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Variable Lebesgue Spaces: Foundations and Harmonic Analysis

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CHAPTER 1

Introduction and Motivation

We begin with an intuitive introduction to the variable Lebesgue spaces, briefly sketch their history, and give some of the contemporary motivations for studying these spaces.

1.1. An intuitive introduction

Recall that given an open set $\Omega \subset \mathbb{R}^n$, the classical Lebesgue space $L^p(\Omega)$, $1 \leq p < \infty$, is defined to be the collection of measurable functions f such that

$$\int_\Omega |f(x)|^p \, dx < +\infty.$$

For our purposes, the key point in this definition is its homogeneity: each point of the space is treated the same as every other. A standard way to weaken this homogeneity assumption is to replace Lebesgue measure with a measure μ, where $\mu = w(x)\,dx$. This leads to the theory of weighted norm inequalities, for which there is an extensive literature. (See [**33, 44**] and their references. There are some surprising and deep connections between weighted spaces and variable Lebesgue spaces: see, for instance, Chapter 4 below and Lerner [**69**].)

On the other hand, in a variable Lebesgue space we vary the exponent, replacing p by a function $p(\cdot)$. More precisely, given a measurable function $p\colon \Omega \to [1,\infty)$, we define $L^{p(\cdot)}(\Omega)$ to be the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < +\infty. \tag{1.1}$$

The factor λ is introduced for technical reasons which will be made clear below; for the moment, think of $\lambda = 1$.

As a simple example on $\mathbb{R}$, consider the function

$$p(x) = \begin{cases} 2 & x \leq 0 \\ 4 & x > 0. \end{cases} \tag{1.2}$$

Then $L^{p(\cdot)}(\mathbb{R})$ consists of all functions f such that

$$\int_{-\infty}^0 |f(x)|^2 \, dx + \int_0^\infty |f(x)|^4 \, dx < +\infty.$$

The lack of homogeneity of this space is immediate: the function $|x|^{-1/3}$ is not in $L^{p(\cdot)}(\mathbb{R})$, but $|x|^{-1/3}\chi_{(-1,0)}$, $|x|^{-1/3}\chi_{(1,\infty)}$ and $|x+1|^{-1/3}\chi_{(-1,1)}$ are.

Clearly, the more complicated $p(\cdot)$, the more delicate the resulting space. For instance, if we partition $\mathbb{R}$ into the union of two sets E and F, and let

$$p(x) = \begin{cases} 2 & x \in E \\ 4 & x \in F, \end{cases}$$

then the resulting space depends heavily on the geometry of the partition. As we shall see, even if we assume $p(\cdot)$ is uniformly continuous, we can still get quite complicated behavior.

We can also consider spaces where the exponent function $p(\cdot)$ is unbounded. For example, on $\mathbb{R}$ we could take $p(x) = 1+|x|$. Such spaces behave quite differently than the classical Lebesgue spaces. For example, in this case we have that $L^\infty(\mathbb{R}) \subset L^{p(\cdot)}(\mathbb{R})$: given $g \in L^\infty$, fix $\lambda > \|g\|_\infty$. Then

$$\int_{\mathbb{R}} \left(\frac{|g(x)|}{\lambda}\right)^{p(x)} dx \leq \int_{\mathbb{R}} \left(\frac{\|g\|_\infty}{\lambda}\right)^{1+|x|} dx < \infty,$$

and so $g \in L^{p(\cdot)}(\mathbb{R})$. Note that here is why we include the factor λ in the definition of $L^{p(\cdot)}$: if did not, then we can easily find g such that $\|g\|_\infty > 1$ and

$$\int_{\mathbb{R}} |g(x)|^{p(x)}\, dx = \infty.$$

Thus, g is not in $L^{p(\cdot)}(\mathbb{R})$ but we would have $cg \in L^{p(\cdot)}(\mathbb{R})$ for c sufficiently small.

1.2. A brief history

The variable Lebesgue spaces have a long history that falls roughly into three overlapping stages. They were introduced by Orlicz [81] in 1931; their properties were further developed by Nakano [77, 78] as special cases of the theory of modular spaces. In the ensuing decades they were primarily considered as important examples of modular spaces or the class of Musielak-Orlicz spaces, concrete examples of modular spaces that are also generalizations of the classical Orlicz spaces. See, for example, [76, 83, 103], and in particular the work of Hudzik [53, 54, 55, 56, 57, 58, 59, 60, 61] that foreshadows many modern developments.

The variable Lebesgue spaces were independently discovered by the Russian mathematician Tsenov [101], and extensively developed first by Sharapudinov [94, 95, 96, 97] and then by Zhikov [104, 105, 106, 107, 108, 109, 110, 112, 111]. Russian mathematicians were the first to consider applications of variable Lebesgue spaces to problems in harmonic analysis and the calculus of variations.

The third stage in the study of variable Lebesgue spaces is usually thought to begin with the foundational paper by Kováčik and Rákosník [64] in 1991. Following its publication a number of mathematicians became interested in these spaces. Without being comprehensive, we mention the work of: Fan and Zhao [36, 37, 38, 39] on the calculus of variations; Edmunds [34, 35] on variable Sobolev spaces (i.e., the space of functions whose distributional derivatives up to order k are in $L^{p(\cdot)}$); and Samko and Ross [84, 90, 92, 93] on fractional differential and integral operators of variable order.

In 2000 the field began to expand even further. Motivated by problems in the study of electrorheological fluids (see below) Diening [25] raised the question of when the Hardy-Littlewood maximal operator and other classical operators in harmonic analysis are bounded on the variable Lebesgue spaces. These and related problems are still the subject of active research to this day.

1.3. Motivation

The variable Lebesgue spaces are interesting not only in their own right, but as we have indicated above, for their application to a wide variety of problems. We first consider a very simple example given in [20]. By a classical result of Calderón-Zygmund [11], given a bounded domain Ω with a smooth boundary and $f \in L^p(\Omega)$, $1 \leq p < \infty$, if u is a solution to $\triangle u = f$, then $u \in L^q(\Omega)$, where $1/p - 1/q = 2/n$.

Note, however, that the conclusion, while global, is affected by the local behavior of f. Thus, f might be badly behaved only on the small subset $A \subset \Omega$ but this affects u on all of Ω. (See Figure 1.)

However, this result remains true if we replace the constant exponent p with a variable exponent $p(\cdot)$ with modest smoothness assumptions (see [20]). Therefore, if we choose an exponent function that more precisely reflects behavior of f on the bad set A and the good set B, we get a correspondingly sharper estimate for u. Similar results hold for other kinds of PDEs.

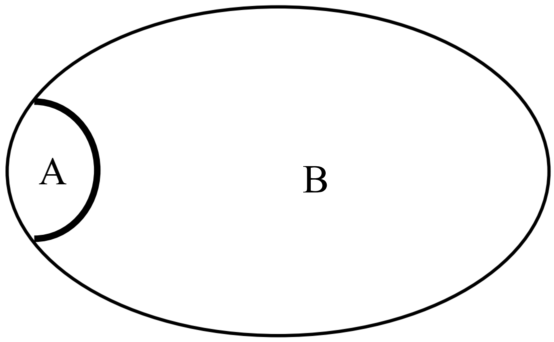

FIGURE 1. Ω and the good and bad parts for f.

More generally, we can consider problems that incorporate the variability from the beginning. For example, in the calculus of variations Zhikov was interested in minimizers of functionals of the form

$$F(u) = \int_\Omega f(x, \nabla u)\, dx,$$

where the Lagrangian satisfies the non-standard growth condition

$$-c_0 + c_1|\xi|^a \le f(x,\xi) \le c_0 + c_2|\xi|^b, \qquad 0 < a < b.$$

An important example of such a Lagrangian is $f(x,\xi) = |\xi|^{p(x)}$, where $a \le p(x) \le b$. The Euler-Lagrange equation associated to this function is the $p(\cdot)$-Laplacian

$$\triangle_{p(\cdot)} u = -\operatorname{div}(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u) = 0.$$

The appropriate function spaces for analyzing the solutions of these equations are the variable Lebesgue spaces $L^{p(\cdot)}$ and the associated variable Sobolev spaces $W^{k,p(\cdot)}$. These problems have been studied by a number of authors and continue to be an active area of research; for further details see the survey articles by Harjulehto, *et al.* [51] and Mingione [74].

In the past decade, the one application that provided the most impetus for the study of the variable Lebesgue spaces is the modeling of electrorheological fluids. These are liquids whose viscosity changes (often dramatically) when exposed to an electric field. (See [48, 98] for further information on their physical properties and potential for wide-ranging applications.) While broadly understood experimentally, a comprehensive theoretical model is still lacking. Extensive work has been done on modeling these as non-Newtonian fluids; in one extensively studied model the energy is given by an expression of the form

$$\int_\Omega |Du(x)|^{p(x)}\, dx,$$

where Du is the symmetric part of the gradient of the velocity field of the fluid, and the exponent function $p(\cdot)$ is a function of the electric field. (Similar energy expressions have appeared in the study of other kinds of fluids. See, for example, [**111**].) This model has been extensively studied by Růžička [**3**, **89**] and Acerbi and Mingioni [**4**, **88**]. As we noted above, this problem contributed to extensive development of harmonic analysis on the variable Lebesgue spaces.

The variable Lebesgue spaces have also emerged in the study of image processing. In 1997 Blomgren, *et al.* [**8**] suggested that smoother images could be obtained by an interpolation technique that uses a variable exponent: the appropriate norm is

$$\int_\Omega |\nabla u(x)|^{p(\nabla u)}\,dx,$$

where the exponent monotonically decreases from 2 to 1 as ∇u increases. These and related ideas has been explored by a number of authors [**1**, **2**, **9**, **13**, **14**, **102**] in recent years.

1.4. Organization of this monograph

The remainder of this monograph is organized as follows. In Chapter 2 we present the fundamental function space properties of the variable Lebesgue spaces, concentrating primarily on the case when the exponent $p(\cdot)$ is bounded. There are several approaches to this. The first is to treat them as examples of abstract Banach function spaces, using the machinery developed by, for instance, Bennett and Sharpley [**7**]. A second approach is to follow their historical development and use the machinery of Musielak-Orlicz spaces [**76**]. This approach was adopted by Diening *et al.* [**30**]. However, we prefer to take a more direct approach, proving everything "with our bare hands." While at times not as elegant as other approaches, we believe that this has the singular advantage of making clear the similarities and differences between the classical and variable Lebesgue spaces.

In Chapter 3 we turn to the behavior of the Hardy-Littlewood maximal operator. We prove that sufficient conditions for the maximal operator to be bounded are the log-Hölder continuity conditions,

$$|p(x)-p(y)| \le \frac{C_0}{-\log(|x-y|)}, \quad |x-y|<1/2,$$
$$|p(x)-p_\infty| \le \frac{C_\infty}{\log(e+|x|)}, \quad x\in\mathbb{R}^n.$$

These conditions are not necessary, but are the sharpest possible pointwise continuity conditions possible. Our understanding of the boundedness of the maximal operator is still incomplete, and we will briefly consider some current areas of research.

In Chapter 4 we consider the boundedness of other classical operators in the variable Lebesgue spaces: convolution operators, singular integrals, Riesz potentials. As motivation and to illustrate a key difference between the classical and variable Lebesgue spaces we consider convolution operators, the failure of Young's inequality, and the convergence of approximate identities. Then, rather than treating the other operators individually, we develop a powerful generalization of the Rubio de Francia extrapolation theorem from the theory of weighted norm inequalities. As a consequence, we show that if an operator satisfies weighted norm inequalities, then it is bounded on variable Lebesgue spaces given reasonable assumptions on the exponent $p(\cdot)$ (e.g., log-Hölder continuity). These results are closely related to recent developments in the study of Rubio de Francia extrapolation, and we refer the reader to [**24**] for more information. For completeness we will provide a brief introduction to the theory of Muckenhoupt A_p weights.

Throughout this monograph we assume that the reader is familiar with basic real and functional analysis; we refer the reader to the standard books by Royden [85], Rudin [86, 87] and Brezis [10]. For brevity we will cite many results from classical harmonic analysis without proof; for complete details the reader may consult the books by Duoandikoetxea [33], García-Cuerva and Rubio de Francia [44] or Grafakos [46, 47].

We have attempted to provide copious references throughout the text, both to standard results and to the original proofs of many results about variable Lebesgue spaces. However, given the long and complex history, many results have been discovered independently, often with slightly different hypotheses. Therefore, our notes will often fail to be comprehensive, and we apologize in advance for any omissions. As general references, we recommend the papers by Kováčik and Rákosník [64] and Fan and Zhao [40], the recent book by Diening, *et al.* [30], and the forthcoming book by the authors of the present monograph [19].

CHAPTER 2

Properties of Variable Lebesgue Spaces

In this chapter we develop the function space properties of variable Lebesgue spaces. We begin with the basic properties and notation for exponent functions. We then define the modular and the norm, and prove that $L^{p(\cdot)}$ is a Banach space. We prove a version of Hölder's inequality, define the associate norm, and then characterize the dual space when $p_+ < \infty$. We conclude with a version of the Lebesgue differentiation theorem.

We want preface this chapter, however, by making a general comment about theorems and proofs in this context. The variable Lebesgue spaces closely resemble the classical L^p spaces, especially when $p_+ < \infty$. However, while this often suggests what should be true, the proofs can range from nearly identical to the corresponding proof in the classical case to completely different.

The situation is very reminiscent of the scene in Lewis Carroll's *Alice in Wonderland*, in which Alice is invited to play croquet with the Queen of Hearts. However, instead of the traditional mallet and ball, she is given a flamingo and hedgehog. (See Figure 1 below.) The flamingo is uncooperative, and when she is finally ready to take a shot, the hedgehog has unrolled and wandered off. In the same way, a proof in the variable Lebesgue spaces can be equally uncooperative, and corralling the various pieces at times takes patience and ingenuity.

2.1. Exponent functions

Throughout, Ω will be a subset of $\mathbb{R}^n$ with positive measure. It is helpful to think of it as an open connected set, and occasionally we will make these or other assumptions on Ω.

Definition 2.1. Given a set Ω, let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(\cdot)\colon \Omega \to [1,\infty]$. The elements of $\mathcal{P}(\Omega)$ are called exponent functions.

In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\cdot)$. To measure the oscillation in $p(\cdot) \in \mathcal{P}(\Omega)$, given a set $E \subset \Omega$, let

$$p_-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \quad p_+(E) = \operatorname*{ess\,sup}_{x\in E} p(x).$$

If the domain is clear we will simply write $p_- = p_-(\Omega)$, $p_+ = p_+(\Omega)$. We define three canonical subsets of Ω:

$$\Omega_\infty^{p(\cdot)} = \{x \in \Omega : p(x) = \infty\},$$

FIGURE 1. Alice, the flamingo and the hedgehog.

$$\begin{aligned}\Omega_1^{p(\cdot)} &= \{x \in \Omega : p(x) = 1\},\\ \Omega_*^{p(\cdot)} &= \{x \in \Omega : 1 < p(x) < \infty\}.\end{aligned}$$

We will omit the superscript $p(\cdot)$ if there is no possibility of confusion. Below, the value of certain constants will depend on whether these sets have positive measure; if they do we will use the fact that, for instance, $\|\chi_{\Omega_1^{p(\cdot)}}\|_\infty = 1$.

Given $p(\cdot)$, we define the conjugate exponent function $p'(\cdot)$ by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad x \in \Omega,$$

with the convention that $1/\infty = 0$. Since $p(\cdot)$ is a function, the notation $p'(\cdot)$ can be mistaken for the derivative of $p(\cdot)$, but we will never use the symbol " $'$ " in this sense.

The notation p' will also be used to denote the conjugate of a constant exponent. The operation of taking the supremum/infimum of an exponent does not commute with forming the conjugate exponent. In fact, a straightforward computation shows that

$$\big(p'(\cdot)\big)_+ = \big(p_-\big)', \quad \big(p'(\cdot)\big)_- = \big(p_+\big)'.$$

For simplicity we will omit one set of parentheses and write the left-hand side of each equality as $p'(\cdot)_+$ and $p'(\cdot)_-$. We will always avoid ambiguous expressions such as p'_+.

The function space theory of variable Lebesgue spaces only requires that $p(\cdot)$ be a measurable function, but in subsequent chapters we will need $p(\cdot)$ to have some additional regularity. In particular, there are two continuity conditions that are of such importance that we want to establish notation for them.

Definition 2.2. Given Ω and a function $r(\cdot)\colon \Omega \to \mathbb{R}$, we say that $r(\cdot)$ is locally log-Hölder continuous, and denote this by $r(\cdot) \in LH_0(\Omega)$, if there exists a constant C_0 such that for all x, $y \in \Omega$, $|x-y| < 1/2$,

$$|r(x) - r(y)| \le \frac{C_0}{-\log(|x-y|)}.$$

We say that $r(\cdot)$ is log-Hölder continuous at infinity, and denote this by $r(\cdot) \in LH_\infty(\Omega)$, if there exist constants C_∞ and r_∞ such that for all $x \in \Omega$,

$$|r(x) - r_\infty| \le \frac{C_\infty}{\log(e+|x|)}.$$

If $r(\cdot)$ is log-Hölder continuous locally and at infinity, we will denote this by writing $r(\cdot) \in LH(\Omega)$.

Remark 2.3. Local log-Hölder continuity was first considered for the variable Lebesgue spaces by Sharapudinov [96]; log-Hölder continuity at infinity was introduced in [22].

One nice property of log-Hölder continuity is the following extension theorem. For a proof, see [20].

Lemma 2.4. *Given a set $\Omega \subset \mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p(\cdot) \in LH(\Omega)$, there exists a function $\tilde{p}(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that:*

1. *$\tilde{p} \in LH$;*
2. *$\tilde{p}(x) = p(x)$, $x \in \Omega$;*
3. *$\tilde{p}_- = p_-$ and $\tilde{p}_+ = p_+$.*

2.2. The modular and the norm

Intuitively, given an exponent function $p(\cdot) \in \mathcal{P}(\Omega)$, we want to define the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions f such that

$$\int_\Omega |f(x)|^{p(x)}\,dx < \infty.$$

There are two problems with this approach: first, as we noted in the Introduction, there is a problem with homogeneity if $p_+ = \infty$. Moreover, if Ω_∞ has positive measure, then the integral no longer makes sense. We therefore make a more careful definition.

Definition 2.5. Given Ω, $p(\cdot) \in \mathcal{P}(\Omega)$ and a measurable function f, define the modular functional (or simply the modular) associated with $p(\cdot)$ by

$$\rho_{p(\cdot)}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)}.$$

If there is no ambiguity, we will write simply $\rho(f)$.

Remark 2.6. There are two other definitions of the modular in the literature. One immediate alternative is to define it as

$$\rho(f) = \max\left(\int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx,\ \|f\|_{L^\infty(\Omega_\infty)}\right).$$

This was done by Edmunds and Rákosnik [35]. Clearly this is equivalent to our definition and yields the same norm. A very different approach motivated by the theory of Musielak-Orlicz spaces is to define

$$\rho(f) = \int_\Omega |f(x)|^{p(x)}\,dx,$$

with the convention that $t^\infty = \infty \cdot \chi_{(1,\infty)}(t)$. This modular (or, more precisely, semi-modular) is no longer equivalent to ours, but the resulting norm is equivalent to ours. See Diening *et al.* [30] for further information about this approach.

The modular has the following properties.

Proposition 2.7. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, then:*

(1) *for all f, $\rho(f) \geq 0$ and $\rho(|f|) = \rho(f)$.*
(2) *$\rho(f) = 0$ if and only if $f(x) = 0$ for a.e. $x \in \Omega$.*
(3) *If $\rho(f) < \infty$, then $f(x) < \infty$ for a.e. $x \in \Omega$.*
(4) *ρ is convex: given α, $\beta \geq 0$, $\alpha + \beta = 1$,*
$$\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g).$$
(5) *If $|f(x)| \geq |g(x)|$ a.e., then $\rho(f) \geq \rho(g)$.*
(6) *If for some $\Lambda > 0$, $\rho(f/\Lambda) < \infty$, then the function $\lambda \mapsto \rho(f/\lambda)$ is continuous and decreasing on $[\Lambda, \infty)$. Further, $\rho(f/\lambda) \to 0$ as $\lambda \to \infty$.*

An immediate consequence of the convexity of ρ is that if $\alpha > 1$, then $\alpha\rho(f) \leq \rho(\alpha f)$, and if $0 < \alpha < 1$, then $\rho(\alpha f) \leq \alpha\rho(f)$. We will often invoke this property by referring to the convexity of the modular.

Proof. Property (1) is immediate from the definition of the modular, and Properties (2), (3) and (5) follow from the properties of the L^1 and L^∞ norms.

