangle

$$i_* F[0] \longrightarrow \tau_{\le -1}(\pi_*\mathbb{Q}_{X'}[2]) \longrightarrow j_{!*}\mathbb{Q}_X[2] \longrightarrow i_*F[1] \ ;$$

in fact, as in the proof of Lemma 1.2, one sees that F is a sheaf (concentrated in Z). More precisely, the restriction to any point P of Z of this sheaf equals the kernel of the composition

$$i'^* i'_* : H^2_{\pi^{-1}(P)}(X'(\mathbb{C}),\mathbb{Q}) \longrightarrow H^2(X'(\mathbb{C}),\mathbb{Q}) \longrightarrow H^2(\pi^{-1}(P),\mathbb{Q}) \ .$$

We thus get the following.

Lemma 1.3. *There is a canonical exact sequence*

$$0 \longrightarrow i_*F[0] \longrightarrow \tau_{\le -1}(\pi_*\mathbb{Q}_{X'}[2]) \longrightarrow j_{!*}\mathbb{Q}_X[2] \longrightarrow 0$$

of perverse sheaves on X^*.

Proof of Theorem 1.1. We shall show that the composition

$$i'^*i'_* : H^2_{D(\mathbb{C})}(X'(\mathbb{C}),\mathbb{Q}) \longrightarrow H^2(D(\mathbb{C}),\mathbb{Q})$$

is in fact an isomorphism. This implies that the sheaf F is zero. It also implies injectivity of

$$i'_* : H^2_{D(\mathbb{C})}(X'(\mathbb{C}),\mathbb{Q}) \longrightarrow H^2(X'(\mathbb{C}),\mathbb{Q}) \ ,$$

as well as surjectivity of

$$i'^* : H^2(X'(\mathbb{C}),\mathbb{Q}) \longrightarrow H^2(D(\mathbb{C}),\mathbb{Q}) \ .$$

Hence the statement of our theorem.

In order to prove bijectivity of $i'^*i'_*$, note that we may assume that D is a divisor, whose irreducible components are smooth. Indeed, if $f : X'' \to X'$ is a further blow-up, such that $f^{-1}(D)$ has the required property [Hi, Thm. $I_2^{N,n}$], then the push-forward f_* is a left inverse of the pull-back f^*, and the diagrams involving cohomology of $D(\mathbb{C})$ and $f^{-1}(D(\mathbb{C}))$, and cohomology with support in $D(\mathbb{C})$ and $f^{-1}(D(\mathbb{C}))$, respectively, commute thanks to proper base change. Therefore, bijectivity on the level of X' follows from bijectivity on the level of X''.

If D_m are the irreducible components of D, then the closed covering $D = \cup_m D_m$ induces canonical isomorphisms

$$\bigoplus_m H^2_{D_m(\mathbb{C})}(X'(\mathbb{C}),\mathbb{Q}) \xrightarrow{\ \sim\ } H^2_{D(\mathbb{C})}(X'(\mathbb{C}),\mathbb{Q})$$

and

$$H^2(D(\mathbb{C}),\mathbb{Q}) \xrightarrow{\ \sim\ } \bigoplus_m H^2(D_m(\mathbb{C}),\mathbb{Q}) \ .$$

Purity identifies each $H^2_{D_m(\mathbb{C})}(X'(\mathbb{C}),\mathbb{Q})$ with $H^0(D_m(\mathbb{C}),\mathbb{Q})(-1)$ (it is here that we use that the D_m are smooth). The induced morphism

$$i'^*i'_* : \bigoplus_m H^0(D_m(\mathbb{C}),\mathbb{Q}) \longrightarrow \bigoplus_m H^2(D_m(\mathbb{C}),\mathbb{Q})(1)$$

corresponds to the intersection pairing on the components of D. This pairing is well known to be negative definite [Mm, p. 6]. In particular, it is non-degenerate. **q.e.d.**

Remark 1.4. The analogue of Theorem 1.1 holds for $\overline{\mathbb{Q}}_\ell$-coefficients, and when k is a finite field of characteristic unequal to ℓ. The proof is exactly the same.