Property (4) follows since the L^∞ norm is convex and since for almost every $x \in \Omega \setminus \Omega_\infty$, the function $t \mapsto t^{p(x)}$ is convex.

To prove (6), note that by Property (5), if $\lambda \geq \Lambda$, then $\rho(f/\lambda)$ is a decreasing function, and by the dominated convergence theorem (applied to the integral) it is continuous and tends to 0 as $\lambda \to \infty$. □

With the modular in hand we define the variable Lebesgue spaces.

Definition 2.8. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, define $L^{p(\cdot)}(\Omega)$ to be the set of Lebesgue measurable functions f such that $\rho(f/\lambda) < \infty$ for some $\lambda > 0$.

While this more technical definition is necessary when $p(\cdot)$ is unbounded, we can simplify the definition when $p_+ < \infty$.

Proposition 2.9. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_+ < \infty$, then $f \in L^{p(\cdot)}(\Omega)$ if and only if*

$$\rho(f) = \int_\Omega |f(x)|^{p(x)}\,dx < \infty.$$

Proof. Since $p_+ < \infty$, we can drop the L^∞ term in the modular. Clearly, if $\rho(f) < \infty$, then $f \in L^{p(\cdot)}$. Conversely, by Property (5) in Proposition 2.7, we have that $\rho(f/\lambda) < \infty$ for some $\lambda > 1$. But then

$$\rho(f) = \int_\Omega \left(\frac{|f(x)|\lambda}{\lambda}\right)^{p(x)} dx \leq \lambda^{p_+(\Omega)}\rho(f/\lambda) < \infty. \qquad \square$$

In the proof we "pulled" a constant out of the modular. The ability to do so is very useful, and makes the study of variable Lebesgue spaces in this case much simpler. The proof of Proposition 2.9 is easily modified to prove the following inequalities.

PROPOSITION 2.10. *Given* Ω *and* $p(\cdot) \in \mathcal{P}(\Omega)$*:*

(1) *if* $p_+ < \infty$, *then for all* $\lambda \geq 1$,
$$\lambda^{p_-}\rho(f) \leq \rho(\lambda f) \leq \lambda^{p_+}\rho(f).$$
When $0 < \lambda < 1$ *the reverse inequalities are true.*

(2) *If* $p_+(\Omega \setminus \Omega_\infty) < \infty$, *then for all* $\lambda \geq 1$,
$$\rho(\lambda f) \leq \lambda^{p_+(\Omega\setminus\Omega_\infty)}\rho(f).$$

We now want to prove that $L^{p(\cdot)}(\Omega)$ is a Banach space. Here we will prove that it is a normed vector space; we defer the proof that it is complete to the next section.

THEOREM 2.11. *Given* Ω *and* $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ *is a vector space.*

PROOF. Since the set of all Lebesgue measurable functions is itself a vector space, and since $0 \in L^{p(\cdot)}(\Omega)$, it will suffice to show that for all α, $\beta \in \mathbb{R}$, not both 0, if f, $g \in L^{p(\cdot)}(\Omega)$, then $\alpha f + \beta g \in L^{p(\cdot)}(\Omega)$. By Property (5) in Proposition 2.7, there exists $\lambda > 0$ such that $\rho(f/\lambda)$, $\rho(g/\lambda) < \infty$. Therefore, by Properties (1), (3) and (4) of the same proposition, if we let $\mu = \big(|\alpha| + |\beta|\big)\lambda$, then
$$\begin{aligned}\rho\left(\frac{\alpha f + \beta g}{\mu}\right) = \rho\left(\frac{|\alpha f + \beta g|}{\mu}\right) &\leq \rho\left(\frac{|\alpha|}{|\alpha| + |\beta|}\frac{|f|}{\lambda} + \frac{|\beta|}{|\alpha| + |\beta|}\frac{|g|}{\lambda}\right) \\ &\leq \frac{|\alpha|}{|\alpha| + |\beta|}\rho(f/\lambda) + \frac{|\beta|}{|\alpha| + |\beta|}\rho(g/\lambda) < \infty. \qquad \square\end{aligned}$$

Next, we define the norm. On the classical Lebesgue spaces, if $1 \leq p < \infty$, then the norm is gotten directly from the modular:
$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}.$$
Such a definition obviously fails since we cannot replace the constant exponent $1/p$ outside the integral with the exponent function $1/p(\cdot)$. Instead, we use the Luxemburg norm, similar to that used to define Orlicz spaces (cf. [**7**, **66**]).

DEFINITION 2.12. Given Ω, $p(\cdot) \in \mathcal{P}(\Omega)$, and a measurable function f, define
$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),\Omega}(f/\lambda) \leq 1\right\}.$$
If there is no ambiguity over the domain Ω, we will often write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\Omega)}$.

When $p(\cdot) = p$, $1 \leq p \leq \infty$, Definition 2.12 is equivalent to the classical norm on $L^p(\Omega)$: if $p < \infty$ and
$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^p dx = 1,$$
then $\lambda = \|f\|_{L^p(\Omega)}$; the same is true if $p = \infty$.

THEOREM 2.13. *Given* Ω *and* $p(\cdot) \in \mathcal{P}(\Omega)$, *the functional* $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ *defines a norm on* $L^{p(\cdot)}(\Omega)$.

PROOF. We will prove that $\|\cdot\|_{p(\cdot)}$ has the following properties:

(1) $\|f\|_{p(\cdot)} = 0$ if and only if $f \equiv 0$;
(2) for all $\alpha \in \mathbb{R}$, $\|\alpha f\|_{p(\cdot)} = |\alpha|\|f\|_{p(\cdot)}$;
(3) $\|f + g\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} + \|g\|_{p(\cdot)}$.

If $f \equiv 0$, then $\rho(f/\lambda) = 0 \le 1$ for all $\lambda > 0$, and so $\|f\|_{p(\cdot)} = 0$. Conversely, if $\|f\|_{p(\cdot)} = 0$, then for all $\lambda > 0$,

$$1 \ge \rho(f/\lambda) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \|f/\lambda\|_{L^\infty(\Omega_\infty)}.$$

We consider each term of the modular separately. It is immediate that we have $\|f\|_{L^\infty(\Omega_\infty)} \le \lambda$; hence, $f(x) = 0$ for almost every $x \in \Omega_\infty$. Similarly, if $\lambda < 1$, we have

$$1 \ge \lambda^{-p_-} \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\, dx.$$

Therefore, $\|f(\cdot)^{p(\cdot)}\|_{L^1(\Omega\setminus\Omega_\infty)} = 0$, and so $f(x) = |f(x)|^{p(x)} = 0$ for almost every $x \in \Omega \setminus \Omega_\infty$. Thus $f \equiv 0$ and we have proved (1).

To prove (2), note that if $\alpha = 0$, this follows from (1). Fix $\alpha \neq 0$; then by a change of variables,

$$\begin{aligned}\|\alpha f\|_{p(\cdot)} &= \inf\{\lambda > 0 : \rho(|\alpha| f/\lambda) \le 1\}\\ &= |\alpha| \inf\{\lambda/|\alpha| > 0 : \rho(f/(\lambda/|\alpha|)) \le 1\}\\ &= |\alpha| \inf\{\mu > 0 : \rho(f/\mu)) \le 1\} = |\alpha|\|f\|_{p(\cdot)}.\end{aligned}$$

Finally, to prove (3), fix $\lambda_f > \|f\|_{p(\cdot)}$ and $\lambda_g > \|g\|_{p(\cdot)}$; then $\rho(f/\lambda_f) \le 1$ and $\rho(g/\lambda_g) \le 1$. Now let $\lambda = \lambda_f + \lambda_g$. Then by Property (3) of Proposition 2.7,

$$\rho\left(\frac{f+g}{\lambda}\right) = \rho\left(\frac{\lambda_f}{\lambda}\frac{f}{\lambda_f} + \frac{\lambda_g}{\lambda}\frac{g}{\lambda_g}\right) \le \frac{\lambda_f}{\lambda}\rho(f/\lambda_f) + \frac{\lambda_g}{\lambda}\rho(g/\lambda_g) \le 1.$$

Hence, $\|f+g\|_{p(\cdot)} \le \lambda_f + \lambda_g$. Taking the infimum over all such λ_f and λ_g we get the desired inequality. □

REMARK 2.14. There is an equivalent norm on $L^{p(\cdot)}(\Omega)$ that is usually referred to as the Amemiya norm. For $p_+ < \infty$, define

$$\|f\|^A_{p(\cdot)} = \inf\{\lambda > 0 : \lambda + \lambda\rho_{p(\cdot)}(f/\lambda)\}.$$

Then

$$\|f\|_{p(\cdot)} \le \|f\|^A_{p(\cdot)} \le 2\|f\|_{p(\cdot)}.$$

For a proof, see Samko [91].

Though the norm is defined by an infimum, if f is non-trivial, then the infimum is always attained. (If $f \equiv 0$, then clearly the infimum is zero and is not attained.) In Proposition 2.15 below we will prove that $\rho(f/\|f\|_{p(\cdot)}) \le 1$, so $\lambda = \|f\|_{p(\cdot)}$ is always an element of the set $\{\lambda : \rho(f/\lambda) \le 1\}$.

PROPOSITION 2.15. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in L^{p(\cdot)}(\Omega)$ and $\|f\|_{p(\cdot)} > 0$, then $\rho(f/\|f\|_{p(\cdot)}) \le 1$. If $p_+ < \infty$, then $\rho(f/\|f\|_{p(\cdot)}) = 1$ for all non-trivial $f \in L^{p(\cdot)}(\Omega)$.*

PROOF. Fix a decreasing sequence $\{\lambda_n\}$ such that $\lambda_n \to \|f\|_{p(\cdot)}$. Then by Fatou's lemma and the definition of the modular,

$$\rho(f/\|f\|_{p(\cdot)}) \le \liminf_{n\to\infty} \rho(f/\lambda_n) \le 1.$$

Now suppose that $p_+ < \infty$ but $\rho(f/\|f\|_{p(\cdot)}) < 1$. Then for all λ, $0 < \lambda < \|f\|_{p(\cdot)}$, by Proposition 2.10,

$$\rho(f/\lambda) = \rho\left(\frac{\|f\|_{p(\cdot)}}{\lambda}\frac{f}{\|f\|_{p(\cdot)}}\right) \le \left(\frac{\|f\|_{p(\cdot)}}{\lambda}\right)^{p_+} \rho\left(\frac{f}{\|f\|_{p(\cdot)}}\right).$$

Therefore, we can find λ sufficiently close to $\|f\|_{p(\cdot)}$ such that $\rho(f/\lambda) < 1$. But by the definition of the norm, we must have $\rho(f/\lambda) \geq 1$. From this contradiction we see that equality holds. $\square$

Corollary 2.16. *Fix Ω and $p(\cdot) \in \mathcal{P}(\Omega)$. If $\|f\|_{p(\cdot)} \leq 1$, then $\rho(f) \leq \|f\|_{p(\cdot)}$; if $\|f\|_{p(\cdot)} > 1$, then $\rho(f) \geq \|f\|_{p(\cdot)}$.*

Proof. If $\|f\|_{p(\cdot)} = 0$, then $f \equiv 0$ and so $\rho(f) = 0$. If $0 < \|f\|_{p(\cdot)} \leq 1$, then by the convexity of the modular (Property (4) of Proposition 2.7) and Proposition 2.15,

$$\rho(f) = \rho(\|f\|_{p(\cdot)}\, f/\|f\|_{p(\cdot)}) \leq \|f\|_{p(\cdot)}\rho(f/\|f\|_{p(\cdot)}) \leq \|f\|_{p(\cdot)}.$$

If $\|f\|_{p(\cdot)} > 1$, then $\rho(f) > 1$: for if $\rho(f) \leq 1$, then by the definition of the norm we would have $\|f\|_{p(\cdot)} \leq 1$. But then we have that

$$\begin{aligned}\rho\big(f/\rho(f)\big) &= \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\rho(f)}\right)^{p(x)} dx + \rho(f)^{-1}\|f\|_{L^\infty(\Omega_\infty)} \\ &\leq \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\rho(f)^{-1}\, dx + \rho(f)^{-1}\|f\|_{L^\infty(\Omega_\infty)} = 1.\end{aligned}$$

It follows that $\|f\|_{p(\cdot)} \leq \rho(f)$. $\square$

The previous result can be strengthened by the following result due to Fan and Zhao [40].

Corollary 2.17. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. If $\|f\|_{p(\cdot)} > 1$, then*

$$\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}.$$

If $0 < \|f\|_{p(\cdot)} \leq 1$, then

$$\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+}.$$

If $p(\cdot)$ is constant, Corollary 2.17 reduces to the identity

$$\|f\|_p = \left(\int_\Omega |f(x)|^p\, dx\right)^{1/p}.$$

Proof. We prove the first pair of inequalities; the proof of the second is essentially the same. If $p_+ < \infty$, by Proposition 2.10,

$$\frac{\rho(f)}{\|f\|_{p(\cdot)}^{p_+}} \leq \rho\left(\frac{f}{\|f\|_{p(\cdot)}}\right) \leq \frac{\rho(f)}{\|f\|_{p(\cdot)}^{p_-}}.$$

By Proposition 2.15, $\rho(f/\|f\|_{p(\cdot)}) = 1$, and we are done. $\square$

2.3. Convergence and completeness

To prove that variable Lebesgue spaces are Banach spaces, we first consider norm convergence. The following results are all of interest in their own right; in addition the first two are necessary for the proof of completeness.

Theorem 2.18. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a sequence of non-negative functions such that f_k increases to a function f pointwise a.e. Then either $f \in L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$, or $f \notin L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \to \infty$.*

In the context of Banach function spaces, Theorem 2.18 is referred to as the Fatou property of the norm. To emphasize the connection with the classical Lebesgue spaces, we will refer to it as the monotone convergence theorem. This result was first proved in [12].

Proof. Since $\{f_k\}$ is an increasing sequence, so is $\{\|f_k\|_{p(\cdot)}\}$; thus, it either converges or diverges to ∞. If $f \in L^{p(\cdot)}(\Omega)$, since $f_k \leq f$, $\|f_k\|_{p(\cdot)} \leq \|f\|_{p(\cdot)}$; otherwise, since $f_k \in L^{p(\cdot)}(\Omega)$, $\|f_k\|_{p(\cdot)} < \infty = \|f\|_{p(\cdot)}$. In either case it will suffice to show that for any $\lambda < \|f\|_{p(\cdot)}$, for all k sufficiently large $\|f_k\|_{p(\cdot)} > \lambda$.

Fix such a λ; by the definition of the norm, $\rho(f/\lambda) > 1$. Therefore, by the monotone convergence theorem on the classical Lebesgue spaces,

$$\begin{aligned}\rho(f/\lambda) &= \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|f\|_{L^\infty(\Omega_\infty)} \\ &= \lim_{k\to\infty} \left(\int_{\Omega\setminus\Omega_\infty} \left(\frac{|f_k(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|f_k\|_{L^\infty(\Omega_\infty)}\right) \\ &= \lim_{k\to\infty} \rho(f_k/\lambda).\end{aligned}$$

Hence, for all k sufficiently large, $\rho(f_k/\lambda) > 1$, and so $\|f_k\|_{p(\cdot)} > \lambda$. □

The next result is the analog of Fatou's Lemma. It is proved in [19].

Theorem 2.19. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ is such that $f_k \to f$ pointwise a.e. If*

$$\liminf_{k\to\infty} \|f_k\|_{p(\cdot)} < \infty,$$

then $f \in L^{p(\cdot)}(\Omega)$ and

$$\|f\|_{p(\cdot)} \leq \liminf_{k\to\infty} \|f_k\|_{p(\cdot)}.$$

Proof. Define a new sequence

$$g_k(x) = \inf_{m\geq k} |f_m(x)|.$$

Then for all $m \geq k$, $g_k(x) \leq |f_m(x)|$, and so $g_k \in L^{p(\cdot)}(\Omega)$. Further, by definition $\{g_k\}$ is an increasing sequence and

$$\lim_{k\to\infty} g_k(x) = \liminf_{m\to\infty} |f_m(x)| = |f(x)|, \quad \text{a.e. } x \in \Omega.$$

Therefore, by Theorem 2.18,

$$\|f\|_{p(\cdot)} = \lim_{k\to\infty} \|g_k\|_{p(\cdot)} \leq \lim_{k\to\infty} \big(\inf_{m\geq k} \|f_m\|_{p(\cdot)}\big) = \liminf_{k\to\infty} \|f_k\|_{p(\cdot)} < \infty,$$

and $f \in L^{p(\cdot)}(\Omega)$. □

Unlike the previous two results, to prove a version of the dominated convergence theorem we need to assume $p_+ < \infty$. This result was first proved in [19] The proof requires a lemma relating convergence in norm to convergence in modular.

Lemma 2.20. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. For any sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ and $f \in L^{p(\cdot)}(\Omega)$, $\|f_k - f\|_{p(\cdot)} \to 0$ if and only if $\rho(f - f_k) \to 0$.*

Proof. Suppose the sequence converges in norm. By Corollary 2.16, for all k sufficiently large,

$$\rho(f - f_k) \leq \|f - f_k\|_{p(\cdot)} \leq 1,$$

and so $\rho(f - f_k) \to 0$.

To prove the converse, fix $\lambda < 1$. By Proposition 2.10,

$$\rho((f - f_k)/\lambda) \leq \left(\frac{1}{\lambda}\right)^{p_+} \rho(f - f_k).$$

Hence, for all k sufficiently large we have that

$$\rho\left(\frac{f - f_k}{\lambda}\right) \leq 1.$$

Equivalently, for all such k, $\|f - f_k\|_{p(\cdot)} \leq \lambda$. Since λ was arbitrary, $f_k \to f$ in norm. $\square$

THEOREM 2.21. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. If the sequence $\{f_k\}$ is such that $f_k \to f$ pointwise a.e., and there exists $g \in L^{p(\cdot)}(\Omega)$ such that $|f_k(x)| \leq g(x)$ a.e., then $f \in L^{p(\cdot)}(\Omega)$ and $\|f - f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$.*

PROOF. By Proposition 2.9,

$$\begin{aligned}|f(x) - f_k(x)|^{p(x)} &\leq 2^{p(x)-1}(|f(x)|^{p(x)} + |f_k(x)|^{p(x)}) \\ &\leq 2^{p_+}|g(x)|^{p(x)} \in L^1(\Omega).\end{aligned}$$

Then by the dominated convergence theorem on L^1, $\rho(f - f_k) \to 0$ as $k \to 0$, and by Lemma 2.20, $\|f - f_k\|_{p(\cdot)} \to 0$. $\square$

The final convergence result shows that norm convergence yields pointwise convergence on subsequences. The proof depends on showing that norm convergence implies convergence in measure; see [19] for details.

THEOREM 2.22. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f_k \to f$ in norm in $L^{p(\cdot)}(\Omega)$, then there exists a subsequence $\{f_{k_j}\}$ that converges pointwise a.e. to f.*

REMARK 2.23. Convergence in norm is not equivalent to convergence in modular when $p_+ = \infty$. We can also consider the relationship between these and convergence in measure. For a careful discussion of all of these ideas, see [**18, 19**].

We can now prove completeness. We do so by first proving that the Riesz-Fischer property holds in variable Lebesgue spaces. This proof is from [19]; a very different proof of completeness appeared in [64].

THEOREM 2.24. *Given Ω and $p(\cdot) \in L^{p(\cdot)}(\Omega)$, let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be such that*

$$\sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

Then there exists $f \in L^{p(\cdot)}(\Omega)$ such that

$$\sum_{k=1}^{i} f_k \to f$$

in norm as $i \to \infty$, and

$$\|f\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)}.$$

PROOF. Define the function F on Ω by

$$F(x) = \sum_{k=1}^{\infty} |f_k(x)|,$$

and define the sequence $\{F_i\}$ by

$$F_i(x) = \sum_{k=1}^{i} |f_k(x)|.$$

Then the sequence $\{F_i\}$ is non-negative and increases pointwise a.e. to F. Further, for each i, $F_i \in L^{p(\cdot)}(\Omega)$, and its norm is uniformly bounded, since

$$\|F_i\|_{p(\cdot)} \leq \sum_{k=1}^{i} \|f_k\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

Therefore, by Theorem 2.18, $F \in L^{p(\cdot)}(\Omega)$.

In particular, F is finite a.e., so the sequence F_k converges pointwise a.e. Hence, if we define the sequence of functions $\{G_i\}$ by

$$G_i(x) = \sum_{k=1}^{i} f_k(x),$$

then this sequence also converges pointwise a.e. since absolute convergence implies convergence. Denote its sum by f.

Now let $G_0 = 0$; then for fixed $j \geq 0$, $G_i - G_j \to f - G_j$ pointwise a.e. Furthermore,

$$\liminf_{i\to\infty} \|G_i - G_j\|_{p(\cdot)} \leq \liminf_{i\to\infty} \sum_{k=j+1}^{i} \|f_k\|_{p(\cdot)} = \sum_{k=j+1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

By Theorem 2.19, if we take $j = 0$, then

$$\|f\|_{p(\cdot)} \leq \liminf_{i\to\infty} \|G_i\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

More generally, for each j the same argument shows that

$$\|f - G_j\|_{p(\cdot)} \leq \liminf_{i\to\infty} \|G_i - G_j\|_{p(\cdot)} \leq \sum_{k=j+1}^{\infty} \|f_k\|_{p(\cdot)};$$

since the sum on the right-hand side tends to 0, we see that $G_j \to f$ in norm, which completes the proof. $\square$

The completeness of $L^{p(\cdot)}(\Omega)$ is a consequence of Theorem 2.24.

Theorem 2.25. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is complete: every Cauchy sequence in $L^{p(\cdot)}(\Omega)$ converges.*

Proof. Let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a Cauchy sequence. Choose k_1 such that $\|f_i - f_j\|_{p(\cdot)} < 2^{-1}$ for i, $j \geq k_1$, choose $k_2 > k_1$ such that $\|f_i - f_j\|_{p(\cdot)} < 2^{-2}$ for i, $j \geq k_2$, and so on. This construction yields a subsequence $\{f_{k_j}\}$, $k_{j+1} > k_j$, such that

$$\|f_{k_{j+1}} - f_{k_j}\|_{p(\cdot)} < 2^{-j}.$$

Define the new sequence $\{g_j\}$ by $g_1 = f_{k_1}$ and for $j > 1$, $g_j = f_{k_j} - f_{k_{j-1}}$. Then for all j we get the telescoping sum

$$\sum_{i=1}^{j} g_i = f_{k_j};$$

further, we have that

$$\sum_{j=1}^{\infty} \|g_j\|_{p(\cdot)} \leq \|f_{k_1}\|_{p(\cdot)} + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Therefore, by Theorem 2.24, there exists $f \in L^{p(\cdot)}(\Omega)$ such that $f_{k_j} \to f$ in norm.

Finally, by the triangle inequality we have that

$$\|f - f_k\|_{p(\cdot)} \leq \|f - f_{k_j}\|_{p(\cdot)} + \|f_{k_j} - f_k\|_{p(\cdot)};$$

since $\{f_k\}$ is a Cauchy sequence, we can make the right-hand side as small as desired. Hence, $f_k \to f$ in norm. $\square$

2.4. Embeddings and dense subsets

In the classical Lebesgue spaces, if $|\Omega| < \infty$, then $L^p(\Omega) \subset L^q(\Omega)$ whenever $p > q$. Similar embeddings holds in the variable Lebesgue spaces.

THEOREM 2.26. *Given Ω and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, suppose $|\Omega| < \infty$. Then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ provided that $p(x) \le q(x)$ almost everywhere. Furthermore, in this case we have that*

$$\|f\|_{p(\cdot)} \le (1 + |\Omega|)\|f\|_{q(\cdot)}. \tag{2.1}$$

PROOF. Suppose that $p(x) \le q(x)$ a.e. By the homogeneity of the norm, it will suffice to show that if $f \in L^{q(\cdot)}(\Omega)$, $\|f\|_{q(\cdot)} \le 1$, then $\|f\|_{p(\cdot)} \le 1 + |\Omega \setminus \Omega_\infty^{p(\cdot)}|$. By the definition of the norm,

$$1 \ge \rho_{q(\cdot)}(f) = \int_{\Omega\setminus\Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})}.$$

In particular, $|f(x)| \le 1$ a.e. on $\Omega_\infty^{q(\cdot)}$. Further, since $p(x) \le q(x)$, $\Omega_\infty^{p(\cdot)} \subset \Omega_\infty^{q(\cdot)}$ up to a set of measure zero. Therefore,

$$\begin{aligned}
\rho_{p(\cdot)}(f) &= \int_{\Omega\setminus\Omega_\infty^{q(\cdot)}} |f(x)|^{p(x)}\,dx + \int_{\Omega_\infty^{q(\cdot)}\setminus\Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \\
&\le |\{x \in \Omega\setminus\Omega_\infty^{q(\cdot)} : |f(x)| \le 1\}| + \int_{\Omega\setminus\Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx \\
&\qquad + |\Omega_\infty^{q(\cdot)} \setminus \Omega_\infty^{p(\cdot)}| + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})} \\
&\le |\Omega| + \rho_{q(\cdot)}(f) \\
&\le |\Omega| + 1.
\end{aligned}$$

Hence, by the convexity of the modular,

$$\rho_{p(\cdot)}\left(\frac{f}{|\Omega|+1}\right) \le \frac{\rho_{p(\cdot)}(f)}{|\Omega|+1} \le 1,$$

and so $\|f\|_{p(\cdot)} \le |\Omega| + 1$. □

As an immediate corollary we get the following embedding relationship between the variable and classical Lebesgue spaces.

COROLLARY 2.27. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $|\Omega| < \infty$. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\|f\|_{p_-} \le \|f\|_{p(\cdot)} \le c_2\|f\|_{p_+}.$$

In particular, given any Ω, if $f \in L^{p(\cdot)}(\Omega)$, then f is locally integrable.

Unlike the classical case, when Ω is unbounded it is possible to get a non-trivial embedding—for instance, as we noted in the Introduction it is possible to embed L^∞ in $L^{p(\cdot)}(\Omega)$. The precise conditions required are given in the following theorem. For a proof, see [**19**, **30**].

THEOREM 2.28. *Given Ω and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ and there exists $K > 1$ such that for all $f \in L^{q(\cdot)}(\Omega)$, $\|f\|_{p(\cdot)} \le K\|f\|_{q(\cdot)}$, if and only if:*

1. *$p(x) \le q(x)$ for almost every $x \in \Omega$;*
2. *there exists $\lambda > 1$ such that*

$$\int_D \lambda^{-r(x)}\,dx < \infty, \tag{2.2}$$

where $D = \{x \in \Omega : p(x) < q(x)\}$ and $r(\cdot)$ is the defect exponent defined by
$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

It is possible to decompose a function $f \in L^{p(\cdot)}(\Omega)$ so that the pieces are contained in Classical Lebesgue spaces. This decomposition complements Corollary 2.27 and is very useful in applications.

THEOREM 2.29. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in L^{p(\cdot)}(\Omega)$, then we can write $f = f_1 + f_2$ where $f_1 \in L^{p_+}(\Omega) \cap L^{p(\cdot)}(\Omega)$ and $f_2 \in L^{p_-}(\Omega) \cap L^{p(\cdot)}(\Omega)$.*

PROOF. By the linearity of the norms we may assume without loss of generality that $\|f\|_{p(\cdot)} = 1$. This implies that $\|f\|_{L^\infty(\Omega_\infty)} \le 1$. Decompose f as $f_1 + f_2$, where
$$f_1 = f\chi_{\{x \in \Omega : |f(x)| \le 1\}}$$
$$f_2 = f\chi_{\{x \in \Omega \setminus \Omega_\infty : |f(x)| > 1\}}.$$
Clearly, $f_1, f_2 \in L^{p(\cdot)}(\Omega)$. If $p_+ < \infty$, $|\Omega_\infty| = 0$, so by Corollary 2.16,
$$\int_\Omega |f_1(x)|^{p_+}\,dx \le \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)}\,dx \le \|f\|_{p(\cdot)} = 1,$$
$$\int_\Omega |f_2(x)|^{p_-}\,dx \le \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)}\,dx \le \|f\|_{p(\cdot)} = 1.$$
Hence,
$$\|f_1\|_{p_+}, \|f_2\|_{p_-} \le 1 = \|f\|_{p(\cdot)}.$$
If $p_+ = \infty$, then we argue as before for f_2 and for f_1 we note that $\|f_1\|_\infty \le 1 = \|f\|_{p(\cdot)}$. $\square$

We now consider the problem of dense subsets in $L^{p(\cdot)}(\Omega)$. A good understanding of dense subsets only exists when $p_+ < \infty$. For the case $p_+ = \infty$, see [19].

THEOREM 2.30. *Given an open set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. Then the set of bounded functions with compact support with $\operatorname{supp}(f) \subset \Omega$ is dense in $L^{p(\cdot)}(\Omega)$. Moreover, the set $C_c^\infty(\Omega)$ of smooth functions with compact support is dense in $L^{p(\cdot)}(\Omega)$.*

PROOF. Fix $f \in L^{p(\cdot)}(\Omega)$. Let K_k be a nested sequence of compact subsets of Ω such that $\Omega = \bigcup_k K_k$. (For instance, let $K_k = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge 1/k\} \cap \overline{B_k(0)}$.) Define the sequence $\{f_k\}$ by
$$f_k(x) = \begin{cases} k & f_k(x) > k \\ f(x) & -k \le f(x) \le k \\ -k & f_k(x) < -k, \end{cases}$$
and let $g_k(x) = f_k(x)\chi_{K_k}(x)$. Since f is finite a.e., $g_k \to f$ pointwise a.e.; since $f \in L^{p(\cdot)}(\Omega)$ and $|g_k(x)| \le |f(x)|$, $g_k \in L^{p(\cdot)}(\Omega)$. Since $p_+ < \infty$, by Theorem 2.21, $g_k \to f$ in norm.

To show that $C_c^\infty(\Omega)$ is dense, fix $\epsilon > 0$; we will find a function $h \in C_c^\infty(\Omega)$ such that $\|f - h\|_{p(\cdot)} < \epsilon$. By the above argument there exists a bounded function of compact support, g, such that $\|f - g\|_{p(\cdot)} < \epsilon/2$. Let $\operatorname{supp}(g) \subset B \cap \Omega$ for some open ball B. Since $p_+ < \infty$, $C_c^\infty(B \cap \Omega)$ is dense in $L^{p_+}(B \cap \Omega)$; thus there exists $h \in C_c^\infty(B \cap \Omega) \subset C_c^\infty(\Omega)$ such that
$$\|g - h\|_{L^{p_+}(\Omega)} = \|g - h\|_{L^{p_+}(B \cap \Omega)} < \frac{\epsilon}{2(1 + |B \cap \Omega|)}.$$

Therefore, by Theorem 2.26,

$$\begin{aligned}\|g-h\|_{L^{p(\cdot)}(\Omega)} &= \|g-h\|_{L^{p(\cdot)}(B\cap\Omega)}\\ &\leq (1+|B\cap\Omega|)\|g-h\|_{L^{p_+}(B\cap\Omega)} < \epsilon/2,\end{aligned}$$

and so

$$\|f-h\|_{p(\cdot)} \leq \|f-g\|_{p(\cdot)} + \|g-h\|_{p(\cdot)} < \epsilon. \qquad \square$$

If $p_+ < \infty$, L^{p_+} is separable, so the proof of Theorem 2.30 can be readily modified to yield the separability of $L^{p(\cdot)}$ in this case. This is false when $p_+ = \infty$; see [19].

THEOREM 2.31. *Given an open set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is separable.*

2.5. Hölder's inequality, the associate norm and duality

In this section we show that the variable Lebesgue space norm satisfies a generalization of Hölder's inequality, and then use this to define an equivalent norm, the associate norm, on $L^{p(\cdot)}(\Omega)$. Based on this we will be able to characterize the dual space when $p_+ < \infty$.

THEOREM 2.32. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \leq K_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

where

$$K_{p(\cdot)} = \frac{1}{p_-} - \frac{1}{p_+} + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty.$$

PROOF. Our proof is adapted from [64]. If $\|f\|_{p(\cdot)} = 0$ or $\|g\|_{p'(\cdot)} = 0$, then $fg \equiv 0$ so there is nothing to prove. Therefore, we may assume that $\|f\|_{p(\cdot)}, \|g\|_{p'(\cdot)} > 0$.

We consider the integral of $|fg|$ on the disjoint sets Ω_∞, Ω_1 and Ω_*. If $x \in \Omega_\infty$, then $p(x) = \infty$ and $p'(x) = 1$, so

$$\begin{aligned}\int_{\Omega_\infty} |f(x)g(x)|\,dx &\leq \|f\chi_{\Omega_\infty}\|_\infty \|g\chi_{\Omega_\infty}\|_1\\ &= \|f\chi_{\Omega_\infty}\|_{p(\cdot)}\|g\chi_{\Omega_\infty}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.\end{aligned}$$

Similarly, if we reverse the roles of $p(\cdot)$ and $p'(\cdot)$, we have that

$$\int_{\Omega_1} |f(x)g(x)|\,dx \leq \|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

To estimate the integral on Ω_* we use Young's inequality:

$$\begin{aligned}\int_{\Omega_*} \frac{|f(x)g(x)|}{\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}}\,dx &\leq \int_{\Omega_*} \frac{1}{p(x)}\left(\frac{|f(x)|}{\|f\|_{p(\cdot)}}\right)^{p(x)} dx + \frac{1}{p'(x)}\left(\frac{|g(x)|}{\|g\|_{p'(\cdot)}}\right)^{p'(x)} dx\\ &\leq \frac{1}{p_-}\rho_{p(\cdot)}(f/\|f\|_{p(\cdot)}) + \frac{1}{p'(\cdot)_-}\rho_{p'(\cdot)}(g/\|g\|_{p'(\cdot)}).\end{aligned}$$

Since

$$\frac{1}{p'(\cdot)_-} = \frac{1}{(p_+)'} = 1 - \frac{1}{p_+},$$

and by Proposition 2.15, $\rho_{p(\cdot)}(f/\|f\|_{p(\cdot)}) \leq 1$ and a similar inequality holds for g, we have that

$$\int_{\Omega_*} \frac{|f(x)g(x)|}{\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}}\,dx \leq \frac{1}{p_-} + 1 - \frac{1}{p_+}.$$

Combining the above terms, and using the fact that each is needed precisely when the L^∞ norm of the corresponding characteristic function equals 1, we have that

$$\int_\Omega |f(x)g(x)|\,dx \leq \left(\|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \frac{1}{p_-} - \frac{1}{p_+} + \|\chi_{\Omega_*}\|_\infty\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

which is the desired inequality. □

As a corollary we get a generalization of Hölder's inequality. For a proof, see Diening [28] and Samko [91, 92].

Corollary 2.33. *Given* Ω *and exponent functions* $r(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$ *define* $p(\cdot) \in \mathcal{P}(\Omega)$ *by*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a constant K *such that for all* $f \in L^{q(\cdot)}(\Omega)$ *and* $g \in L^{r(\cdot)}(\Omega)$, $fg \in L^{p(\cdot)}(\Omega)$ *and*

$$\|fg\|_{p(\cdot)} \leq K\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

Using Hölder's inequality we can define an alternative norm on $L^{p(\cdot)}$, the so-called associate norm.

Definition 2.34. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, and given a measurable function f, define

$$\|f\|'_{p(\cdot)} = \sup \int_\Omega f(x)g(x)\,dx, \tag{2.3}$$

where the supremum is taken over all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{p'(\cdot)} \leq 1$.

Temporarily denote by $M^{p(\cdot)}(\Omega)$ the set of all measurable functions f such that $\|f\|'_{p(\cdot)} < \infty$. We will show that $M^{p(\cdot)}$ and $L^{p(\cdot)}$ are the same space, and $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|'_{p(\cdot)}$ are equivalent norms.

Proposition 2.35. *Given* Ω *and* $p(\cdot) \in \mathcal{P}(\Omega)$, *the set* $M^{p(\cdot)}(\Omega)$ *is a normed vector space with respect to the norm* $\|\cdot\|'_{p(\cdot)}$. *Furthermore, the norm is order preserving: given* $f, g \in M^{p(\cdot)}(\Omega)$ *such that* $|f| \leq |g|$, *then* $\|f\|'_{p(\cdot)} \leq \|g\|'_{p(\cdot)}$.

Proof. It is immediate that $M^{p(\cdot)}(\Omega)$ is a vector space. The fact that $\|\cdot\|'_{p(\cdot)}$ is an order preserving norm is an consequence of the properties of integrals and supremums and the following equivalent characterization of $\|\cdot\|'_{p(\cdot)}$. First note that it is immediate from this definition that for all measurable functions f,

$$\|f\|'_{p(\cdot)} \leq \sup_{\|g\|_{p'(\cdot)}\leq 1} \left|\int_\Omega f(x)g(x)\,dx\right| \leq \sup_{\|g\|_{p'(\cdot)}\leq 1} \int_\Omega |f(x)g(x)|\,dx,$$

but in fact all of these are equal. To see this, note that for any $g \in L^{p'(\cdot)}(\Omega)$, $\|g\|_{p'(\cdot)} \leq 1$, $|f(x)g(x)| = f(x)h(x)$, where $h(x) = \operatorname{sgn} f(x)|g(x)|$ and $\|h\|_{p'(\cdot)} \leq \|g\|_{p'(\cdot)} \leq 1$; hence,

$$\int_\Omega |f(x)g(x)|\,dx = \int_\Omega f(x)h(x)\,dx \leq \|f\|'_{p(\cdot)}.$$ □

Theorem 2.36. *Given* Ω, $p(\cdot) \in \mathcal{P}(\Omega)$, *and a measurable* f, *then* $f \in L^{p(\cdot)}(\Omega)$ *if and only if* $f \in M^{p(\cdot)}(\Omega)$; *furthermore,*

$$k_{p(\cdot)}\|f\|_{p(\cdot)} \leq \|f\|'_{p(\cdot)} \leq K_{p(\cdot)}\|f\|_{p(\cdot)},$$

where

$$K_{p(\cdot)} = \frac{1}{p_-} - \frac{1}{p_+} + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty,$$
$$\frac{1}{k_{p(\cdot)}} = \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty.$$

To motivate the proof of Theorem 2.36, recall the proof of (2.3) if $1 < p < \infty$. By Hölder's inequality, $\|f\|'_p \le \|f\|_p$. To prove the reverse inequality, let

$$g(x) = \left(\frac{|f(x)|}{\|f\|_p}\right)^{p/p'} \operatorname{sgn} f(x).$$

Then $\|g\|_{p'} = 1$, and

$$\int_\Omega f(x)g(x)\,dx = \|f\|_p,$$

and so in fact the supremum is attained.

Lemma 2.37. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $\|f\chi_{\Omega_*}\|'_{p(\cdot)} \le 1$ and $\rho(f\chi_{\Omega_*}) < \infty$, then $\rho(f\chi_{\Omega_*}) \le 1$.*

Proof. Suppose to the contrary that $\rho(f\chi_{\Omega_*}) > 1$. Then by the continuity of the modular (Proposition 2.7, (6)) there exists $\lambda > 1$ such that $\rho(f\chi_{\Omega_*}/\lambda) = 1$. Let

$$g(x) = \left(\frac{|f(x)|}{\lambda}\right)^{p(x)-1} \operatorname{sgn} f(x)\chi_{\Omega_*}(x).$$

Then $\rho_{p'(\cdot)}(g) = \rho_{p(\cdot)}(f\chi_{\Omega_*}/\lambda) = 1$, so $\|g\|_{p'(\cdot)} \le 1$. Therefore, by the definition of the associate norm,

$$\begin{aligned}\|f\chi_{\Omega_*}\|'_{p(\cdot)} &\ge \int_\Omega f(x)\chi_{\Omega_*}(x)g(x)\\ &= \lambda\int_{\Omega_*}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx = \lambda\rho(f\chi_{\Omega_*}/\lambda) > 1.\end{aligned}$$

This contradicts our hypothesis on f, so the desired inequality holds. □

Proof of Theorem 2.36. One implication is immediate: by Theorem 2.32,

$$\|f\|'_{p(\cdot)} \le K_{p(\cdot)}\|f\|_{p(\cdot)}.$$

To prove the converse, we will assume that

$$|\Omega_\infty^{p(\cdot)}|,\ |\Omega_1^{p(\cdot)}|,\ |\Omega_*^{p(\cdot)}| > 0.$$

If any of these sets has measure 0, then the proof can be readily adapted by omitting the terms associated with them. Further, by the definition of the norm we may assume f is non-negative.