2. Construction of the intersection motive

Fix a base field k, and assume given a proper surface $\overline{X}$ over k. The aim of this section is to construct the *Chow motive* modelling intersection cohomology of $\overline{X}$. The discussion preceding Theorem 1.1 showed that intersection cohomology is invariant under passage to the normalization X^* of $\overline{X}$; the same should thus be true for the motive we intend to construct. Fix

$$X \overset{j}{\hookrightarrow} X^* \overset{i}{\hookleftarrow} Z$$

where i is a closed immersion of a finite sub-scheme Z, with smooth complement X. Choose a resolution of singularities. More precisely, consider in addition the following diagram, assumed to be cartesian:

$$\begin{tikzcd} X \arrow[r, hook, "j'"] \arrow[d, equal] & X' \arrow[d, "\pi"'] & D \arrow[l, hook', "i'"'] \arrow[d, "\pi"'] \\ X \arrow[r, hook, "j"] & X^* & Z \arrow[l, hook', "i"'] \end{tikzcd}$$

where π is proper (and birational), X' is smooth (and proper), and D is a divisor with normal crossings, whose irreducible components D_m are smooth (and proper). Note that by Abhyankar's result on resolution of singularities in dimension two [L2, Theorem], X^* can be desingularized; in addition (see the discussion in [L1, pp. 191–194]), by further blowing up possible singularities of the components of the pre-image D of Z, it can be assumed to be of the required form. This discussion also shows that the system of such resolutions is filtering.

Theorem 1.1 suggests how to construct the intersection motive: it should be a direct complement of $h^2(D)$ in $h(X')$. By definition, the Chow motive $h^2(D)$ equals the direct sum of the $h^2(D_m)$. Recall [S2, 1.13] that the $h^2(D_m)$ are canonically defined as quotient objects of the motives $h(D_m)$. Hence there is a canonical morphism

$$i'^* : h(X') \longrightarrow \bigoplus_m h(D_m) \longrightarrow\!\!\!\!\rightarrow \bigoplus_m h^2(D_m)$$

of Chow motives. Similarly [S2, 1.11], there is a canonical morphism

$$i'_* : \bigoplus_m h^0(D_m)(-1) \hookrightarrow \bigoplus_m h(D_m)(-1) \longrightarrow h(X') \; .$$

Here, the twist by (-1) denotes the tensor product with the Lefschetz motive $\mathbb{L} = h^2(\mathbb{P}^1)$.

Theorem 2.1. *(i) The composition $\alpha := i'^* i'_*$ is an isomorphism of Chow motives.*

(ii) The composition $p := i'_* \alpha^{-1} i'^*$ is an idempotent on $h(X')$. Hence so is the difference $\mathrm{id}_{X'} - p$.

(iii) The image $\mathrm{im}\, p$ is canonically isomorphic to $h^2(D) = \oplus_m h^2(D_m)$.

Proof. (ii) and (iii) are formal consequences of (i). The formula "$\phi_* \phi^* = \deg \phi$" for finite morphisms ϕ [S2, 1.10] shows that we may prove our claim after a finite extension of our ground field k. In particular, we may assume that all components D_m are geometrically irreducible, with field of constants equal to k. We then have canonical isomorphisms $h^0(D_m) \cong h(\mathbf{Spec}\, k)$ and $h^2(D_m) \cong \mathbb{L}$. Denote by i_m the closed immersion of D_m into X'. The map α in question equals

$$\bigoplus_{m,n} i_m^* i_{n,*} : \bigoplus_n h^0(D_n)(-1) \longrightarrow \bigoplus_m h^2(D_m)$$

For each pair (m,n), the composition $i_m^* i_{n,*}$ is an endomorphism of $\mathbb{L}$. Now the degree map induces an isomorphism

$$\mathrm{End}(\mathbb{L}) = CH^0(\mathbf{Spec}\, k) \xrightarrow{\sim} \mathbb{Q} \; .$$

We leave it to the reader to show that under this isomorphism, the endomorphism $i_m^* i_{n,*}$ is mapped to the intersection number $D_n \cdot D_m$. Our claim follows from the non-degeneracy of the intersection pairing on the components of D [Mm, p. 6]. **q.e.d.**

Definition 2.2. The *intersection motive* of $\overline{X}$ is defined as

$$h_{!*}(\overline{X}) := (X', \mathrm{id}_{X'} - p, 0) \; .$$

Here, we follow the standard notation for Chow motives (see e.g. [S2, 1.4]). Idempotents on Chow motives have an image; by definition, the image of the idempotent $\mathrm{id}_{X'} - p$ on the Chow motive $(X', \mathrm{id}_{X'}, 0) = h(X')$ is $(X', \mathrm{id}_{X'} - p, 0) = h_{!*}(\overline{X})$. Note that by definition, we have the equality

$h_{!*}(\overline{X}) = h_{!*}(X^*)$.