We will prove that if $\|f\|'_{p(\cdot)} \le 1$ and $\rho_{p(\cdot)}(f\chi_{\Omega_*}) < \infty$, then

$$\rho_{p(\cdot)}(k_{p(\cdot)}f) \le 1. \tag{2.4}$$

Given this, fix any non-negative $f \in M^{p(\cdot)}(\Omega)$; by homogeneity we may assume that $\|f\|'_{p(\cdot)} = 1$. For each $k \ge 1$, define the sets

$$E_k = B_k(0) \cap \big(\Omega \setminus \Omega_* \cup \{x \in \Omega_* : p(x) < k\}\big),$$

and define the functions $f_k = \min(f, k)\chi_{E_k}$. Then $f_k \le f$, so by Proposition 2.35, $\|f_k\|'_{p(\cdot)} \le \|f\|'_{p(\cdot)} \le 1$. Furthermore, the sequence $\{f_k\}$ increases to f pointwise. Finally, $\rho(f_k\chi_{\Omega_*}) < \infty$, and so we can apply (2.4) with f replaced by f_k. Therefore, by Fatou's lemma on classical Lebesgue spaces and (2.4),

$$\rho_{p(\cdot)}(k_{p(\cdot)}f/\|f\|'_{p(\cdot)}) = \rho_{p(\cdot)}(k_{p(\cdot)}f) \le \liminf_{k\to\infty} \rho_{p(\cdot)}(k_{p(\cdot)}f_k) \le 1.$$

Thus, we have that

$$\|f\|_{p(\cdot)} \leq k_{p(\cdot)}^{-1}\|f\|'_{p(\cdot)}.$$

To complete the proof, fix f with $\|f\|'_{p(\cdot)} \leq 1$ and $\rho(f\chi_{\Omega_*}) < \infty$; we will show that (2.4) holds. First note that by Proposition 2.35, $\|f\chi_{\Omega_*^{p(\cdot)}}\|'_{p(\cdot)} \leq 1$. Now fix ϵ, $0 < \epsilon < 1$; then there exists a set $E_\epsilon \subset \Omega_\infty^{p(\cdot)}$ such that $0 < |E_\epsilon| < \infty$, and for each $x \in E_\epsilon$,

$$|f(x)| \geq (1-\epsilon)\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}.$$

Now define the function g_ϵ by

$$g_\epsilon(x) = \begin{cases} k_{p(\cdot)}|f(x)|^{p(x)-1}\operatorname{sgn} f(x) & x \in \Omega_*^{p(\cdot)} = \Omega_*^{p'(\cdot)}, \\ k_{p(\cdot)}\operatorname{sgn} f(x) & x \in \Omega_1^{p(\cdot)} = \Omega_\infty^{p'(\cdot)}, \\ k_{p(\cdot)}|E_\epsilon|^{-1}\chi_{E_\epsilon}(x)\operatorname{sgn} f(x) & x \in \Omega_\infty^{p(\cdot)} = \Omega_1^{p'(\cdot)}. \end{cases}$$

We claim that $\rho_{p'(\cdot)}(g_\epsilon) \leq 1$, so $\|g_\epsilon\|_{p'(\cdot)} \leq 1$. To see this, note that

$$\begin{aligned} \rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) &\leq \int_{\Omega_*^{p'(\cdot)}} |f(x)|^{p(x)}\,dx + \|\operatorname{sgn} f\|_{L^\infty(\Omega_\infty^{p'(\cdot)})} + |E_\epsilon|^{-1}\int_{\Omega_1^{p'(\cdot)}} \chi_{E_\epsilon}(x)\,dx \\ &= \int_{\Omega_*^{p(\cdot)}} |f(x)|^{p(x)}\,dx + \|\operatorname{sgn} f\|_{L^\infty(\Omega_1^{p(\cdot)})} + |E_\epsilon|^{-1}\int_{\Omega_\infty^{p(\cdot)}} \chi_{E_\epsilon}(x)\,dx. \end{aligned}$$

By Lemma 2.37, the first term on the right-hand side is dominated by 1; the second term equals 0 or 1, and the third term always equals 1. Therefore,

$$\rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) \leq \|\chi_{\Omega_*^{p(\cdot)}}\|_\infty + \|\chi_{\Omega_1^{p(\cdot)}}\|_\infty + \|\chi_{\Omega_\infty^{p(\cdot)}}\|_\infty = \frac{1}{k_{p(\cdot)}}.$$

Since $k_{p(\cdot)} \leq 1$, by the convexity of the modular (Proposition 2.7),

$$\rho_{p'(\cdot)}(g_\epsilon) \leq k_{p(\cdot)}\rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) \leq 1,$$

which is what we claimed to be true.

Furthermore, we have that

$$\begin{aligned} \int_\Omega f(x)g_\epsilon(x)\,dx &= k_{p(\cdot)}\int_{\Omega_*^{p(\cdot)}} |f(x)|^{p(x)}\,dx + k_{p(\cdot)}\int_{\Omega_1^{p(\cdot)}} |f(x)|\,dx + k_{p(\cdot)}\fint_{E_\epsilon} |f(x)|\,dx \\ &\geq k_{p(\cdot)}\int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + (1-\epsilon)k_{p(\cdot)}\|f\|_{L^\infty(\Omega_\infty)} \\ &\geq (1-\epsilon)k_{p(\cdot)}\rho_{p(\cdot)}(f). \end{aligned}$$

Therefore, by the definition of the associate norm, since $\|g_\epsilon\|_{p'(\cdot)} \leq 1$,

$$1 \geq \|f\|'_{p(\cdot)} \geq \int_\Omega f(x)g_\epsilon(x)\,dx \geq (1-\epsilon)k_{p(\cdot)}\rho_{p(\cdot)}(f).$$

Since $\epsilon > 0$ was arbitrary, again by the convexity of the modular we have that

$$1 \geq k_{p(\cdot)}\rho_{p(\cdot)}(f) \geq \rho_{p(\cdot)}(k_{p(\cdot)}f). \qquad \square$$

We digress to prove Minkowski's integral inequality as a corollary of Theorem 2.36. This was first proved by Samko [**91, 92**].

Corollary 2.38. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, let $f\colon \Omega\times\Omega \to \mathbb{R}$ be a measurable function (with respect to product measure) such that for almost every $y \in \Omega$, $f(\cdot, y) \in L^{p(\cdot)}(\Omega)$. Then*

$$\left\|\int_\Omega f(\cdot,y)\,dy\right\|_{p(\cdot)} \leq k_{p(\cdot)}^{-1}K_{p(\cdot)}\int_\Omega \|f(\cdot,y)\|_{p(\cdot)}\,dy. \tag{2.5}$$

PROOF. If the right-hand side of (2.5) is infinite, then there is nothing to prove, so we may assume that this integral is finite. Define the function

$$g(x) = \int_\Omega f(x,y)\,dy,$$

and take any $h \in L^{p'(\cdot)}(\Omega)$, $\|h\|_{p'(\cdot)} \leq 1$. Then by Fubini's theorem (see Royden [85]),

$$\begin{aligned}
\int_\Omega |g(x)h(x)|\,dx &\leq \int_\Omega \int_\Omega |f(x,y)|\,dy\,|h(x)|\,dx \\
&= \int_\Omega \int_\Omega |f(x,y)h(x)|\,dx\,dy \\
&\leq K_{p(\cdot)} \int_\Omega \|f(\cdot,y)\|_{p(\cdot)} \|h\|_{p'(\cdot)}\,dy \\
&\leq K_{p(\cdot)} \int_\Omega \|f(\cdot,y)\|_{p(\cdot)}\,dy.
\end{aligned}$$

Therefore, we have that

$$\|g\|'_{p(\cdot)} \leq K_{p(\cdot)} \int_\Omega \|f(\cdot,y)\|_{p(\cdot)}\,dy,$$

and inequality (2.5) follows by Theorem 2.36. □

We finally turn to the dual space $L^{p(\cdot)}(\Omega)^*$ of continuous linear functionals $\Phi\colon L^{p(\cdot)}(\Omega) \to \mathbb{R}$ with norm

$$\|\Phi\| = \sup_{\|f\|_{p(\cdot)} \leq 1} |\Phi(f)|.$$

It follows immediately from Theorem 2.36 that given a measurable function g,

$$\Phi_g(f) = \int_\Omega f(x)g(x)\,dx$$

is a linear functional if and only if $g \in L^{p'(\cdot)}(\Omega)$ and

$$k_{p'(\cdot)}\|g\|_{p'(\cdot)} \leq \|\Phi_g\| \leq K_{p'(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.6}$$

When $p(\cdot)$ is bounded, we get every element of the dual space in this way.

THEOREM 2.39. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_+ < \infty$, then the map $g \mapsto \Phi_g$ is an isomorphism: given any continuous linear functional $\Phi \in L^{p(\cdot)}(\Omega)^*$ there exists a unique $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$ and $\|g\|_{p'(\cdot)} \approx \|\Phi\|$. Moreover, if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive.*

REMARK 2.40. Our proof is taken from [19] and is adapted from the proof for classical Lebesgue spaces in Royden [85]. When $p_+ = \infty$ this result is false. This was proved in [64]; their proof depends on deeper results about Orlicz-Musielak spaces due to Hudzik [61] and Kozek [65]. In [19] we give a direct proof.

PROOF OF THEOREM 2.39. Since $p'(\cdot)_+ = (p_-)'$, reflexivity follows at once from the first part of the theorem. Therefore, we will concentrate on proving the equivalence.

Suppose first that $p_+ < \infty$. Fix $\Phi \in L^{p(\cdot)}(\Omega)^*$; we will find $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$. Note that by (2.6) we immediately get that $\|g\|_{p'(\cdot)} \approx \|\Phi\|$.

We initially consider the case when $|\Omega| < \infty$. Define the set function μ by $\mu(E) = \Phi(\chi_E)$ for all measurable $E \subset \Omega$. Since Φ is linear and $\chi_{E\cup F} = \chi_E + \chi_F$ if $E \cap F = \emptyset$, μ is additive. To see that it is countably additive, let

$$E = \bigcup_{j=1}^{\infty} E_j,$$

where the sets $E_j \subset \Omega$ are pairwise disjoint, and let

$$F_k = \bigcup_{j=1}^{k} E_j.$$

Then by Theorem 2.26,

$$\begin{aligned}\|\chi_E - \chi_{F_k}\|_{p(\cdot)} &\le (1 + |\Omega|)\|\chi_E - \chi_{F_k}\|_{p_+} \\ &= (1 + |\Omega|)|E \setminus F_k|^{1/p_+}.\end{aligned}$$

Since $|E| < \infty$, the last term tends to 0 as $k \to \infty$; thus $\chi_{F_k} \to \chi_E$ in norm. Therefore, by the continuity of Φ, $\Phi(\chi_{F_k}) \to \Phi(\chi_E)$; equivalently,

$$\sum_{j=1}^{\infty} \mu(E_j) = \mu(E),$$

and so μ is countably additive.

In other words μ is a measure on Ω. Further, it is absolutely continuous: if $E \subset \Omega$, $|E| = 0$, then $\chi_E \equiv 0$, and so

$$\mu(E) = \Phi(\chi_E) = 0.$$

By the Radon-Nikodym theorem (see Royden [85]), absolutely continuous measures are gotten from L^1 functions. More precisely, there exists $g \in L^1(\Omega)$ such that

$$\Phi(\chi_E) = \mu(E) = \int_\Omega \chi_E(x) g(x)\, dx.$$

By the linearity of Φ, for every simple function $f = \sum a_j \chi_{E_j}$, $E_j \subset \Omega$,

$$\Phi(f) = \int_\Omega f(x) g(x)\, dx.$$

Arguing as we did in the proof of Theorem 2.30, we have that the simple functions are dense in $L^{p(\cdot)}(\Omega)$. Hence, Φ and Φ_g agree on a dense subset. Thus, by continuity $\Phi = \Phi_g$, and so $g \in L^{p'(\cdot)}(\Omega)$.

Finally, to see that g is unique, it is enough to note that if g, $\tilde{g} \in L^{p'(\cdot)}(\Omega)$ are such that $\Phi_g = \Phi_{\tilde{g}}$, then for all $f \in L^{p(\cdot)}(\Omega)$,

$$\int_\Omega f(x)(g(x) - \tilde{g}(x))\, dx = 0. \tag{2.7}$$

By Corollary 2.27, $g - \tilde{g} \in L^{p'(\cdot)}(\Omega) \subset L^{p'(\cdot)_-}(\Omega) = L^{(p_+)'}(\Omega)$, and since (2.7) holds for all $f \in L^{p_+}(\Omega) \subset L^{p(\cdot)}(\Omega)$, by the duality theorem for the classical Lebesgue spaces, $g - \tilde{g} = 0$ a.e.

We now consider the case when $|\Omega| = \infty$. Write

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k,$$

where for each k, $|\Omega_k| < \infty$ and $\overline{\Omega}_k \subset \Omega_{k+1}$. Given $\Phi \in L^{p(\cdot)}(\Omega)^*$, by restriction Φ induces a bounded linear functional on $L^{p(\cdot)}(\Omega_k)$ for each k. Therefore, by the above argument, there exists $g_k \in L^{p'(\cdot)}(\Omega_k)$ such that for all $f \in L^{p(\cdot)}(\Omega)$, $\operatorname{supp}(f) \subset \overline{\Omega}_k$,

$$\Phi(f) = \int_{\Omega_k} f(x) g_k(x)\, dx.$$

Further, $\|g_k\|_{p'(\cdot)} \le k^{-1}_{p'(\cdot),\Omega_k} \|\Phi\| \le 3\|\Phi\|$. Since the sets Ω_k are nested, we must have that for all f with support in Ω_k,

$$\int_{\Omega_k} f(x) g_k(x)\, dx = \int_{\Omega_{k+1}} f(x) g_{k+1}(x)\, dx.$$

Since the functions g_k are unique, we must have that $g_k = g_{k+1}\chi_{\Omega_k}$. Therefore, we can define g by $g(x) = g_k(x)$ for all $x \in \Omega_k$. Since $\operatorname{supp}(g_k) \subset \overline{\Omega}_k$, the sequence $|g_k|$ increases to $|g|$; hence, by Theorem 2.18,

$$\|g\|_{p'(\cdot)} = \lim_{k\to\infty} \|g_k\|_{p'(\cdot)} \le 3\|\Phi\| < \infty.$$

Thus $g \in L^{p'(\cdot)}(\Omega)$.

Now fix $f \in L^{p(\cdot)}(\Omega)$ and let $f_k = f\chi_{\Omega_k}$. Then $|f - f_k| \le |f|$, so by Theorem 2.21, $f_k \to f$ in norm. Further, $f_k g \to fg$ pointwise, and by Hölder's inequality for variable Lebesgue spaces (Theorem 2.32), $|f_k g| \le |fg| \in L^1(\Omega)$. Therefore, by the classical dominated convergence theorem and the continuity of Φ,

$$\begin{aligned}\int_\Omega f(x) g(x)\, dx &= \lim_{k\to\infty} \int_{\Omega_k} f_k(x) g(x)\, dx \\ &= \lim_{k\to\infty} \int_{\Omega_k} f_k(x) g_k(x)\, dx = \lim_{k\to\infty} \Phi(f_k) = \Phi(f).\end{aligned}$$

Finally, since the restriction of g to each Ω_k is uniquely determined, g itself is the the unique element of $L^{p'(\cdot)}(\Omega)$ with this property. This completes the proof. □

2.6. The Lebesgue differentiation theorem

If $f \in L^1_{\mathrm{loc}}$, then for almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \fint_{B_r(x)} f(y)\, dy = f(x).$$

If $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$, then a stronger result holds: for almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - f(x)|^p\, dy = 0.$$

These results are usually referred to collectively as the Lebesgue differentiation theorem—see [33, 46]. When $p_+ < \infty$ the Lebesgue differentiation theorem holds in the variable Lebesgue spaces; this is due to Harjulehto and Hästö [49]. A slightly weaker results holds when $p_+ = \infty$: see [19].

Proposition 2.41. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p_+ < \infty$, and $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$,*

$$\lim_{r\to 0} \fint_{B_r(x)} \big|\big(f(y) - f(x)\big)\big|^{p(y)}\, dy = 0. \tag{2.8}$$

Proof. Since this is a local result, it will suffice to fix a ball B and prove it for a.e. $x \in B$. Since $f \in L^{p(\cdot)}(B)$, by Proposition 2.9,

$$\int_B |f(y)|^{p(y)}\, dy < \infty.$$

Enumerate the rationals as $\{q_i\}$; then

$$\int_B \big|\big(f(y) - q_i\big)\big|^{p(y)}\,dy \leq 2^{p_+-1} \int_B \big(\big|f(y)\big|^{p(y)} + |q_i|^{p(y)}\big)\,dy < \infty.$$

Therefore, for each i, by the classical Lebesgue differentiation theorem, for almost every $x \in B$,

$$\lim_{r\to 0} \fint_{B_r(x)} \big|\big(f(y) - q_i\big)\big|^{p(y)}\,dy = \big|\big(f(x) - q_i\big)\big|^{p(x)}.$$

Since the countable union of sets of measure 0 again has measure 0, this limit holds for all i and almost every $x \in B$. Fix such an x. Fix ϵ, $0 < \epsilon < 1$, and choose q_i such that

$$\big|\big(f(x) - q_i\big)\big| < \epsilon.$$

Then we have that

$$\begin{aligned}
\limsup_{r\to 0} \fint_{B_r(x)} \big|\big(f(y) - f(x)\big)\big|^{p(y)}\,dy &\leq 2^{p_+-1} \limsup_{r\to 0} \bigg(\fint_{B_r(x)} \big|\big(f(y) - q_i\big)\big|^{p(y)}\,dy \\
&\qquad + \fint_{B_r(x)} \big|\big(f(x) - q_i\big)\big|\,dy \bigg) \\
&= 2^{p_+-1} \Big(\big|\big(f(x) - q_i\big)\big)\big|^{p(x)} + \big|\big(f(x) - q_i\big)\big)\big| \Big) \\
&< 2^{p_+}\epsilon.
\end{aligned}$$

The limit (2.8) follows at once. $\square$

CHAPTER 3

The Hardy-Littlewood Maximal Operator

In this chapter we turn to the study of harmonic analysis on the variable Lebesgue spaces. Our goal is to establish sufficient conditions for the Hardy-Littlewood maximal operator to be bounded on $L^{p(\cdot)}$; in the next chapter we will show how this can be used to prove norm inequalities on $L^{p(\cdot)}$ for the other classical operators of harmonic analysis. We begin with a brief review of the maximal operator on the classical Lebesgue spaces and introduce our principal tool, the Calderón-Zygmund decomposition.

3.1. Basic properties

The results on the maximal operator in this section are well-known; see for example, [**33**, **44**, **46**].

Definition 3.1. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then Mf, the Hardy-Littlewood maximal function of f, is defined for any $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain x and whose sides are parallel to the coordinate axes.

There are several variant definitions of the maximal operator, all of them pointwise equivalent. We could restrict the supremum to cubes centered at x; this is referred to as the centered maximal operator and is denoted by M^c. Clearly, $M^c f(x) \leq Mf(x)$. On the other hand, given any cube Q containing x, there exists a cube $\tilde{Q}$ centered at x and containing Q such that $|\tilde{Q}| \leq 3^n |Q|$. Hence, $Mf(x) \leq 3^n M^c f(x)$. Similarly, the supremum could be taken over all cubes and not just those whose sides are parallel to the coordinate axes; again, this definition is pointwise equivalent to Definition 3.1. Alternatively we could define the maximal operator by taking the supremum over all balls that contain x, or even over balls centered at x. Again, these two operators are equivalent pointwise to one another and to the maximal operator defined with respect to cubes.

The maximal operator is very difficult to compute exactly for most functions, but in certain cases it can be approximated easily. The following example and variations of it occur repeatedly in practice; the proof is a straightforward computation.

Example 3.2. In $\mathbb{R}^n$, let $f(x) = |x|^{-a}$, $0 < a < n$. Then

$$Mf(x) \approx |x|^{-a}.$$

We record some elementary properties of the maximal operator that follow at once from the definition.

Proposition 3.3. *The Hardy-Littlewood maximal operator has the following properties:*

(1) *M is sublinear: $M(f+g)(x) \le Mf(x) + Mg(x)$, and for all $\alpha \in \mathbb{R}$, $M(\alpha f)(x) = |\alpha| Mf(x)$.*
(2) *If f is not identically zero, then on any bounded set Ω there exists $\epsilon > 0$ such that $Mf(x) \ge \epsilon$, $x \in \Omega$.*
(3) *If f is not equal to 0 a.e., then $Mf \notin L^1(\mathbb{R}^n)$.*
(4) *If $f \in L^\infty(\mathbb{R}^n)$, then $Mf \in L^\infty(\mathbb{R}^n)$ and $\|Mf\|_\infty = \|f\|_\infty$.*

A deeper property of the maximal operator is a consequence of the Lebesgue differentiation theorem, which in turn can be proved using the weak $(1,1)$ inequality proved below.

PROPOSITION 3.4. *Given a locally integrable function f, then for a. e. $x \in \mathbb{R}^n$, $|f(x)| \le Mf(x)$.*

3.2. The maximal operator on L^p, $1 \le p < \infty$

In this section we prove the classical norm inequalities for the Hardy-Littlewood maximal operator. We will need these results, and the tools used to prove them, to control the maximal operator on the variable Lebesgue spaces. Further, it is useful to recall these proofs to compare them to the more complicated argument needed in $L^{p(\cdot)}$. In this section we follow the presentation in [**33**, **44**].

THEOREM 3.5. *Given $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, for every $t > 0$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \le \frac{3^n 4^{np}}{t^p} \int_{\mathbb{R}^n} |f(x)|^p \, dx. \tag{3.1}$$

Further, if $1 < p \le \infty$, then

$$\|Mf\|_{L^p(\mathbb{R}^n)} \le C(n)(p')^{1/p} \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.2}$$

REMARK 3.6. The weak (p,p) inequality (3.1) can be rewritten in terms of L^p norms:

$$t\|\chi_{\{x\in\mathbb{R}^n : Mf(x)>t\}}\|_p \le C(n,p)\|f\|_p. \tag{3.3}$$

This is the form which we will generalize to $L^{p(\cdot)}$.

We will prove Theorem 3.5 using the Calderón-Zygmund decomposition, one of the most versatile tools in harmonic analysis.

DEFINITION 3.7. Let $Q_0 = [0,1)^n$, and let Δ_0 be the set of all translates of Q_0 whose vertices are on the lattice $\mathbb{Z}^n$. More generally, for each $k \in \mathbb{Z}$, let $Q_k = 2^{-k} Q_0 = [0, 2^{-k})^n$, and let Δ_k be the set of all translates of Q_k whose vertices are on the lattice $2^{-k}\mathbb{Z}^n$. Define the set of dyadic cubes Δ by

$$\Delta = \bigcup_{z\in\mathbb{Z}} \Delta_k.$$

The dyadic cubes have the following properties which are immediate consequences of the definition.

PROPOSITION 3.8.

(1) *For each $k \in \mathbb{Z}$, if $Q \in \Delta_k$, then $\ell(Q) = 2^{-k}$.*
(2) *For each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, there exists a unique cube $Q \in \Delta_k$ such that $x \in Q$.*
(3) *Given any two cubes $Q_1, Q_2 \in \Delta$, either $Q_1 \cap Q_2 = \emptyset$, $Q_1 \subset Q_2$, or $Q_2 \subset Q_1$.*
(4) *For each $k \in \mathbb{Z}$, if $Q \in \Delta_k$, then there exists a unique cube $\widetilde{Q} \in \Delta_{k-1}$ such that $Q \subset \widetilde{Q}$. $\widetilde{Q}$ is referred to as the dyadic parent of Q.*

(5) *For each $k \in \mathbb{Z}$, if $Q \in \Delta_k$, then there exist 2^n cubes $P_i \in \Delta_{k+1}$ such that $P_i \subset Q$.*

Associated with the dyadic cubes is a corresponding maximal operator.

DEFINITION 3.9. Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define the dyadic maximal operator M^d by

$$M^d f(x) = \sup_{\substack{Q \ni x \\ Q \in \Delta}} \fint_Q |f(y)|\, dy.$$

Somewhat surprisingly, even though the dyadic maximal operator is pointwise smaller than the maximal operator, we can use it to control the maximal operator.

LEMMA 3.10. *If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is such that $\fint_Q |f(y)|\, dy \to 0$ as $|Q| \to \infty$, then for all $t > 0$ there exists a (possibly empty) set of disjoint dyadic cubes $\{Q_j\}$ such that*

$$E^d_t = \{x \in \mathbb{R}^n : M^d f(x) > t\} = \bigcup_j Q_j$$

and

$$t < \fint_{Q_j} |f(x)|\, dx \leq 2^n t. \tag{3.4}$$

Further, for a.e. $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$, $|f(x)| \leq t$.

The cubes $\{Q_j\}$ are referred to as the Calderón-Zygmund cubes of f at height t. As part of the proof we get that the Q_j are the largest dyadic cubes with the property that $\fint_Q |f(y)|\, dy > t$, and any other dyadic cube with this property is contained in one of the Q_j. We refer to this property as the maximality of the Calderón-Zygmund cubes.

By Hölder's inequality, the condition that $\fint_Q |f(y)|\, dy \to 0$ as $|Q| \to \infty$ is satisfied if, for example, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

PROOF. Fix $t > 0$; if E^d_t is empty, then there are no dyadic cubes Q such that $\fint_Q |f(y)|\, dy > t$ so we will let the collection $\{Q_j\}$ be the empty set. Otherwise, take $x \in E^d_t$. By the definition of the dyadic maximal operator, there exists $Q \in \Delta$ such that $x \in Q$ and

$$\fint_Q |f(y)|\, dy > t.$$

Since $\fint_Q |f(y)|\, dy \to 0$ as the size of Q increases, if there is more than one dyadic cube with this property, then there must be a largest such cube. Denote it by Q_x. Since we can do this for every such x,

$$E^d_t \subset \bigcup_{x \in E^d_t} Q_x. \tag{3.5}$$

Conversely, given any other point $x' \in Q_x$,

$$M^d f(x') \geq \fint_{Q_x} |f(y)|\, dy > t,$$

and so $x' \in E^d_t$. Therefore, $Q_x \subset E^d_t$ and equality holds in (3.5).

Since Δ is countable, the set $\{Q_x : x \in E^d_t\}$ is at most countable. Re-index this set as $\{Q_j\}$. The cubes Q_j are pairwise disjoint; for if there exist two different cubes that intersect, then by Proposition 3.8 one is contained in the other. However, this contradicts the way in which these cubes were chosen since each was supposed to be the largest such cube.

The left-hand inequality in (3.4) follows from our choice of the Q_j; furthermore, since each Q_j was chosen to be the largest cube containing a point x with this property, if we let $\widetilde{Q}_j$ be its dyadic parent,

$$t \geq \fint_{\widetilde{Q}_j} |f(y)|\,dy \geq 2^{-n} \fint_{Q_j} |f(y)|\,dy.$$

Finally, for every $x \in \mathbb{R}^n \setminus E_t^d$, $M^d f(x) \leq t$. Therefore, for a.e. such x, by the Lebesgue differentiation theorem,

$$|f(x)| = \lim_{\substack{x \in Q \in \Delta \\ |Q| \to 0}} \left| \fint_Q f(y)\,dy \right| \leq M^d f(x) \leq t. \qquad \square$$

Lemma 3.11. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be such that $\fint_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$. Then for any $t > 0$, if $\{Q_j\}$ is the set of Calderón-Zygmund cubes of f at height $t/4^n$,*

$$E_t = \{x \in \mathbb{R}^n : Mf(x) > t\} \subset \bigcup_j 3Q_j.$$

Proof. Fix $x \in E_t$; then there exists a cube Q containing x such that

$$\fint_Q |f(y)|\,dy > t.$$

Let $k \in \mathbb{Z}$ be such that $2^{-k-1} \leq \ell(Q) < 2^{-k}$. Then Q intersects at most $M \leq 2^n$ dyadic cubes in Δ_k; denote them by $P_1, \ldots, P_M$. Since $\ell(P_i) = 2^{-k} \leq 2\ell(Q)$, we have that

$$t < \fint_Q |f(y)|\,dy \leq |Q|^{-1} \sum_{i=1}^{M} \int_{P_i} |f(y)|\,dy \leq 2^n \sum_{i=1}^{M} \fint_{P_i} |f(y)|\,dy.$$

Therefore, there must exist at least one index i such that

$$\fint_{P_i} |f(y)|\,dy > \frac{t}{2^n M} \geq \frac{t}{4^n}.$$

In particular, $P_i \subset E^d_{t/4^n}$; since it is a dyadic cube, by the maximality of the Calderón-Zygmund cubes, $P_i \subset Q_j$ for some j. Further, P_i and Q intersect, so $x \in Q \subset 3P_i \subset 3Q_j$. This is true for every $x \in E_t$, so we get the desired inclusion. $\square$

Proof of Theorem 3.5. We will first prove inequality (3.1) and then prove (3.2) for $1 < p < \infty$. We have already shown that the maximal operator is bounded on L^∞: by Proposition 3.3 we have that $\|Mf\|_\infty = \|f\|_\infty$.

Fix p, $1 \leq p < \infty$, and $f \in L^p(\mathbb{R}^n)$. For any $t > 0$, by Lemma 3.10, there exist the disjoint Calderón-Zygmund cubes $\{Q_j\}$ of f at height $t/4^n$. By Lemma 3.11 and Hölder's inequality (when $p > 1$),

$$\begin{aligned}
|\{x \in \mathbb{R}^n : Mf(x) > t\}| &\leq \left| \bigcup_j 3Q_j \right| \\
&\leq \sum_{j=1}^{\infty} |3Q_j| \leq \sum_{j=1}^{\infty} 3^n |Q_j| \left(\frac{4^n}{t} \fint_{Q_j} |f(x)|\,dx \right)^p \\
&\leq \sum_{j=1}^{\infty} 3^n |Q_j| \frac{4^{np}}{t^p} \fint_{Q_j} |f(x)|^p\,dx \leq \frac{3^n 4^{np}}{t^p} \int_{\mathbb{R}^n} |f(x)|\,dx.
\end{aligned}$$

Now fix p, $1 < p < \infty$, and $f \in L^p(\mathbb{R}^n)$. The heart of the proof is an appeal to Marcinkiewicz interpolation: see [33]. To make clear why the proof will not extend to variable Lebesgue spaces, we include instead adapt the proof of interpolation to this particular problem.

For each $t > 0$ we can decompose f as $f_0^t + f_1^t$, where

$$f_0^t = f\chi_{\{x\in\mathbb{R}^n : |f(x)|>t/2\}}, \qquad f_1^t = f\chi_{\{x\in\mathbb{R}^n : |f(x)|\leq t/2\}}.$$

Since $\|f_1^t\|_\infty \leq t/2$, we have by Proposition 3.3 that

$$Mf(x) \leq Mf_0^t(x) + Mf_1^t(x) \leq Mf_0^t(x) + t/2.$$

Given a function $h \in L^p(\mathbb{R}^n)$,

$$\|h\|_p^p = p\int_0^\infty t^{p-1}|\{x \in \mathbb{R}^n : |h(x)| > t\}|\,dt. \tag{3.6}$$

(See [71, 86].) Therefore, by the weak $(1,1)$ inequality and Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} Mf(x)^p\,dx &= p\int_0^\infty t^{p-1}|\{x \in \mathbb{R}^n : Mf(x) > t\}|\,dt \\
&\leq p\int_0^\infty t^{p-1}|\{x \in \mathbb{R}^n : Mf_0^t(x) > t/2\}|\,dt \\
&\leq 2p\cdot 12^n\int_0^\infty t^{p-2}\int_{\mathbb{R}^n}|f_0^t(x)|\,dx\,dt \\
&= 2p\cdot 12^n\int_0^\infty t^{p-2}\int_{\{x\in\mathbb{R}^n : |f(x)|>t/2\}}|f(x)|\,dx\,dt \\
&= 2p\cdot 12^n\int_{\mathbb{R}^n}|f(x)|\int_0^{2|f(x)|} t^{p-2}\,dt\,dx \\
&= 2p'\cdot 12^n\int_{\mathbb{R}^n}|f(x)|^p\,dx.
\end{aligned}$$

□

Remark 3.12. This proof will fail in the variable Lebesgue spaces because (3.6) does not hold: this inequality reflects in a fundamental way the fact that the L^p spaces are rearrangement invariant. The variable Lebesgue spaces do not have this property: in fact, they are not even translation invariant: see Theorem 4.5 below.

3.3. The maximal operator on variable Lebesgue spaces

The maximal operator is well-defined on any variable Lebesgue space. The easiest way to see this is by using the embedding theorems in Section 2.4. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then by Theorem 2.26 f is locally integrable, so Mf is defined. Moreover, by Theorem 2.29, $f = f_1 + f_2$ where $f_1 \in L^{p_-}$ and $f_2 \in L^{p_+}$. Then $Mf \leq Mf_1 + Mf_2$, and by Theorem 3.5 the right-hand side is finite a.e.

The central result of this chapter is that log-Hölder continuity is sufficient for the maximal operator to be bounded.

Theorem 3.13. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $1/p(\cdot) \in LH(\mathbb{R}^n)$, then*

$$\|t\chi_{\{x:Mf(x)>t\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.7}$$

If in addition $p_- > 1$, then

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.8}$$

In both inequalities the constant depends on the dimension n, the log-Hölder constants of $1/p(\cdot)$, p_-, and p_∞ (if this value is finite).

Theorem 3.13 is due to a number of people. It was first proved by Diening [25] when $p_+ < \infty$ and $p(\cdot)$ is constant outside a large ball. The full result, including the LH_∞ condition, but again when $p_+ < \infty$, was proved in [22]. Independently, Nekvinda [79] proved it with a slightly different condition at infinity: see below.

The case $p_+ = \infty$ is due to Diening [28]; see also [29]. The proof given here is adapted from [15].

We will only prove the strong-type inequality (3.8) in the special case $1 < p_- \leq p_+ < \infty$. This proof reveals the essential ideas of what is going on without getting obscured by technical details. The weak-type inequality when $p_- = 1$ is gotten by modifying this proof. For the complete proof when $p_+ = \infty$, see [15, 19].

For the proof we need three lemmas. The first is a geometric characterization of local log-Hölder continuity due to Diening [25].

Lemma 3.14. *Given $p(\cdot)\colon \mathbb{R}^n \to [0,\infty)$ such that $p_+ < \infty$, the following are equivalent:*

(1) *$p(\cdot) \in LH_0(\mathbb{R}^n)$;*
(2) *there exists a constant C depending on n such that given any cube Q and $x \in Q$,*

$$|Q|^{p(x)-p_+(Q)} \leq C \quad \text{and} \quad |Q|^{p_-(Q)-p(x)} \leq C.$$

Proof. Suppose $p(\cdot) \in LH_0(\mathbb{R}^n)$. We will prove the first inequality in (2); the proof of the second is identical. If $\ell(Q) \geq (2\sqrt{n})^{-1}$, then

$$|Q|^{p(x)-p_+(Q)} \leq (2\sqrt{n})^{n(p_+-p_-)} = C(n, p(\cdot)).$$

If $\ell(Q) < (2\sqrt{n})^{-1}$, then for all $y \in Q$, $|x-y| < 1/2$. In particular, since $p(\cdot)$ is continuous, there exists $y \in Q$ such that $p(y) = p_+(Q)$. Therefore, by the definition of LH_0,

$$\begin{aligned} |Q|^{p(x)-p_+(Q)} &\leq \big(n^{-1/2}|x-y|\big)^{-n|p(x)-p(y)|} \\ &\leq \exp\left(\frac{C_0(\log(n^{1/2}) - \log|x-y|)}{-\log|x-y|}\right) \leq C(n, p(\cdot)). \end{aligned}$$

Now suppose that (2) holds. Fix x, $y \in \mathbb{R}^n$ such that $|x-y| < 1/2$; then there exists a cube Q such that x, $y \in Q$ and $\ell(Q) \leq |x-y|$ (and so $|Q| < 1$). Combining the two inequalities in (2) we have that

$$\begin{aligned} C \geq |Q|^{p_-(Q)-p_+(Q)} &\geq |Q|^{-|p(x)-p(y)|} \\ &\geq |x-y|^{-n|p(x)-p(y)|} = \exp\big(-n|p(x)-p(y)|\log(|x-y|)\big). \end{aligned}$$

If we take the logarithm we get that

$$|p(x)-p(y)| \leq \frac{C}{-\log(|x-y|)},$$

where C does not depend on x, y. Hence $p(\cdot) \in LH_0(\mathbb{R}^n)$. □

The second lemma shows that given log-Hölder continuity at infinity, we can work with modular inequalities by replacing the variable exponent with a constant one at the price of an error term. Versions of this inequality appeared in [12, 15, 22].

Lemma 3.15. *Let $p(\cdot) : \mathbb{R}^n \to [0,\infty)$ be such that $p(\cdot) \in LH_\infty(\mathbb{R}^n)$ and $0 < p_\infty < \infty$, and let $R(x) = (e+|x|)^{-N}$, $N > n/p_\infty$. Then there exists a constant C depending on n, N and the LH_∞ constant of $p(\cdot)$ such that given any set E and any function F with $0 \leq F(y) \leq 1$ for $y \in E$,*

$$\int_E F(y)^{p(y)}\,dy \leq C\int_E F(y)^{p_\infty}\,dy + C\int_E R(y)^{p_\infty}\,dy, \tag{3.9}$$

$$\int_E F(y)^{p_\infty}\,dy \leq C\int_E F(y)^{p(y)}\,dy + C\int_E R(y)^{p_\infty}\,dy. \tag{3.10}$$

Proof. We will prove (3.9); the proof of the second inequality is essentially the same. By the LH_∞ condition,

$$R(y)^{-|p(y)-p_\infty|} = \exp\big(N\log(e+|y|)|p(y)-p_\infty|\big) \le \exp(NC_\infty).$$

Write the set E as $E_1 \cup E_2$, where $E_1 = \{x \in E : F(y) \le R(y)\}$ and $E_2 = \{x \in E : R(y) < F(y)\}$. Then

$$\begin{aligned}\int_{E_1} F(y)^{p(y)}\,dy &\le \int_{E_1} R(y)^{p(y)}\,dy \\ &\le \int_{E_1} R(y)^{p_\infty} R(y)^{-|p(y)-p_\infty|}\,dy \le \exp(NC_\infty)\int_{E_1} R(y)^{p_\infty}\,dy.\end{aligned}$$

Similarly, since $F(y) \le 1$,

$$\begin{aligned}\int_{E_2} F(y)^{p(y)}\,dy &\le \int_{E_2} F(y)^{p_\infty} F(y)^{-|p(y)-p_\infty|}\,dy \\ &\le \int_{E_2} F(y)^{p_\infty} R(y)^{-|p(y)-p_\infty|}\,dy \le \exp(NC_\infty)\int_{E_2} F(y)^{p_\infty}\,dy.\end{aligned}$$

□

Our third lemma allows us to apply the Calderón-Zygmund decomposition to functions in $L^{p(\cdot)}(\mathbb{R}^n)$. This result is from [19].

Lemma 3.16. *Given $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, suppose $p_+<\infty$. Then for all $f\in L^{p(\cdot)}(\mathbb{R}^n)$, $⨍_Q |f(y)|\,dy \to 0$ as $|Q| \to \infty$. In particular, the conclusion of Lemma 3.10 holds.*

Proof. Fix $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and a cube Q with $|Q| > 2$. Then by Theorem 2.32,

$$⨍_Q |f(y)|\,dy \le K_{p(\cdot)}|Q|^{-1}\|\chi_Q\|_{p'(\cdot)}\|f\|_{p(\cdot)}.$$

We will show that $|Q|^{-1}\|\chi_Q\|_{p'(\cdot)} \to 0$. By the definition of the norm, since $|Q| > 2$,

$$\begin{aligned}\|\chi_Q\|_{p'(\cdot)} &= \inf\{\lambda>0 : \int_{Q\setminus\Omega_\infty^{p'(\cdot)}} \lambda^{-p'(x)}\,dx + \lambda^{-1}\|\chi_Q\|_{L^\infty(\Omega_\infty^{p'(\cdot)})} \le 1\} \\ &\le \inf\{\lambda>1 : \int_{Q\setminus\Omega_\infty^{p'(\cdot)}} \lambda^{-p'(x)}\,dx + \lambda^{-1} \le 1\} \\ &\le \inf\{\lambda>1 : \lambda^{-p'(\cdot)_-}|Q| + \lambda^{-1} \le 1\} \\ &\le \inf\{\lambda>2^{1/p'(\cdot)_-} : \lambda^{-p'(\cdot)_-}|Q| + \lambda^{-1} \le 1\}.\end{aligned}$$

The last infimum is obtained when λ satisfies $\lambda^{-p'(\cdot)_-}|Q| + \lambda^{-1} = 1$. Fix this value of λ. Then $1 < \lambda^{-p'(\cdot)_-}|Q| + 2^{-1/p'(\cdot)_-}$, and so

$$\lambda < \frac{|Q|^{1/p'(\cdot)_-}}{(1-2^{-1/p'(\cdot)_-})^{1/p'(\cdot)_-}}.$$

Hence,

$$|Q|^{-1}\|\chi_Q\|_{p'(\cdot)} \le |Q|^{-1}\lambda < \frac{|Q|^{1/p'(\cdot)_- -1}}{(1-2^{-1/p'(\cdot)_-})^{1/p'(\cdot)_-}}.$$

Since $p_+ < \infty$, $p'(\cdot)_- > 1$, and so the right-hand term tends to 0 as $|Q| \to \infty$. □

Remark 3.17. As an alternative to using this lemma, we can prove norm inequalities by first proving them for bounded functions of compact support, and then use an approximation argument with the monotone convergence theorem and the density of such functions (Theorems 2.18 and 2.30). This is the approach used in [15].