Theorem 2.1 shows that there is a canonical decomposition

$$h(X') = h_{!*}(\overline{X}) \oplus h^2(D) .$$

By Theorem 1.1 and Remark 1.4, the Betti, resp. ℓ-adic realization of the intersection motive (for the base fields for which this realization exists) coincides with intersection cohomology of $\overline{X}$ (and of X^*).

Proposition 2.3. *As before, denote by X^* the normalization of $\overline{X}$. The definition of $h_{!*}(\overline{X})$ is independent of the choices of the finite sub-scheme Z containing the singularities X^*, and of the desingularization X' of X^*.*

This statement is going to be proved together with the functoriality properties of the intersection motive, whose formulation we prepare now. Consider a dominant morphism $f : \overline{X} \to \overline{Y}$ of proper surfaces over k. By the universal property of the normalization Y^* of $\overline{Y}$, it induces a morphism, still denoted f, between X^* and Y^*. It is generically finite. Hence we can find a finite closed subscheme W of Y^* containing the singularities, and such that the pre-image under f of $Y := Y^* - W$ is dense, and smooth. The closed sub-scheme $f^{-1}(W)$ of X contains the singularities of X^*. We thus can find a morphism F of desingularizations of X^* and Y^* of the type considered before:

$$\begin{tikzcd} X' \arrow[d, "F"'] & D \arrow[l, hook', "i_D"'] \arrow[d, "F"] \\ Y' & C \arrow[l, hook', "i_C"'] \end{tikzcd}$$

This means that X' and Y' are smooth, and D and C are divisors with normal crossings, whose irreducible components are smooth, and lying over finite closed sub-schemes of X^* and Y^*, respectively. Choose and fix such a diagram. Note that if the original morphism $f : \overline{X} \to \overline{Y}$ is finite, then the diagram F can be chosen to be cartesian.

Proposition 2.4. *(i) The pull-back $F^* : h(Y') \to h(X')$ maps the sub-object $h_{!*}(\overline{Y})$ of $h(Y')$ to the sub-object $h_{!*}(\overline{X})$ of $h(X')$.*

(ii) The push-forward $F_* : h(X') \to h(Y')$ maps the quotient $h_{!*}(\overline{X})$ of $h(X')$ to the quotient $h_{!*}(\overline{Y})$ of $h(Y')$.

(iii) The composition $F_*F^* : h_{!*}(\overline{Y}) \to h_{!*}(\overline{Y})$ equals multiplication with the degree of f.

(iv) If f is finite, and if the morphism F is chosen to be cartesian, then both F^* and F_* respect the decompositions

$$h(Y') = h_{!*}(\overline{Y}) \oplus h^2(C)$$

and

$$h(X') = h_{!*}(\overline{X}) \oplus h^2(D)$$

of $h(Y')$ and of $h(X')$, respectively.

Proof. By definition, there are (split) exact sequences

$$0 \longrightarrow h_{!*}(\overline{X}) \longrightarrow h(X') \xrightarrow{i_D^*} h^2(D) \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_m h^0(D_m)(-1) \xrightarrow{i_{D,*}} h(X') \longrightarrow h_{!*}(\overline{X}) \longrightarrow 0\ ;$$

similarly for Y' and C. Obviously, the first sequence is contravariant, and the second is covariant. This proves parts (i) and (ii). Part (iii) follows from this, and from the corresponding formula for F_*F^* on the motive of Y' [S2, 1.10]; note that the degree of F equals the one of f. If F is cartesian, then the above sequences are both co- and contravariant thanks to the base change formulae $F_*i_D^* = i_C^*F_*$ and $F^*i_{C,*} = i_{D,*}F^*$. This proves part (iv).
q.e.d.

Proof of Proposition 2.3. First, let us show that for a fixed choice of Z, the definition of $h_{!*}(\overline{X})$ is independent of the choice of the desingularization X' of X^*. Using that the system of such desingularizations is filtering, we reduce ourselves to the situation considered in Proposition 2.4, with $f = \mathrm{id}$. We thus have a cartesian diagram

$$\begin{tikzcd} X' \arrow[d, "F"'] & D \arrow[l, hook', "i_D"'] \arrow[d, "F"'] \\ X'' & C \arrow[l, hook', "i_C"'] \end{tikzcd}$$

Let us denote by $h'_{!*}(\overline{X})$ and $h''_{!*}(\overline{X})$ the two intersection motives formed with respect to X' and X'', respectively. We want to show that $F^* : h''_{!*}(\overline{X}) \to h'_{!*}(\overline{X})$ is an isomorphism. The scheme X'' is normal, and the morphism F is proper. By the valuative criterion of properness, the locus of points of X'' where F^{-1} is not defined is of dimension zero. Let P be a point in this locus. If the fibre over P were finite, then F would be quasi-finite near P. Since it is proper, it would be finite. But since both its source and target are normal, it would be an isomorphism near P, contrary to our assumption. This shows that the fibre over P is of dimension one. Since the fibre is connected [EGA3, Cor. (4.3.12)], it is pure of dimension one, i.e., it is a divisor. By the universal property of the blow-up, X' dominates the blow-up of X'' in the points $P_1, \ldots, P_r$ where F is not an isomorphism. This blow-up lies between X' and X'', and satisfies the same conditions on desingularizations. Repeating this argument and using the fact that X' is Noetherian, one sees that this process stops at some point; F is therefore the composition of blow-ups in points. By induction, we may assume that F equals the blow-up of X'' in one point P. The exceptional divisor $E := F^{-1}(P)$ is a projective bundle (of rank one) over P. It is also one of the irreducible components of D; in fact, the morphism F induces a bijection between the components of D other than E and the components of C. Denote by i_E the closed immersion of E into X'. By Manin's computation of the motive of a blow-up [S2, Thm. 2.8], the sequence

$$0 \longrightarrow h(X'') \xrightarrow{F^*} h(X') \xrightarrow{i_E^*} h^2(E) \longrightarrow 0$$

is (split) exact. But obviously, so is

$$0 \longrightarrow h^2(C) \xrightarrow{F^*} h^2(D) \xrightarrow{i_E^*} h^2(E) \longrightarrow 0 \, .$$

Hence F^* maps the kernel $h''_{!*}(\overline{X})$ of i_C^* isomorphically to the kernel $h'_{!*}(\overline{X})$ of i_D^*.

In the same way, one shows that enlarging Z by adding non-singular points of X^* does not change the value of $h_{!*}(\overline{X})$. **q.e.d.**

Recall the definition of the *dual* of a Chow motive [S2, 1.15]. For example, for any desingularization X' of X^*, the dual of $(X', \mathrm{id}_{X'}, 0) = h(X')$ is given by $(X', \mathrm{id}_{X'}, 2) = h(X')(2)$.

Proposition 2.5. *The dual of the intersection motive $h_{!*}(\overline{X})$ is canonically isomorphic to $h_{!*}(\overline{X})(2)$.*

Proof. By definition, the dual of $(X', \mathrm{id}_{X'} - p, 0)$ equals $(X', {}^t(\mathrm{id}_{X'} - p), 2)$, where ^{t} denotes the transposition of cycles in $X' \times X'$. But p is symmetric: in fact, ${}^t(i'^*) = i'_*$, and ${}^t(i'_*) = i'^*$.

One checks as in the proof of Proposition 2.3 that this identification of $h_{!*}(\overline{X})^*$ with $h_{!*}(\overline{X})(2)$ does not depend on the choice of X'. **q.e.d.**

3. Künneth decompositions of the intersection motive

We continue to consider the situation of Section 2. Thus, $\overline{X}$ is a proper surface over the base field k with normalization X^*, and we fix

$$X \xhookrightarrow{j} X^* \xhookleftarrow{i} Z$$

where i is a closed immersion of a finite sub-scheme Z, with smooth complement X. In addition, we consider the following cartesian diagram:

$$\begin{tikzcd} X \arrow[r, hook, "j'"] \arrow[d, equal] & X' \arrow[d, "\pi"'] & D \arrow[l, hook', "i'"'] \arrow[d, "\pi"'] \\ X \arrow[r, hook, "j"] & X^* & Z \arrow[l, hook', "i"'] \end{tikzcd}$$

where π is proper, X' is smooth and projective, and D is a divisor with normal crossings, whose irreducible components D_m are smooth.