PROOF OF INEQUALITY (3.8). We begin the proof with some reductions. First, without loss of generality we may assume f is non-negative. Second, given the assumption that $p_+ < \infty$, then $1/p(\cdot) \in LH_0$ is equivalent to assuming $p(\cdot) \in LH_0$, since

$$\left|\frac{p(x)-p(y)}{(p_+)^2}\right| \leq \left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \left|\frac{p(x)-p(y)}{(p_-)^2}\right|.$$

The same computation with p_∞ in place of $p(y)$ shows that $1/p(\cdot) \in LH_\infty$ is equivalent to $p(\cdot) \in LH_\infty$.

Third, to prove (3.8) we will need to pass between a norm inequality and a modular inequality. In the classical Lebesgue spaces this is trivial, since norm and modular inequalities are equivalent. This is no longer the case in the variable Lebesgue spaces: in fact, as we will see below in Theorem 3.36, the modular inequality

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx$$

is always false unless $p(\cdot)$ is constant.

To avoid this, we will use the following approach that can be adapted to many other operators. By homogeneity, it is enough to prove (3.8) with the additional assumption that $\|f\|_{p(\cdot)} = 1$; in this case, Corollary 2.16 implies that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \leq 1.$$

Then by Theorem 2.9 it will suffice to prove that

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \leq C,$$

since then we have that

$$\|Mf\|_{p(\cdot)} \leq C = C\|f\|_{p(\cdot)}.$$

We now argue as follows. Decompose f as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x:f(x)>1\}}, \qquad f_2 = f\chi_{\{x:f(x)\leq 1\}};$$

then $\rho(f_i) \leq \|f_i\|_{p(\cdot)} \leq 1$. Further, since $Mf \leq Mf_1 + Mf_2$, it will suffice to show that for $i = 1, 2$, that $\|Mf_i\|_{p(\cdot)} \leq C(n, p(\cdot))$; since $p_+ < \infty$, as we argued above it will in turn suffice to show that

$$\int_{\mathbb{R}^n} Mf_i(x)^{p(x)}\,dx \leq C.$$

The estimate for f_1. Let $A = 4^n$, and for each $k \in \mathbb{Z}$ let

$$\Omega_k = \{x \in \mathbb{R}^n : Mf_1(x) > A^k\}.$$

Since $f_1 \in L^{p(\cdot)}(\mathbb{R}^n)$, as we noted at the beginning of Section 3.3, $Mf_1(x) < \infty$ almost everywhere; similarly, without loss of generality we may assume f_1 is non-zero on a set of positive measure, and so by Proposition 3.3, $Mf(x) > 0$ for all x. Therefore, up to a set of measure 0, $\mathbb{R}^n = \bigcup_k \Omega_k \setminus \Omega_{k+1}$. Further, by Lemma 3.16 for each k we can apply Lemma 3.10 to form the Calderón-Zygmund decomposition of f at height A^{k-1}: pairwise disjoint cubes $\{Q_j^k\}_j$ such that

$$\Omega_k \subset \bigcup_j 3Q_j^k \quad \text{and} \quad \fint_{Q_j^k} f_1(y)\,dy > A^{k-1}.$$

From the second inequality we get that

$$\fint_{3Q_j^k} f_1(y)\,dy > 3^{-n}A^{k-1}.$$

Define the sets E_j^k inductively: $E_1^k = (\Omega_k \setminus \Omega_{k+1}) \cap 3Q_1^k$, $E_2^k = ((\Omega_k \setminus \Omega_{k+1}) \cap 3Q_2^k) \setminus E_1^k$, $E_3^k = ((\Omega_k \setminus \Omega_{k+1}) \cap 3Q_3^k) \setminus (E_1^k \cup E_2^k)$, etc. Then the sets E_j^k are pairwise disjoint for all j and k and $\Omega_k \setminus \Omega_{k+1} = \bigcup_j E_j^k$.

We now estimate as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} Mf_1(x)^{p(x)}\,dx &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} Mf_1(x)^{p(x)}\,dx \\
&\leq \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} [A^{k+1}]^{p(x)}\,dx \\
&\leq A^{2p_+} 3^{np_+} \sum_{k,j} \int_{E_j^k} \left(\fint_{3Q_j^k} f_1(y)\,dy \right)^{p(x)} dx.
\end{aligned}$$

To estimate the last sum, note that since $f_1(x) = 0$ or $f_1(x) \geq 1$ a.e., if we let $p_{jk} = p_-(3Q_j^k)$,

$$\int_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \leq \int_{3Q_j^k} f_1(y)^{p(y)}\,dy \leq 1. \tag{3.11}$$

Further, since $p(\cdot) \in LH_0(\mathbb{R}^n)$ and $p_+ < \infty$, by Lemma 3.14 there exists a constant C depending on $p(\cdot)$ and n such that for $x \in 3Q_j^k$,

$$|3Q_j^k|^{-p(x)} \leq C|3Q_j^k|^{-p_{jk}}. \tag{3.12}$$

Therefore, since for $x \in E_j^k \subset 3Q_j^k$, $p(x) \geq p_{jk} \geq p_-$, by (3.11), (3.12) and Hölder's inequality with exponent p_{jk}/p_-,

$$\begin{aligned}
\sum_{k,j} \int_{E_j^k} \left(\fint_{3Q_j^k} f_1(y)\,dy \right)^{p(x)} dx &\leq \sum_{k,j} \int_{E_j^k} |3Q_j^k|^{-p(x)} \left(\int_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \right)^{p(x)} dx \\
&\leq C \sum_{k,j} \int_{E_j^k} |3Q_j^k|^{-p_{jk}} \left(\int_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \right)^{p_{jk}} dx \\
&\leq C \sum_{k,j} \int_{E_j^k} \left(\fint_{3Q_j^k} f_1(y)^{p(y)/p_{jk}}\,dy \right)^{p_{jk}} dx \\
&\leq C \sum_{k,j} \int_{E_j^k} \left(\fint_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy \right)^{p_-} dx \\
&\leq C \sum_{k,j} \int_{E_j^k} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \\
&\leq C \int_{\mathbb{R}^n} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx.
\end{aligned}$$

Since $p_- > 1$, by Theorem 3.5 the maximal operator is bounded on $L^{p_-}(\mathbb{R}^n)$. Hence,

$$\int_{\mathbb{R}^n} M(f_1(\cdot)^{p(\cdot)/p_-})(x)^{p_-}\,dx \leq C \int_{\mathbb{R}^n} f_1(x)^{p(x)}\,dx \leq C.$$

If we combine the above inequalities we get the desired result.

The estimate for f_2 Since $0 \leq f_2(x) \leq 1$, we have that $0 \leq Mf_2(x) \leq 1$. Since $1 < p_\infty < \infty$, if we set $R(x) = (e + |x|)^{-n}$, then by inequality (3.9),

$$\int_{\mathbb{R}^n} Mf_2(x)^{p(x)}\,dx \leq C \int_{\mathbb{R}^n} Mf_2(x)^{p_\infty}\,dx + C \int_{\mathbb{R}^n} R(x)^{p_\infty}\,dx.$$

The second integral is a constant depending only on n and p_∞. To bound the first integral, note that since $p_\infty \geq p_- > 1$, by Theorem 3.5 and (3.10),

$$\begin{aligned}\int_{\mathbb{R}^n} Mf_2(x)^{p_\infty}\,dx &\leq C\int_{\mathbb{R}^n} f_2(x)^{p_\infty}\,dx \\ &\leq C\int_{\mathbb{R}^n} f_2(x)^{p(x)}\,dx + C\int_{\mathbb{R}^n} R(x)^{p_\infty}\,dx \leq C.\end{aligned}$$

Combining these two inequalities we get the desired estimate for f_2. This completes the proof. □

3.4. The necessity of the hypotheses in Theorem 3.13

Since in the classical case the maximal operator is bounded on L^p for both p finite and $p = \infty$, it is makes sense that Theorem 3.13 includes the case $p_+ = \infty$. At the other end of the scale of Lebesgue spaces, by Proposition 3.3 the maximal operator is not bounded on L^1. Initially it was conjectured that if $p(x) > 1$ everywhere and is "far" from 1 except on a small set—for example, if $p(x) = 1 + |\log(x)|^{-1}$ near the origin—then the maximal operator could be bounded on $L^{p(\cdot)}$. However, this is never the case. The following result first appeared in [22] with the additional assumption that $p(\cdot)$ is upper semicontinuous; this hypothesis was removed by Diening [28]; see also [19, 30].

THEOREM 3.18. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $p_- = 1$, then the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

PROOF. For each $k \in \mathbb{N}$, choose s_k such that

$$1 < s_k < n\left(n - \frac{1}{k+1}\right)^{-1}.$$

Then for each k, since $p_- = 1$ the set

$$E_k = \{x : p(x) < s_k\}$$

has positive measure. By the Lebesgue differentiation theorem, for each function χ_{E_k} there exists a point $x_k \in E_k$ such that

$$\lim_{r\to 0^+} \frac{|B_r(x_k) \cap E_k|}{|B_r(x_k)|} = 1.$$

In particular, there exists R_k, $0 < R_k < 1$, such that if $0 < r \leq R_k$, then

$$\frac{|B_r(x_k) \cap E_k|}{|B_r(x_k)|} > 1 - 2^{-n(k+1)}. \tag{3.13}$$

Let $B_k = B_{R_k}(x_k)$ and define

$$f_k(x) = |x - x_k|^{-n+\frac{1}{k+1}}\chi_{B_k\cap E_k}(x).$$

We now must prove that $f_k \in L^{p(\cdot)}(\mathbb{R}^n)$, and that

$$\|Mf_k\|_{p(\cdot)} \geq c(n)(k+1)\|f_k\|_{p(\cdot)}.$$

To see the first, note that since $R_k < 1$ and $-n + \frac{1}{k+1} < 0$,

$$\begin{aligned}\rho(f_k) &= \int_{B_k\cap E_k} |x - x_k|^{(-n+\frac{1}{k+1})p(x)}\,dx \\ &\leq \int_{B_k\cap E_k} |x - x_k|^{(-n+\frac{1}{k+1})s_k}\,dx < \infty.\end{aligned}$$

For the second, we will use the equivalent definition of the maximal operator and consider averages over balls. Fix $x \in B_k \cap E_k$ and let $r = |x - x_k| \le R_k$. Then

$$Mf_k(x) \ge \frac{1}{|B_r(x_k)|} \int_{B_r(x_k)\cap E_k} |y - x_k|^{-n+\frac{1}{k+1}}\, dy.$$

Let $\delta_k = 2^{-(k+1)}$; then

$$|\{y : \delta_k r < |y - x_k| < r\}| = (1 - 2^{-n(k+1)})|B_r(x_k)|.$$

Therefore, since $|x - x_k|^{-n+\frac{1}{k+1}}$ is radially decreasing and since by (3.13) $|B_r(x_k) \cap E_k| \ge (1 - 2^{-n(k+1)})|B_r(x_k)|$, we have that

$$\begin{aligned}
Mf_k(x) &\ge \frac{1}{|B_r(x_k)|} \int_{B_r(x_k)\cap E_k} |y - x_k|^{-n+\frac{1}{k+1}}\, dy \\
&\ge c(n)r^{-n} \int_{\{\delta_k r<|y-x_k|<r\}} |y - x_k|^{-n+\frac{1}{k+1}}\, dy \\
&= c(n)(k+1)(1 - \delta_k^{\frac{1}{k+1}})|x - x_k|^{-n+\frac{1}{k+1}} \\
&\ge c(n)(k+1)f_k(x).
\end{aligned}$$

Trivially, this inequality also holds if $x \notin B_k \cap E_k$; hence, $\|Mf_k\|_{p(\cdot)} \ge c(n)(k+1)$ $\|f_k\|_{p(\cdot)}$, and this completes the proof. □

We now turn to the regularity assumption in Theorem 3.13. The next example shows that a simple jump discontinuity will cause the maximal operator to be unbounded. We first saw this example in the Introduction.

Example 3.19. Let $\Omega = \mathbb{R}$, and let

$$p(x) = \begin{cases} 2 & x \le 0 \\ 4 & x > 0. \end{cases}$$

Let $f(x) = |x|^{-2/5}\chi_{(-1,0)}(x)$. Since $|x|^{-4/5}\chi_{(-1,0)} \in L^1(\mathbb{R})$, by Proposition 2.9, $f \in L^{p(\cdot)}(\mathbb{R})$. On the other hand, $Mf \notin L^{p(\cdot)}(\mathbb{R})$: if $0 < x < 1$, then

$$Mf(x) \ge \frac{1}{2x}\int_{-x}^{x} |f(y)|\, dy = \frac{1}{2x}\int_0^x y^{-2/5}\, dy = \frac{5}{6}x^{-2/5} \notin L^4((0,1));$$

hence $\rho(Mf) = \infty$, so again by Proposition 2.9, $Mf \notin L^{p(\cdot)}(\Omega)$. Further, from this inequality we get that for any $t > 0$,

$$t^4|\{x \in \mathbb{R} : Mf(x) > t\}| \ge t^4 \left(\frac{5}{6t}\right)^{5/2};$$

hence, for t large, by Corollary 2.17 we have that

$$\|t\chi_{\{x:Mf(x)>t\}}\|_{p(\cdot)} \ge \rho\big(t\chi_{\{x:Mf(x)>t\}}\big)^{1/4} \ge t\left(\frac{5}{6t}\right)^{5/8}.$$

Since the right-hand side is unbounded as $t \to \infty$, (3.7) does not hold.

The next example shows that local regularity is also not sufficient: there must be some control at infinity.

Example 3.20. Let $p(x) = 3 + \sin(x)$. Then the maximal operator is unbounded on $L^{p(\cdot)}(\mathbb{R})$.

PROOF. For all $k \in \mathbb{N}$, define the sets

$$A_k = \left[\frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + 2k\pi\right],$$
$$B_k = \left[\frac{5\pi}{4} + 2k\pi, \frac{7\pi}{4} + 2k\pi\right].$$

If we let $a = 3 + \sqrt{2}/2$ and $b = 3 - \sqrt{2}/2$, then if $x \in A_k$, $p(x) \geq a$ and if $x \in B_k$, $p(x) \leq b$.

We now define the function

$$f(x) = \sum_{k=1}^{\infty} |x|^{-1/3}\chi_{A_k}(x).$$

Since $a/3 > 1$,

$$\rho(f) = \sum_{k=1}^{\infty} \int_{A_k} |x|^{-p(x)/3}\,dx \leq \int_{\pi/4+2\pi}^{\infty} |x|^{-a/3}\,dx < \infty,$$

so by Proposition 2.9, $f \in L^{p(\cdot)}(\mathbb{R})$. On the other hand, given $x \in [2k\pi, 2(k+1)\pi]$,

$$Mf(x) \geq \frac{1}{2\pi}\int_{2k\pi}^{2(k+1)\pi} f(y)\,dy \geq c|x|^{-1/3}.$$

Therefore, since $b/3 < 1$,

$$\begin{aligned}\rho(Mf) &\geq \sum_{k=1}^{\infty} c\int_{B_k} |x|^{-p(x)/3}\,dx \\ &\geq \sum_{k=1}^{\infty} c\int_{B_k} |x|^{-b/3}\,dx \geq c\sum_{k=1}^{\infty}\left(\frac{7\pi}{4} + 2k\pi\right)^{-b/3} = \infty,\end{aligned}$$

and so again by Proposition 2.9, $Mf \notin L^{p(\cdot)}(\mathbb{R})$. $\square$

As we will see in Section 3.5, the log-Hölder continuity conditions are not necessary. However, as the next two results show, they are sharp in the sense that if we replace the right-hand side by any larger modulus of continuity, we can find an exponent function $p(\cdot)$ such that the maximal operator is not bounded. The following two examples are from [22] and [82].

EXAMPLE 3.21. Fix p_∞, $1 < p_\infty < \infty$, and let $\phi : [0, \infty) \to [0, 1)$ be such that $\phi(0+) = 0$, $\phi_+ < p_\infty - 1$, ϕ is decreasing on $[1, \infty)$, $\phi(x) \to 0$ as $x \to \infty$, and

$$\lim_{x\to\infty} \phi(x)\log(x) = \infty. \tag{3.14}$$

Define $p(\cdot) \in \mathcal{P}(\mathbb{R})$ by

$$p(x) = \begin{cases} p_\infty & x \leq 0 \\ p_\infty - \phi(x) & x > 0. \end{cases}$$

Then $p(\cdot) \notin LH_\infty(\mathbb{R})$ and the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R})$.

REMARK 3.22. A family of functions that satisfy the hypotheses of Example 3.21 is

$$p(x) = \begin{cases} p_0 & x \in (-\infty, 0] \\ p_0 - \frac{x}{\log(e+1)^a} & x \in (0, 1) \\ p_0 - \frac{1}{\log(e+x)^a} & x \in [1, \infty), \end{cases}$$

where $p_0 > 2$ and $0 < a < 1$.

PROOF. It is immediate from (3.14) that $p(\cdot)$ does not satisfy the $LH_\infty(\mathbb{R})$ condition, so we only have to construct a function f such that $f \in L^{p(\cdot)}(\mathbb{R})$ but $Mf \notin L^{p(\cdot)}(\mathbb{R})$. By inequality (3.14) we have that

$$\lim_{x\to\infty} \left(1 - \frac{p_\infty}{p(2x)}\right) \log(x) = -\infty,$$

which in turn implies that

$$\lim_{x\to\infty} x^{1-p_\infty/p(2x)} = 0.$$

Hence, we can form a sequence $\{c_n\} \subset (-\infty, -1)$ such that $c_{n+1} < 2c_n$ and

$$|c_n|^{1-p_\infty/p(2|c_n|)} \le 2^{-n}.$$

Let $d_n = 2c_n$ and define the function f by

$$f(x) = \sum_{n=1}^{\infty} |c_n|^{-1/p(|d_n|)} \chi_{(d_n,c_n)}(x).$$

Since $p_+ < \infty$, by Proposition 2.9 it will suffice to show that $\rho(f) < \infty$ and $\rho(Mf) = \infty$. First,

$$\begin{aligned}\rho(f) &= \sum_{n=1}^{\infty} \int_{d_n}^{c_n} |c_n|^{-p(x)/p(|d_n|)}\,dx = \sum_{n=1}^{\infty} \int_{d_n}^{c_n} |c_n|^{-p_\infty/p(|d_n|)}\,dx \\ &= \sum_{n=1}^{\infty} |c_n|^{1-p_\infty/p(|d_n|)} \le \sum_{n=1}^{\infty} 2^{-n} = 1.\end{aligned}$$

On the other hand, if $x \in (|c_n|, |d_n|)$, then

$$\begin{aligned}Mf(x) &\ge \frac{1}{2|d_n|} \int_{d_n}^{|d_n|} f(y)\,dy \\ &\ge \frac{1}{2|d_n|} \int_{d_n}^{c_n} |c_n|^{-1/p(|d_n|)}\,dy = \frac{1}{4}|c_n|^{-1/p(|d_n|)}.\end{aligned}$$

Therefore, since $p(\cdot)$ is an increasing function on $(1, \infty)$ and $|c_n| \ge 1$,

$$\begin{aligned}\rho(Mf) &\ge \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(x)/p(|d_n|)}\,dx \\ &\ge \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} \int_{|c_n|}^{|d_n|} |c_n|^{-p(|d_n|)/p(|d_n|)}\,dx = \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} 1 = \infty. \qquad \square\end{aligned}$$

EXAMPLE 3.23. Fix p_0, $1 < p_0 < \infty$, and let $\phi : [0, \infty) \to [0, 1]$ be such that ϕ is increasing, $\phi(0) = 0$, $\phi(x) \to 0$ as $x \to 0^+$, and

$$\lim_{x\to 0^+} \phi(x) \log(x) = -\infty. \tag{3.15}$$

Let $\Omega = (-1, 1)$ and define $p(\cdot) \in \mathcal{P}(\Omega)$ by

$$p(x) = \begin{cases} p_0 + \phi(x) & x \ge 0 \\ p_0 & x < 0. \end{cases}$$

Then $p(\cdot) \notin LH_0(\Omega)$ and the maximal operator is not bounded on $L^{p(\cdot)}(\Omega)$.

REMARK 3.24. A particular family of exponent functions $p(\cdot)$ that satisfy the hypotheses of Example 3.23 is

$$p(x) = \begin{cases} 2 & x \in (-1, 0] \\ 2 + \frac{1}{\log(e/x)^a} & x \in (0, 1), \end{cases}$$

where $0 < a < 1$.

PROOF. The construction of this example is very similar to the construction of Example 3.21. It is immediate from (3.15) that $p(\cdot)$ does not satisfy the $LH_0(\Omega)$ condition at the origin, so we only have to construct a function f such that $f \in L^{p(\cdot)}(\Omega)$ but $Mf \notin L^{p(\cdot)}(\Omega)$. Intuitively, we will generalize Example 3.19 by showing that $f(x) = |x|^{-1/p(|x|)}\chi_{(-1,0)}(x)$ is in $L^{p(\cdot)}(\Omega)$ but Mf is not. However, to simplify the calculations we replace this f by a discrete analog.