We fix a further set of data, namely a *Künneth decomposition* of the motive of X',

$$(*) \qquad h(X') = h^0(X') \oplus h^1(X') \oplus h^2(X') \oplus h^3(X') \oplus h^4(X') \; .$$

Our aim is to deduce from $(*)$ a Künneth decomposition of the intersection motive of $h_{!*}(\overline{X}) \subset h(X')$,

$$h_{!*}(\overline{X}) = h^0_{!*}(\overline{X}) \oplus h^1_{!*}(\overline{X}) \oplus h^2_{!*}(\overline{X}) \oplus h^3_{!*}(\overline{X}) \oplus h^4_{!*}(\overline{X}) \; .$$

Consider the ascending filtration of $h(X')$ by sub-motives induced by $(*)$:

$$0 \subset h^0(X') \subset h^{\le 1}(X') \subset h^{\le 2}(X') \subset h^{\le 3}(X') \subset h^{\le 4}(X') = h(X') \; ,$$

where we set $h^{\le r}(X') := \oplus_{n=0}^r h^n(X')$. Since these sub-objects are direct factors, the quotients $h^{\ge r}(X') := h(X')/h^{\le r-1}(X')$ exist. Recall the quotient $\oplus_m h^2(D_m)$ and the sub-object $\oplus_m h^0(D_m)$ of $\oplus_m h(D_m)$.

Remark 3.1. The reader should note the implicit abuse of notation: for the curves D_m, we do not fix Künneth decompositions, i.e., we do not identify complements of the $h^2(D_m)$ and $h^0(D_m)$ inside $h(D_m)$. For the surface X', we do consider the $h^n(X')$ as sub-objects of $h(X')$, together with a fixed choice of complement (i.e., the direct sum of the $h^j(X')$, $j \ne n$).

Definition 3.2. The decomposition $(*)$ of $h(X')$ is called *admissible* if it satisfies the following conditions:

(1) duality $h(X')^\vee \xrightarrow{\sim} h(X')(2)$ induces an isomorphism

$$h^{\le 1}(X')^\vee \xrightarrow{\sim} h^{\ge 3}(X')(2) ,$$

(2) the composition of morphisms

$$h^{\le 1}(X') \hookrightarrow h(X') \xrightarrow{i'^*} \bigoplus_m h(D_m) \twoheadrightarrow \bigoplus_m h^2(D_m)$$

equals zero.

Of course, condition (1) is fulfilled if the decomposition ($*$) is auto-dual in the obvious sense: $h^n(X')^\vee \cong h^{4-n}(X')(2)$, $0 \le n \le 4$ under the duality $h(X')^\vee \cong h(X')(2)$.

Proposition 3.3 (Murre–Scholl)**.** *Admissible Künneth decompositions of $h(X')$ exist.*

Proof. We sketch the construction by Murre [Mr1], following Scholl's presentation [S2, Sect. 4], as well as his modification ensuring auto-duality. Fix a projective embedding of X'. Fix (i) a hyperplane section $C \subset X'$ that is a smooth connected curve. As explained in [S2, 4.3], the embedding of C into X' induces an isogeny $P \to J$ from the Picard variety to the Albanese variety of X'. Fix (ii) an isogeny $\beta : J \to P$ such that the composition of the two isogenies equals multiplication by $n > 0$. Finally, fix (iii) a 0-cycle Z of degree one on C. Then by [S2, Thm. 3.9], β corresponds to a symmetric cycle class

$$\widetilde{\beta} \in CH^1(X' \times X')$$

satisfying the condition $p_{X',*}(\widetilde{\beta} \cdot [X' \times Z]) = 0 \in CH^1(X')$, where $p_{X'}$ is the first projection from the product $X' \times X'$ to X'.

One then defines [S2, 4.3] projectors $\pi_0 := [Z \times X']$ and $\pi_4 := {}^t\pi_0 = [X' \times Z]$, as well as $p_1 := \frac{1}{n}\widetilde{\beta} \cdot [C \times X']$ and $p_3 := {}^t p_1$. All orthogonality relations are satisfied (including $p_3 p_1 = 0$), except that $p_1 p_3$ is not necessarily equal to zero. This is why a modification is necessary: one puts $\pi_1 := p_1 - \frac{1}{2} p_1 p_3$ and $\pi_3 := {}^t\pi_1 = p_3 - \frac{1}{2} p_1 p_3$. [1] This, together with $\pi_2 := \mathrm{id}_{X'} - \pi_0 - \pi_1 - \pi_3 - \pi_4$, gives a full auto-dual set of orthogonal projectors. We thus get a Künneth decomposition of $h(X')$:

$$h^n(X') := (X', \pi_n, 0) \subset (X', \mathrm{id}_{X'}, 0) = h(X') .$$

[1]The construction differs from Murre's solution [Mr1, Rem. 6.5], who takes $p_1 - p_1 p_3$ and p_3 instead of π_1 and π_3.

The decomposition satisfies (1). In order to check (2), we need to compute the composition of correspondences

$$h(X') \xrightarrow{\pi_n} h(X') \xrightarrow{i'^*} \bigoplus_m h(D_m) \xrightarrow{pr} \bigoplus_m h^2(D_m) \ ,$$

for $n = 0, 1$. The composition is zero if and only if it zero after base change to a finite field extension. Hence we may assume that all D_m are geometrically irreducible, with field of constants k. Then the $h^2(D_m)$ equal $\mathbb{L}$, and the composition $pr \circ i'^*$ corresponds to the cycle class

$$([D_m])_m \in \bigoplus_m CH^1(X')$$

on $\coprod_m X' \times \mathbf{Spec}\, k$. By definition of the composition of correspondences, we then find

$$pr \circ i'^* \circ \pi = \big(p_{X',*}(\pi \cdot [X' \times D_m])\big)_m \in \bigoplus_m CH^1(X') \ ,$$

for any $\pi \in CH^2(X' \times X')$. Here as before, $p_{X'}$ is the first projection from the product $X' \times X'$ to X'. Let us fix m. We need to show that for $n = 0, 1$, the cycle class

$$p_{X',*}(\pi_n \cdot [X' \times D_m]) \in CH^1(X')$$

is zero. For $n = 0$, this is easy: the intersection

$$\pi_0 \cdot [X' \times D_m] = [Z \times X'] \cdot [X' \times D_m] = [Z \times D_m]$$

has one-dimensional fibres under $p_{X'}$. Therefore, its push-forward under $p_{X'}$ is zero.

For $n = 1$, observe first that by definition of π_1, and by associativity of composition of correspondences, it suffices to show that

$$p_{X',*}(p_1 \cdot [X' \times D_m]) = 0 \ .$$

By definition, the intersection $p_1 \cdot [X' \times D_m]$ is a non-zero multiple of

$$\widetilde{\beta} \cdot [C \times X'] \cdot [X' \times D_m] \ .$$

By the projection formula, the image under $p_{X',*}$ of this cycle equals the image under the push-forward $CH^0(C) \to CH^1(X')$ of

$$p_{1,*}(\widetilde{\beta}_C \cdot [C \times D_m]) \ ,$$

where $\widetilde{\beta}_C$ denotes the pull-back of $\widetilde{\beta}$ to $C \times X'$, and p_1 the projection from $C \times X'$ to C. Denote by p_2 the projection from this product to X'. Now symmetry of $\widetilde{\beta}$ and the condition $p_{X',*}(\widetilde{\beta} \cdot [X' \times Z]) = 0$ imply that

$$p_{2,*}(\widetilde{\beta}_C \times [Z \times X']) = 0 \in CH^1(X') \; .$$

It follows that

$$p_{2,*}(\widetilde{\beta}_C \times [Z \times D_m]) = 0 \in CH^1(D_m) \; .$$

In particular, the degree a of this 0-cycle is zero. But since Z is of degree one, we have

$$p_{1,*}(\widetilde{\beta}_C \cdot [C \times D_m]) = a[C] \in CH^0(C) \; .$$

q.e.d.