By (3.15) we have that

$$\lim_{x\to 0^+} \left(1 - \frac{p_0}{p(x/2)}\right) \log(x) = -\infty;$$

equivalently,

$$\lim_{x\to 0^+} x^{1-\frac{p_0}{p(x/2)}} = 0.$$

Hence, we can form a sequence $\{a_n\} \subset (-1, 0)$ such that $a_n/2 < a_{n+1}$ and

$$|a_n|^{1-p_0/p(|a_n|/2)} \le 2^{-n}.$$

Let $b_n = a_n/2$ and define the function f by

$$f(x) = \sum_{n=1}^{\infty} |a_n|^{-1/p(|b_n|)}\chi_{(a_n,b_n)}(x).$$

Since $p_+ < \infty$, by Proposition 2.9 it will suffice to show that $\rho(f) < \infty$ and $\rho(Mf) = \infty$. First, we have that

$$\begin{aligned}\rho(f) &= \sum_{n=1}^{\infty} \int_{a_n}^{b_n} |a_n|^{-p_0/p(|b_n|)}\,dx \\ &= \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^{1-p_0/p(|b_n|)} \le \sum_{n=1}^{\infty} 2^{-n-1} < \infty.\end{aligned}$$

On the other hand, if $x \in (|b_n|, |a_n|)$, then

$$\begin{aligned}Mf(x) &\ge \frac{1}{2|a_n|}\int_{a_n}^{|a_n|} f(y)\,dy \\ &\ge \frac{1}{2|a_n|}\int_{a_n}^{b_n} |a_n|^{-1/p(|b_n|)}\,dy = \frac{1}{4}|a_n|^{-1/p(|b_n|)}.\end{aligned}$$

Therefore, since $p(\cdot)$ is an increasing function and $|a_n| \le 1$,

$$\begin{aligned}\rho(Mf) &\ge \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} \int_{|b_n|}^{|a_n|} |a_n|^{-p(x)/p(|b_n|)}\,dx \\ &\ge \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} \int_{|b_n|}^{|a_n|} |a_n|^{-p(|b_n|)/p(|b_n|)}\,dx = \left(\frac{1}{4}\right)^{p_+} \sum_{n=1}^{\infty} \frac{1}{2} = \infty. \qquad \square\end{aligned}$$

3.5. Weakening the hypotheses in Theorem 3.13

While the log-Hölder continuity conditions are sufficient and sharp as pointwise conditions, they are not necessary. In this section we review some recent work on weaker sufficient conditions and conclude with a necessary and sufficient condition due to Diening that has important theoretical implications.

To see that the log-Hölder continuity is not necessary, we give three examples. Since the details of their proof are quite complicated, we omit them and refer the reader to the literature. The first example shows that LH_∞ is not necessary. For a proof, see Nekvinda [80].

EXAMPLE 3.25. On the real line, if

$$p(x) = p_0 + \frac{1}{\log(e + |x|)^a},$$

$p_0 > 1$ and $0 < a < 1$, then M is bounded on $L^{p(\cdot)}(\mathbb{R})$.

The second shows that the LH_0 condition is not necessary. For a proof, see [19].

EXAMPLE 3.26. Given a, $0 < a < 1$, let $I_a = (-e^{-3^{1/a}}, e^{-3^{1/a}}) \subset \mathbb{R}$. Then the exponent $p(\cdot) \in \mathcal{P}(I_a)$ defined by

$$\frac{1}{p(x)} = \frac{1}{2} + \frac{1}{\log(1/|x|)^a},$$

then M is bounded on $L^{p(\cdot)}(I_a)$.

If we compare these examples with Examples 3.21 and 3.23, we see that in the latter examples the proof depends in an essential way on the asymmetry of $p(\cdot)$. In fact, if we maintain symmetry, then it is possible to construct a discontinuous exponent such that the maximal operator is bounded. The following remarkable example is due to Lerner [68].

EXAMPLE 3.27. Given $p_0 > 1$ and $\mu \in \mathbb{R}$, define $p(\cdot) \in \mathcal{P}(\mathbb{R})$ by

$$p(x) = p_0 - \mu \sin(\log\log(1 + \max(|x|, |x|^{-1}))).$$

Then for μ sufficiently close to 0, the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R})$, but $p(\cdot)$ does not have a limit at 0 or infinity.

REMARK 3.28. The following interesting question is suggested by the previous examples: does there exist an even, continuous exponent function $p(\cdot)$ on $\mathbb{R}$ such that $1 < p_- \le p_+ < \infty$ and the maximal operator is not bounded on $L^{p(\cdot)}(\mathbb{R})$?

Motivated by these examples, there has been an effort to find weaker sufficient conditions, both locally and infinity. Much more is known locally. At infinity, the only known condition that can replace the LH_∞ condition is due to Nekvinda [79], who used it to independently prove Theorem 3.13.

DEFINITION 3.29. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we say that $p(\cdot) \in N_\infty(\mathbb{R}^n)$ if there exist constants $\Lambda_\infty > 0$ and $p_\infty \in [1, \infty]$ such that

$$\int_{\Omega_+} \exp\left(-\Lambda_\infty \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|^{-1}\right) dx < \infty,$$

where

$$\Omega_+ = \left\{x \in \mathbb{R}^n : \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| > 0\right\}.$$

The N_∞ condition implies that the exponent $p(\cdot)$ satisfies the log-Hölder continuity in some average sense at infinity. In fact, the proof of the strong-type inequality above goes through with LH_∞ replaced by N_∞: a version of Lemma 3.15 is still true. This is essentially Nekvinda's argument. The N_∞ condition, however, is not necessary; this is shown by Example 3.25 above.

An interesting replacement for local log-Hölder continuity is the K_0 condition introduced by Kopaliani [62].

DEFINITION 3.30. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then $p(\cdot) \in K_0(\mathbb{R}^n)$ if there exists a constant C_K such that

$$\sup_Q |Q|^{-1}\|\chi_Q\|_{L^{p(\cdot)}(\Omega)}\|\chi_Q\|_{L^{p'(\cdot)}(\Omega)} \leq C_K < \infty, \tag{3.16}$$

where the supremum is taken over all cubes Q.

The K_0 condition is very similar to the Muckenhoupt A_p condition for weighted norm inequalities: see Chapter 4 below. Furthermore, it is a necessary condition.

PROPOSITION 3.31. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then $p(\cdot) \in K_0(\mathbb{R}^n)$.*

The K_0 condition is not sufficient: there exist examples due to Kopaliani [63] and Diening [28] (see also [30]) of exponents $p(\cdot)$ such that $p(\cdot) \in K_0$, but the maximal operator is not bounded on $L^{p(\cdot)}$. On the other hand, it is a replacement for the LH_0 condition, as the next result due to Kopaliani [62] and Lerner [69] shows. It is possible to prove this result by adapting the proof of Theorem 3.13 and using some ideas of Lerner: see [19].

THEOREM 3.32. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in K_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$. Then*

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.17}$$

We conclude this section with a necessary and sufficient condition due to Diening [27, 30]. Though not easy to check in practice, it has important theoretical implications. To state it, we give two definitions.

DEFINITION 3.33. Let $\mathcal{Q} = \{Q_j\}$ be a collection of pairwise disjoint cubes. Given a locally integrable function f, define the averaging operator $\mathcal{A}_\mathcal{Q}$ by

$$\mathcal{A}_\mathcal{Q} f(x) = \sum_j A_{Q_j} f(x) \sum_j \fint_{Q_j} f(y)\,dy\chi_{Q_j}(x).$$

DEFINITION 3.34. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then $p(\cdot) \in \mathcal{A}$ if there exists a constant $C_\mathcal{A}$ such that given any set $\mathcal{Q}$ of disjoint cubes and any function $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|\mathcal{A}_\mathcal{Q} f\|_{p(\cdot)} \leq C_\mathcal{A}\|f\|_{p(\cdot)}$.

THEOREM 3.35. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $1 < p_- \leq p_+ < \infty$. Then the following are equivalent:*

(1) *$p(\cdot) \in \mathcal{A}$.*
(2) *The maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*
(3) *The maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$.*
(4) *There exists $s>1$ such that the maximal operator is bounded on $L^{p(\cdot)/s}(\mathbb{R}^n)$.*

A direct proof of the equivalence of (2) and (4) was given by Lerner and Ombrosi [70]. As we will see in the next chapter, the equivalence of (2) and (3) plays a major role in the application of extrapolation in the variable Lebesgue spaces. It would be very interesting to have a direct proof of this result, even in the case of the classical Lebesgue spaces.

3.6. Modular inequalities

We close this chapter by considering a different approach to norm inequalities for the maximal operator. In the classical Lebesgue spaces, norm inequalities are equivalent to modular inequalities, so if we consider the particular case when $p_+ < \infty$, then corresponding to inequalities (3.7) and (3.8) are the modular inequalities

$$\int_{\{x: Mf(x)>t\}} t^{p(x)}\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \tag{3.18}$$

$$\int_{\mathbb{R}^n} Mf(x)^{p(x)}\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \tag{3.19}$$

By the definition of the norm, these modular inequalities imply the corresponding norm inequalities, so these inequalities would be stronger results. However, they are neve true unless $p(\cdot)$ is constant. This was proved by Lerner [67]. His proof is interesting because it reveals a deep connection between the Muckenhoupt A_p weights (defined in the next Chapter) and the variable Lebesgue spaces.

THEOREM 3.36. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$. Then the modular inequalities (3.18) and (3.19) are true if and only if there is a constant p_0 such that $p(\cdot) = p_0$ a.e.*

There is a weaker formulation of a modular weak-type inequality; somewhat surprisingly, given the above results, it is true with extremely weak assumptions on $p(\cdot)$.

THEOREM 3.37. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $|\Omega_\infty| = 0$, then there exists a constant C such that for all $t > 0$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq C \int_{\mathbb{R}^n} \left(\frac{4|f(x)|}{t}\right)^{p(x)} dx.$$

A version of this inequality was first proved in [22] with a much more complicated proof. This version and its elegant proof are due to Aguilar Cañestro and Ortega Salvador [5].

PROOF. Fix $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $t > 0$. Define

$$f_1 = f\chi_{\{x\in\mathbb{R}^n : |f(x)|>t/2\}}, \qquad f_2 = f\chi_{\{x\in\mathbb{R}^n : |f(x)|\leq t/2\}},$$

By Proposition 3.3, $Mf_2(x) \leq t/2$. Therefore,

$$\begin{aligned} |\{x \in \mathbb{R}^n : Mf(x) > t\}| &\leq |\{x \in \mathbb{R}^n : Mf_1(x) + Mf_2(x) > t\}| \\ &\leq |\{x \in \mathbb{R}^n : Mf_1(x) > t/2\}|. \end{aligned}$$

We estimate the last term: since $|4t^{-1}f_1| \geq 1$, by the weak (p_-, p_-) inequality for the maximal operator (Theorem 3.5),

$$\begin{aligned} |\{x \in \mathbb{R}^n : Mf_1(x) > t/4\}| &= |\{x \in \mathbb{R}^n : M(4t^{-1}f_1)(x) > 1\}| \\ &\leq |\{x \in \mathbb{R}^n : M((4t^{-1}f_1)^{p(\cdot)/p_-})(x) > 1\}| \\ &\leq C \int_{\mathbb{R}^n} \left(\frac{4|f_1(x)|}{t}\right)^{p(x)} dx \\ &\leq C \int_{\mathbb{R}^n} \left(\frac{4|f(x)|}{t}\right)^{p(x)} dx. \end{aligned}$$

□

CHAPTER 4

Extrapolation in Variable Lebesgue Spaces

In this chapter we develop a general theory for proving norm inequalities for the other classical operators in harmonic analysis. Our main result is a powerful generalization of the Rubio de Francia extrapolation theorem. This approach, developed in detail in [**20**, **24**], lets us use the theory of weighted norm inequalities to prove the corresponding estimates on variable Lebesgue spaces. This greatly reduces the work required since it lets us use the well-developed theory of weights. The underlying philosophy might best be described by paraphrasing Antonio Cordoba's pithy summary of extrapolation theory [**43**]:

There are no variable Lebesgue spaces: only weighted L^2.

In the first three sections we discuss convolution operators and the convergence of approximate identities. We begin by reviewing the basic properties of convolutions on the classic Lebesgue spaces. We then show that these properties fail to extend to the variable setting by proving that variable Lebesgue spaces are not translation invariant, and, as a consequence, that Young's inequality fails spectacularly. On the other hand we are able to prove that approximate identities converge given reasonable assumptions on the exponent functions. These results are of interest in their own right, and the proof provides a motivation for the theory of extrapolation.

In the final three sections we develop the theory of extrapolation. We first digress briefly to present some basic facts about the Muckenhoupt A_p weights and weighted norm inequalities. We then prove the extrapolation theorem, and give some examples of the kinds of inequalities that can be proved using this theory. We develop in detail one particular example: convolution type singular integrals. While not the most general, this example makes clear the technical considerations that arise when applying the extrapolation theorem.

4.1. Convolution operators and approximate identities

We begin by recalling some basic results about convolutions. For further details and proofs of these results, see [**33**, **46**, **99**].

Definition 4.1. Given two locally integrable functions f and g, their convolution is the function $f * g$ defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy,$$

wherever this integral is finite.

It is immediate from the definition that convolutions are linear and commute. They also satisfy the following norm inequality referred to as Young's inequality. Though we omit the details, we recall the fact that the proof of Proposition 4.2 depends in a crucial way on the translation invariance of L^p.

PROPOSITION 4.2. *Given measurable functions f and g, and given p, q, r, $1 \leq p, q, r \leq \infty$, such that*

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q},$$

if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{4.1}$$

An important application of convolutions is the technique of approximate identities. Given a function $\phi \in L^1$, for each $t > 0$ let $\phi_t(x) = t^{-n}\phi(x/t)$. Then $\|\phi_t\|_1 = \|\phi\|_1$. Define the radial majorant of ϕ to be the function

$$\Phi(x) = \sup_{|y| \geq |x|} |\phi(y)|.$$

The function Φ is radial and decreasing as $|x|$ increases; however, it need not be in L^1 even though ϕ is. In important cases, e.g. when ϕ is bounded and has compact support, Φ is in L^1. It follows from the definitions that

$$|\phi_t * f(x)| \leq (\Phi_t * |f|)(x), \tag{4.2}$$

so in practice we can often replace ϕ by its radial majorant.

DEFINITION 4.3. Given $\phi \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$, the set $\{\phi_t\} = \{\phi_t : t > 0\}$ is called an approximate identity. If the radial majorant of ϕ is also in $L^1(\mathbb{R}^n)$, $\{\phi_t\}$ is called a potential type approximate identity.

THEOREM 4.4. *Given an approximate identity $\{\phi_t\}$, then for all p, $1 \leq p < \infty$, if $f \in L^p(\mathbb{R}^n)$, then $\|\phi_t * f - f\|_p \to 0$ as $t \to 0$. Further, if $\{\phi_t\}$ is a potential type approximate identity, then for all p, $1 \leq p \leq \infty$, $\phi_t * f(x) \to f(x)$ pointwise a.e. as $t \to 0$.*

4.2. The failure of Young's inequality in $L^{p(\cdot)}$

As we noted previously, Young's inequality fails to hold on the variable Lebesgue space: the proof depends fundamentally on the fact that the classical Lebesgue spaces are translation invariant. More precisely, given a function and $h \in \mathbb{R}^n$, define the translation operator τ_h by $\tau_h f(x) = f(x-h)$. Then for any p, if $f \in L^p(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, $\tau_h f \in L^p(\mathbb{R}^n)$ and $\|f\|_p = \|\tau_h f\|_p$. This property is never universally true on the variable Lebesgue spaces; this was first proved by Kováčik and Rákosník [64].

THEOREM 4.5. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, each of the translation operators τ_h, $h \in \mathbb{R}^n$, is a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $p(\cdot)$ is constant. Moreover, if $p(\cdot)$ is non-constant, there exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ such that $\tau_h f \notin L^{p(\cdot)}(\mathbb{R}^n)$.*

PROOF. If $p(\cdot)$ is constant, then this is immediate. To prove the converse, suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is such that for all h, $\|\tau_h f\|_{p(\cdot)} \leq C_h \|f\|_{p(\cdot)}$. By a change of variables we have that $\|\tau_h f\|_{p(\cdot)} = \|f\|_{\tau_{-h}p(\cdot)}$. More generally, fix h and a ball B. If $f \in L^{p(\cdot)}(B)$ and $f = 0$ on $\mathbb{R}^n \setminus B$, then $\tau_h f \in L^{p(\cdot)}(B+h)$, where $B + h = \{x + h : x \in B\}$, and $\|f\|_{L^{\tau_{-h}p(\cdot)}(B)} = \|\tau_h f\|_{L^{p(\cdot)}(B-h)}$. Hence, by our assumption on τ_h,

$$\|f\|_{L^{\tau_{-h}p(\cdot)}(B)} \leq \|\tau_h f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_h \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = C_h \|f\|_{L^{p(\cdot)}(B)}.$$

Therefore, by Theorem 2.28, $\tau_{-h}p(x) \leq p(x)$ for almost every $x \in B$. If we replace h by $-h$ and repeat the argument, we get the reverse inequality. Thus, $\tau_h p(x) = p(x)$ a.e. in B. Since B and h are arbitrary, this implies that $p(\cdot)$ is constant.

Given a non-constant $p(\cdot)$, to construct the desired function f, fix $h \in \mathbb{R}^n$ such that τ_h is not a bounded operator. Then there exists a sequence of functions $f_k \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $\|f_k\|_{p(\cdot)} \leq 1$ but $\|\tau_h f_k\|_{p(\cdot)} \geq 4^k$. If for some k, $\tau_h f_k \notin L^{p(\cdot)}(\mathbb{R}^n)$, we are done. Otherwise, let

$$f = \sum_{k=1}^{\infty} 2^{-k}|f_k|.$$

Then

$$\|f\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} 2^{-k}\|f_k\|_{p(\cdot)} \leq 1,$$

but for every k, $f \geq 2^{-k}|f_k|$, and so

$$\|\tau_h f\|_{p(\cdot)} \geq 2^{-k}\|\tau_h f_k\|_{p(\cdot)} \geq 2^k.$$

Hence, $\|\tau_h f\|_{p(\cdot)} = \infty$ and $\tau_h f \notin L^{p(\cdot)}(\mathbb{R}^n)$. □

The failure of Young's inequality is a direct consequence of Theorem 4.5; it was first proved by Diening [25] in a somewhat different form.

THEOREM 4.6. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the inequality*

$$\|f * g\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}\|g\|_1 \tag{4.3}$$

is true for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ if and only if $p(\cdot)$ is constant.

PROOF. If $p(\cdot) = p$ is constant, then (4.3) becomes (4.1).

Now suppose that $p(\cdot)$ is not constant but (4.3) holds for all f and g. By Theorem 4.5 there exists $h \in \mathbb{R}^n$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $\tau_h f \notin L^{p(\cdot)}(\Omega)$. If we replace f by $|f|/\|f\|_{p(\cdot)}$ we may assume f is non-negative and $\|f\|_{p(\cdot)} = 1$. For each $N > 0$, let $g_N(x) = \min(f(x), N)\chi_{B_N(0)}$. Then $\|g_N\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} \leq 1$. Further, since g_N is a bounded function of compact support, for each N, and $\tau_h g_N \in L^{p(\cdot)}(\mathbb{R}^n)$. Since $g_N \to f$ pointwise, by Theorem 2.19, $\|\tau_h g_N\|_{p(\cdot)} \to \infty$ as $N \to \infty$. Therefore, for every $k \geq 1$ we can construct a new sequence $\{f_k\}$ such that $f_k \in L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \leq 1$, but $\|\tau_h f_k\|_{p(\cdot)} \geq 2^k$.