Remark 3.4. Condition (2) of admissibility is a consequence of Murre's Conjecture B [Mr2, Sect. 1.4] on the triviality of the action of the ℓ-th Künneth projector on $CH^j(Y)$, for $\ell > 2j$; see [J, Prop. 5.8]. Here, Y equals the product of X' and D_m, $j = 2$, and $\ell = 5, 6$. Note that for products of a surface and a curve, the conjecture is known to hold [Mr3, Lemma 8.3.2]. This implies that any auto-dual Künneth decomposition of $h(X')$ is admissible. This would have allowed to shorten the proof of Proposition 3.3. But since the argument proving (2) is rather explicit, we decided to give it for the convenience of the reader.

For the rest of this section, assume that the decomposition ($*$) is admissible. Given that duality $h(D_m)^\vee \xrightarrow{\sim} h(D_m)(1)$ induces an isomorphism

$$h^0(D_m)^\vee \xrightarrow{\sim} h^2(D_m)(1) \; ,$$

it is easy to see that the morphism i'_* dual to the one from condition (2)

$$\bigoplus_m h^0(D_m) \hookrightarrow \bigoplus_m h(D_m) \xrightarrow{i'_*} h(X')(1) \twoheadrightarrow h^{\geq 3}(X')(1)$$

is zero, i.e., the map $i'_* : \oplus_m h^0(D_m) \to h(X')(1)$ factors through the submotive $h^{\leq 2}(X')(1)$. On the other hand, by condition (2), the inverse image $i'^* : h(X') \to \oplus_m h^2(D_m)$ factors through the quotient motive $h^{\geq 2}(X')$. It follows that the composition

$$\alpha = i'^* i'_* : \bigoplus_m h^0(D_m)(-1) \longrightarrow \bigoplus_m h^2(D_m)$$

considered in Section 2 factors naturally through $h^{\leq 2}(X')/h^{\leq 1}(X') \cong h^2(X')$. Thus we get the following.

Theorem 3.5. *(i) Any admissible Künneth decomposition*

$$(*) \qquad h(X') = h^0(X') \oplus h^1(X') \oplus h^2(X') \oplus h^3(X') \oplus h^4(X')$$

of $h(X')$ induces a Künneth decomposition

$$(*)_{!*} \qquad h_{!*}(\overline{X}) = h^0_{!*}(\overline{X}) \oplus h^1_{!*}(\overline{X}) \oplus h^2_{!*}(\overline{X}) \oplus h^3_{!*}(\overline{X}) \oplus h^4_{!*}(\overline{X})$$

of $h_{!*}(\overline{X})$. If $(*)$ is auto-dual, then so is $(*)_{!*}$ The projection from $h(X')$ to $h_{!*}(\overline{X})$ maps $h^n(X')$ isomorphically to $h^n_{!*}(\overline{X})$, for all $n \neq 2$.

(ii) Assume in addition that $(*)$ satisfies the following condition: the composition of morphisms

$$h^n(X') \hookrightarrow h(X') \xrightarrow{i'^*} \bigoplus_m h(D_m) \twoheadrightarrow \bigoplus_m h^2(D_m)$$

equals zero for $n \neq 2$. Then the canonical decomposition

$$h(X') = h_{!*}(\overline{X}) \oplus h^2(D)$$

from Section 2 is compatible with the Künneth decompositions of $h(X')$ and of $h_{!*}(\overline{X})$.

Proof. By Theorem 2.1 (i), the morphism α is an isomorphism. Hence the composition $i'_*\alpha^{-1}i'^*$ is an idempotent on $h^2(X')$. Define $h^2_{!*}$ as its kernel. With this definition, the direct sum

$$h^0(X') \oplus h^1(X') \oplus h^2_{!*} \oplus h^3(X') \oplus h^4(X') \subset h(X')$$

projects isomorphically to $h_{!*}(\overline{X})$ under $h(X') \twoheadrightarrow h_{!*}(\overline{X})$. **q.e.d.**

Remark 3.6. (a) It is not clear to me whether the additional hypothesis of Theorem 3.5 (ii) can be satisfied using the recipe of Murre–Scholl, and if D has more than one component. The problem is the relation

$$p_{X',*}(p_3 \cdot [X' \times D_m]) = 0 \, .$$

(We use the same notation as in the proof of Proposition 3.3.) The cycle class in question is a non-zero multiple of

$$p_{X',*}(\widetilde{\beta} \cdot [X' \times C \cdot D_m]) \, .$$

For any m, the Murre–Scholl Künneth decomposition of $h(X')$ can be *chosen* such that this cycle class vanishes: take Z to be equal to $\frac{1}{d}[C \cdot D_m]$, where d is the degree of $C \cdot D_m$. If there are more than one D_m, and if they represent different classes in $CH^1(X')$, then Theorem 3.5 (i) shows how to get a

Künneth decomposition of $h(X')$ *a posteori*: first take the decomposition of $h_{!*}(\overline{X})$, then apply the canonical decomposition

$$h(X') = h_{!*}(\overline{X}) \oplus h^2(D)$$

from Section 2 to get a decomposition of $h(X')$ trivially satisfying the additional hypothesis of Theorem 3.5 (ii). The result does not seem to be obtained in an obvious way from the recipe of Murre–Scholl.

(b) At this point, the reader may have guessed that I consider the filtration

$$0 \subset h^0(X') \subset h^{\leq 1}(X') \subset h^{\leq 2}(X') \subset h^{\leq 3}(X') \subset h^{\leq 4}(X') = h(X')$$

as more canonical data associated to the motive of X' than its possible Künneth decompositions. I do not know whether this filtration depends on the choice of Künneth decomposition; again, this question is related to Murre's Conjecture B [Mr2, Sect. 1.4], this time on the triviality of the action of the ℓ-th Künneth projector on $CH^2(X' \times X')$, for $5 \leq \ell \leq 8$, independently of the choice of Künneth decomposition of $h(X' \times X')$. A fortiori, I do not know whether the analogous filtration

$$0 \subset h^0_{!*}(\overline{X}) \subset h^{\leq 1}_{!*}(\overline{X}) \subset h^{\leq 2}_{!*}(\overline{X}) \subset h^{\leq 3}_{!*}(\overline{X}) \subset h^{\leq 4}_{!*}(\overline{X}) = h(\overline{X})_{!*}$$

is an invariant associated to the intersection motive of $\overline{X}$.

References

[BBD] A.A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, in: B. Teissier, J.L. Verdier (eds.), *Analyse et topologie sur les espaces singuliers (I)*, Astérisque **100**, Soc. Math. France (1982).

[EGA3] A. Grothendieck, *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie*, Publ. Math. IHES **11** (1961).

[GHaMr] B.B. Gordon, M. Hanamura, J.P. Murre, *Relative Chow–Künneth projectors for modular varieties*, J. Reine Angew. Math. **558** (2003), 1–14.

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II, Ann. of Math. **79** (1964), 109–203, 205–326.

[J] U. Jannsen, *Motivic sheaves and filtrations on Chow groups*, in U. Jannsen, S. Kleiman, J.-P. Serre (eds.), *Motives. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference, held at the University of Washington, Seattle, July 20–August 2, 1991*, Proc. of Symp. in Pure Math. **55**, Part 1, AMS (1994), 245–302.

[L1] J. Lipman, *Introduction to resolution of singularities*, in R. Hartshorne (ed.), *Algebraic Geometry. Proceedings of the Symposium in Pure Mathematics of the AMS, held at Humboldt State University, Arcata, California, July 29–August 16, 1974*, Proc. of Symp. in Pure Math. **29**, AMS (1975), 187–230.

[L2] J. Lipman, *Desingularization of two-dimensional schemes*, Ann. of Math. **107** (1978), 151–207.

[Mm] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math. IHES **9** (1961), 5–22.

[Mr1] J.P. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.

[Mr2] J.P. Murre, *On a conjectural filtration on the Chow groups of an algebraic variety. Part I*, Indag. Mathem. **4** (1993), 177–188.

[Mr3] J.P. Murre, *On a conjectural filtration on the Chow groups of an algebraic variety. Part II*, Indag. Mathem. **4** (1993), 189–201.

[N] A. Nair, *Intersection cohomology, Shimura varieties, and motives*, preprint, Tata Institute of Fundamental Research, version of July 2003, 55 pages.

[S1] A.J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.

[S2] A.J. Scholl, *Classical Motives*, in U. Jannsen, S. Kleiman, J.-P. Serre (eds.), *Motives. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference, held at the University of Washington, Seattle, July 20–August 2, 1991*, Proc. of Symp. in Pure Math. **55**, Part 1, AMS (1994), 163–187.

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