Let ϕ be a bounded, non-negative function of compact support such that $\|\phi\|_1 = 1$. For every $t > 0$, let $\psi_{t,h}(x) = t^{-n}\phi((x-h)/t)$. Then by a change of variables,

$$\begin{aligned}\psi_{t,h} * f_k(x) &= t^{-n}\int_{\mathbb{R}^n} \phi\left(\frac{x-y-h}{t}\right) f_k(y)\,dy \\ &= t^{-n}\int_{\mathbb{R}^n} \phi\left(\frac{x-y}{t}\right) f_k(y-h)\,dy = \phi_t * (\tau_h f_k)(x).\end{aligned}$$

By assumption, $\tau_h f_k \in L^{p(\cdot)}(\mathbb{R}^n)$, so by Theorem 4.9 below, $\phi_t * (\tau_h f_k) \to \tau_h f_k$ pointwise a.e. Therefore, again by Theorem 2.19 and by (4.3),

$$\begin{aligned}2^k \leq \|\tau_h f_k\|_{p(\cdot)} &\leq \liminf_{t\to 0} \|\phi_t * (\tau_h f_k)\|_{p(\cdot)} \\ &= \liminf_{t\to 0} \|\psi_{t,h} * f_k\|_{p(\cdot)} \leq C\|f_k\|_{p(\cdot)}\|\psi_{t,h}\|_1 \leq C.\end{aligned}$$

This is impossible for arbitrary k, so we get a contradiction. Hence, inequality (4.3) holds if and only if $p(\cdot)$ is constant. □

As a consequence of Lemma 4.14 below we can prove a weak version of Young's inequality. This was first noted in [17].

PROPOSITION 4.7. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Then for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and every non-negative, radially decreasing function $g \in L^1(\mathbb{R}^n)$,*

$$\|f * g\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}\|g\|_1.$$

However, even given the restrictive hypotheses of Proposition 4.7, Young's inequality does not hold for general exponents. This example is also from [17].

EXAMPLE 4.8. Let $p(\cdot) \in \mathcal{P}(\mathbb{R})$ be a smooth function such that $p(x) = 2$ if $x \in \mathbb{R}^n \setminus [-2, 2]$, and $p(x) = 4$ on $[-1, 1]$. Then $p(\cdot) \in LH(\mathbb{R})$, so the maximal operator is bounded. Define

$$f(x) = |x - 3|^{-1/3}\chi_{[2,4]},$$
$$g(x) = |x|^{-2/3}\chi_{[-1,1]}.$$

Since $f^2 \in L^1(\mathbb{R})$, by Proposition 2.9, $f \in L^{p(\cdot)}(\mathbb{R})$. Similarly, since $p'(x) = 4/3$ on $[-1, 1]$ and $g^{4/3} \in L^1(\mathbb{R})$, $g \in L^{p'(\cdot)}(\mathbb{R})$. However, we do not have that

$$\|f * g\|_\infty \leq C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

since $f * g$ is unbounded in a neighborhood of 3. To show this, let $E_x = [2, 4] \cap [x - 1, x + 1]$. Then by Fatou's lemma on the classical Lebesgue spaces,

$$\begin{aligned}
\liminf_{x\to 3} f * g(x) &= \liminf_{x\to 3} \int_{\mathbb{R}^n} |x - y|^{-2/3}|y - 3|^{-1/3}\chi_{E_x}(y)\, dy \\
&\geq \int_{\mathbb{R}^n} \lim_{x\to 3} \left(|x - y|^{-2/3}|y - 3|^{-1/3}\chi_{E_x}(y)\right) dy \\
&= \int_2^4 |y - 3|^{-1}\, dy = \infty.
\end{aligned}$$

4.3. Approximate identities on variable Lebesgue spaces

While the failure of Young's inequality might suggest that no property of convolution operators can be salvaged in the variable Lebesgue spaces, the convergence of approximate identities is preserved if we assume that the exponent function $p(\cdot)$ has some regularity: in particular, if the maximal operator is bounded.

We consider both pointwise convergence and norm convergence. In the L^p spaces, the norm convergence of an approximate identity is relatively straightforward to prove, but pointwise convergence requires a more sophisticated argument using the maximal operator. For variable Lebesgue spaces the opposite holds: pointwise convergence is an immediate consequence of the classical result, but norm convergence requires the boundedness of the maximal operator. The following two results were proved by Diening [25] assuming that $1 < p_- \leq p_+ < \infty$ and the maximal operator is bounded on $L^{p(\cdot)}$. The general version of Theorem 4.9 was proved in [17]. Theorem 4.11 was also proved there assuming that $p(\cdot) \in LH$. This proof depended on a pointwise estimate for approximate identities. The proof of Theorem 4.11 given here is from [19].

THEOREM 4.9. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, let $f \in L^{p(\cdot)}(\mathbb{R}^n)$. If $\{\phi_t\}$ is any potential type approximate identity, then for all $t > 0$, $\phi_t * f$ is finite a.e., and $\phi_t * f \to f$ pointwise a.e.*

PROOF. By Theorem 2.29, write $f = f_1 + f_2$, where $f_1 \in L^{p_+}(\mathbb{R}^n)$ and $f_2 \in L^{p_-}(\mathbb{R}^n)$. Since $\phi_t * f = \phi_t * f_1 + \phi_t * f_2$, and $\phi_t \in L^1(\mathbb{R}^n)$, by Young's inequality (4.1) each term is finite a.e., and the desired limit follows at once from Theorem 4.4. □

Remark 4.10. Though a simple application, the proof illustrates the utility of Theorem 2.29: by decomposing f in this way we can immediately apply known results in the classical L^p spaces. Thus we do not have to work directly with functions in $L^{p(\cdot)}$; we will use this idea repeatedly below.

We now consider the convergence in norm of approximate identities. To achieve this we need a stronger assumption on $p(\cdot)$.

Theorem 4.11. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. If $\{\phi_t\}$ is a potential type approximate identity, then*

$$\sup_{t>0} \|\phi_t * f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}, \tag{4.4}$$

and $\phi_t * f \to f$ in norm on $L^{p(\cdot)}(\mathbb{R}^n)$. The constant C in (4.4) depends on n, $p(\cdot)$, $\|M\|_{\mathcal{B}(L^{p'(\cdot)}(\mathbb{R}^n))}$ and $\|\Phi\|_1$.

Remark 4.12. If we replace $\mathbb{R}^n$ by a bounded set Ω, then we can modify the proof to show that this result remains true if we only assume that $p(\cdot) \in LH_0(\Omega)$. This fact is often useful in applications: for instance, in proving the density of smooth functions of compact support in the variable Sobolev spaces. See [**19, 30**].

Remark 4.13. The assumption $p_+ < \infty$ is redundant: if the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$, then $p'(\cdot)_- > 1$, and so $p_+ = (p'(\cdot)_-)' < \infty$.

The proof of Theorem 4.11 requires the following lemma which is adapted from [**33**].

Lemma 4.14. *Let $\{\phi_t\}$ be a potential type approximate identity and let Φ be the radial majorant of ϕ. Then for every locally integrable function f and every x,*

$$\sup_{t>0} |\phi_t * f(x)| \leq C(n)\|\Phi\|_1 Mf(x).$$

Proof. By (4.2) and the discussion in Section 3.1, it will suffice to prove that given any non-negative $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, for all $t > 0$,

$$\Phi_t * f(x) \leq \|\Phi\|_1 Mf(x),$$

where here we take the maximal operator to be the supremum of averages over balls. For each j, $k \geq 1$ let $B_j^k = B_{j2^{-k}}(0)$. Since Φ is radial, we abuse notation and let $\Phi(|x|) = \Phi(x)$. Define the function Φ_k by

$$\Phi_k(x) = \sum_{j=1}^{\infty} \big(\Phi(j2^{-k}) - \Phi((j+1)2^{-k})\big)\chi_{B_j^k}(x) = \sum_{j=1}^{\infty} a_j^k \chi_{B_j^k}(x).$$

Since Φ is decreasing, $a_j^k \geq 0$. Let $A_j^k = B_j^k \setminus B_{j-1}^k$; then for $x \in A_j^k$,

$$\Phi_k(x) = \sum_{i=j}^{\infty} \big(\Phi(i2^{-k}) - \Phi((i+1)2^{-k})\big) = \Phi(j2^{-k}) \leq \Phi(x).$$

The middle sum converges since Φ is a non-negative function that decreases to 0 as $|x| \to \infty$. Further, $\{\Phi_k\}$ increases to Φ pointwise a.e. Hence, by the monotone convergence theorem on $L^1(\mathbb{R}^n)$, if f is non-negative, for each $t > 0$, $(\Phi_k)_t * f$ increases to $\Phi_t * f$ pointwise as $k \to \infty$. Therefore, it will suffice to prove that for all $k \geq 1$ and $t > 0$,

$$(\Phi_k)_t * f(x) \leq \|\Phi\|_1 Mf(x).$$

We first consider the case $t = 1$. Since for all x,

$$|B_j^k|^{-1}\chi_{B_j^k} * f(x) = \fint_{B_j^k} f(x-y)\,dy = \fint_{B_{j2^{-k}}(x)} f(y)\,dy \leq Mf(x),$$

$$\Phi_k * f(x) = \sum_j a_j^k |B_j^k| \cdot |B_j^k|^{-1} \chi_{B_j^k} * f(x) \le \|\Phi_k\|_1 Mf(x) \le \|\Phi\|_1 Mf(x).$$

We can now repeat this argument with Φ_k replaced by $(\Phi_k)_t$; since $\|(\Phi_k)_t\|_1 = \|\Phi_k\|_1$, we get the desired inequality for all $t > 0$. □

Proof of Theorem 4.11. Fix $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $t > 0$. Let Φ be the radial majorant of ϕ. Then by (4.2) and Theorem 2.36 there exists $h \in L^{p'(\cdot)}(\mathbb{R}^n)$, $\|h\|_{p'(\cdot)} = 1$, such that

$$\|\phi_t * f\|_{p(\cdot)} \le \|\Phi_t * |f|\|_{p(\cdot)} \le 2k_{p(\cdot)}^{-1} \int_{\mathbb{R}^n} \Phi_t * |f|(x) h(x)\, dx.$$

Since Φ_t is a radial function, by Fubini's theorem, Theorem 2.32, Lemma 4.14 and our assumption on $p'(\cdot)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} (\Phi_t * |f|)(x) h(x)\, dx &= \int_{\mathbb{R}^n} |f(x)| \Phi_t * h(x)\, dx \\
&\le C(n)\|\Phi\|_1 \int_{\mathbb{R}^n} |f(x)| Mh(x)\, dx \\
&\le C(n)\|\Phi\|_1 K_{p(\cdot)} \|f\|_{p(\cdot)} \|Mh\|_{p'(\cdot)} \\
&\le C\|M\|_{\mathcal{B}(L^{p'(\cdot)}(\mathbb{R}^n))} \|f\|_{p(\cdot)} \|h\|_{p'(\cdot)} \\
&= C\|f\|_{p(\cdot)}.
\end{aligned}$$

Since the constants do not depend on t, inequality (4.4) follows at once.

To prove that $\phi_t * f$ converges to f in norm on $L^{p(\cdot)}(\mathbb{R}^n)$, we use an approximation argument. Fix $\epsilon > 0$. By Theorem 2.30 there exists a function g, bounded with compact support and not identically zero, such that $\|f - g\|_{p(\cdot)} < \epsilon$. Then by (4.4),

$$\begin{aligned}
\|\phi_t * f - f\|_{p(\cdot)} &\le \|\phi_t * (f - g)\|_{p(\cdot)} \\
&\qquad + \|\phi_t * g - g\|_{p(\cdot)} + \|f - g\|_{p(\cdot)} \\
&\le C\epsilon + \|\phi_t * g - g\|_{p(\cdot)}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, to complete the proof it will suffice to show that

$$\lim_{t \to 0} \|\phi_t * g - g\|_{p(\cdot)} = 0;$$

since $p_+ < \infty$, by Lemma 2.20 it will suffice to show that

$$\lim_{t \to 0} \int_{\mathbb{R}^n} |\phi_t * g(x) - g(x)|^{p(x)}\, dx = 0.$$

Let $g_0(x) = g(x)/(2\|\phi\|_1 \|g\|_\infty)$; since $\|\phi\|_1 \ge 1$, $\|g_0\|_\infty \le 1/2$. Furthermore,

$$|\phi_t * g_0(x)| \le \int_{\mathbb{R}^n} |\phi_t(x - y)||g_0(y)|\, dy \le \|g_0\|_\infty \int_{\mathbb{R}^n} |\phi_t(x - y)|\, dy \le 1/2.$$

Therefore, $\|\phi_t * g_0 - g_0\|_\infty \le 1$, and so

$$\begin{aligned}
&\lim_{t \to 0} \int_{\mathbb{R}^n} |\phi_t * g(x) - g(x)|^{p(x)}\, dx \\
&\qquad = \lim_{t \to 0} \int_{\mathbb{R}^n} (2\|\phi\|_1 \|g\|_\infty)^{p(x)} |\phi_t * g_0(x) - g_0(x)|^{p(x)}\, dx \\
&\qquad \le (2\|\phi\|_1 \|g\|_\infty + 1)^{p_+} \lim_{t \to 0} \int_{\mathbb{R}^n} |\phi_t * g_0(x) - g_0(x)|^{p_-}\, dx.
\end{aligned}$$

Since $g_0 \in L^{p_-}(\mathbb{R}^n)$ and $1 \le p_- < \infty$, by Theorem 4.4 the last term equals 0. This completes the proof. □

In the proof of Theorem 4.11 we actually only need the hypothesis that $p_+ < \infty$ to prove norm convergence; inequality (4.4) remains true if we assume $p_+ = \infty$, $p_- > 1$ and M is bounded on $L^{p(\cdot)}$. In fact, slightly weaker hypotheses suffice: see [30]. However, to prove norm convergence this hypothesis is necessary. For an example, see [19].

We conclude this section by showing that the classical solutions to the Laplacian and the heat equation extend to the variable Lebesgue spaces. The Poisson and Gauss-Weierstrass kernels are defined as follows: for $t > 0$ and $x \in \mathbb{R}^n$, let

$$P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}, \qquad W_t(x) = t^{-n} e^{-\pi|x|^2/t^2}.$$

Clearly, $\{P_t\}$ and $\{W_t\}$ are potential type approximate identities.

PROPOSITION 4.15. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then $u(x,t) = P_t * f(x)$ is the solution of the boundary value problem*

$$\begin{cases} \triangle u(x,t) = 0, & (x,t) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where the second equality is understood in the sense that $u(x,t)$ converges to $f(x)$ as $t \to 0$ pointwise a.e. and in $L^{p(\cdot)}(\mathbb{R}^n)$ norm.

PROPOSITION 4.16. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that $p_+ < \infty$ and the maximal operator is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$. Given $f \in L^{p(\cdot)}(\mathbb{R}^n)$, define $w(x,t) = W_t * f(x)$ and $\bar{w}(x,t) = w(x, \sqrt{4\pi t})$. Then $\bar{w}$ is the solution of the initial value problem*

$$\begin{cases} \frac{\partial \bar{w}}{\partial t}(w,t) - \triangle \bar{w}(x,t) = 0, & (x,t) \in \mathbb{R}^{n+1}_+, \\ \bar{w}(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where the second equality is understood in the sense that $\bar{w}(x,t)$ converges to $f(x)$ as $t \to 0$ pointwise a.e. and in $L^{p(\cdot)}(\mathbb{R}^n)$ norm.

Propositions 4.15 and 4.16 were first proved in [17]. Sharapudinov [97] proved similar results on the unit circle.

PROOF. We sketch the proof of Proposition 4.15; the proof of Proposition 4.16 is identical. First, we show that u is a solution. By Theorem 2.29 write $f = f_1 + f_2$ with $f_1 \in L^{p_-}(\mathbb{R}^n)$ and $f_2 \in L^{p_+}(\mathbb{R}^n)$. By the classical theory (see [45]), $u_1 = P_t * f_1$ and $u_2 = P_t * f_2$ are solutions, and so $u = u_1 + u_2$ is also a solution. The identity $u(x,0) = f(x)$ follows from Theorems 4.9 and 4.11 since $\{P_t\}$ is a potential type approximate identity. □

4.4. Muckenhoupt weights and weighted norm inequalities

In this section we give some basic definitions and state without proof some fundamental results from the theory of weighted norm inequalities. For further information and proofs of all the results, see the books by Duoandikoetxea [33], García-Cuerva and Rubio de Francia [44] and Grafakos [47].

Hereafter, by a weight we mean a non-negative, locally integrable function such that $0 < w(x) < \infty$ a.e. For $1 < p < \infty$, a weight w is in the Muckenhoupt class A_p—or simply, $w \in A_p$—if

$$[w]_{A_p} = \sup_Q \left(\fint_Q w(x)\,dx \right) \left(\fint_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty, \tag{4.5}$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes. For $p = 1$, we say that $w \in A_1$ if

$$[w]_{A_1} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty, \tag{4.6}$$

where M is the Hardy-Littlewood maximal operator. It follows from this definition that if $w \in A_1$, then for almost every x,

$$Mw(x) \le [w]_{A_1} w(x). \tag{4.7}$$

In turn, this implies that for every cube Q,

$$\fint_Q w(y)\,dy \le [w]_{A_1} \operatorname*{ess\,inf}_{x\in Q} w(x). \tag{4.8}$$

The collection of all the A_p weights is denoted by A_∞:

$$A_\infty = \bigcup_{p\ge 1} A_p.$$

Lemma 4.17. *The A_p classes are nested: given p, q, $1 \le p < q < \infty$, $A_p \subset A_q$.*

Remark 4.18. In the definition of A_p weights we can substitute balls for cubes. In Section 3.1 we showed that the maximal operator can be defined using either balls or cubes, and the same reasoning applies here: given any ball B, there exist two cubes Q_1, Q_2 with the same center such that $Q_1 \subset B \subset Q_2$ and $|Q_2|/|Q_1| = n^{n/2}$, and a similar relationship holds with the roles of balls and cubes reversed.

The theory of extrapolation requires that we construct A_1 weights using arbitrary functions in $L^{p(\cdot)}$. We do so using an iteration technique referred to as the Rubio de Francia iteration algorithm.

Lemma 4.19. *Given $p(\cdot)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, for each $h \in L^{p(\cdot)}(\mathbb{R}^n)$ define*

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}^n))}},$$

where for $k \ge 1$, $M^k = M \circ M \circ \cdots \circ M$ denotes k iterations of the maximal operator and $M^0 f = |f|$. Then

- (a) *for all $x \in \mathbb{R}^n$, $|h(x)| \le \mathcal{R}h(x)$;*
- (b) *$\mathcal{R}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and $\|\mathcal{R}h\|_{p(\cdot)} \le 2\|h\|_{p(\cdot)}$;*
- (c) *$\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \le 2\|M\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}^n))}$.*

Proof. Property (a) follows immediately from the definition. Property (b) follows from the subadditivity of the norm:

$$\|\mathcal{R}h\|_{p(\cdot)} \le \sum_{k=0}^{\infty} \frac{\|M^k h\|_{p(\cdot)}}{2^k \|M\|^k_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}^n))}} \le \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} 2^{-k} = 2\|h\|_{p(\cdot)}.$$

Property (c) follows by the subadditivity and homogeneity of the maximal operator:

$$\begin{aligned} M(\mathcal{R}h)(x) &\le \sum_{k=0}^{\infty} \frac{M^{k+1} h(x)}{2^k \|M\|^k_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}^n))}} \\ &\le 2\|M\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}^n))} \sum_{k=0}^{\infty} \frac{M^{k+1} h(x)}{2^{k+1} \|M\|^{k+1}_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}^n))}} \le 2\|M\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}^n))} \mathcal{R}h(x). \end{aligned}$$ □

REMARK 4.20. The iteration algorithm is an extremely powerful tool. For example, it can be used to give an elementary proof of Diening's result (see Theorem 3.35) that if M is bounded on $L^{p(\cdot)}$, there exists $s > 1$ such that M is bounded on $L^{p(\cdot)/s}$. For a proof, see Lerner and Ombrosi [70].

There is a close connection between Muckenhoupt A_p weights and the maximal operator: the following result is fundamental in the study of weighted norm inequalities.

THEOREM 4.21. *Given p, $1 \leq p < \infty$, then $w \in A_p$ if and only if for every $f \in L^p(w)$ and every $t > 0$,*

$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq \frac{C(n, p, [w]_{A_p})}{t^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx. \tag{4.9}$$

Furthermore, if $p > 1$, then $w \in A_p$ if and only if

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\, dx \leq C(n, p, [w]_{A_p}) \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx. \tag{4.10}$$

We end this section by briefly describing a generalization of A_p weights to variable Lebesgue spaces. To motivate our definition we need to recast the definition of A_p weights. In (4.5) if we replace w by w^p we can rewrite the definition of A_p as

$$\sup_Q |Q|^{-1} \|w\chi_Q\|_p \|w^{-1}\chi_Q\|_{p'} < \infty.$$

Then inequality (4.10) becomes

$$\|(Mf)w\|_p \leq C\|fw\|_p.$$

In this formulation, we treat w not as a measure (i.e., as $w\,dx$) but as a multiplier.

The advantage of this reformulation is that it extends immediately to the variable Lebesgue spaces. We say that a weight $w \in A_{p(\cdot)}$ if

$$\sup_Q |Q|^{-1} \|w\chi_Q\|_{p(\cdot)} \|w^{-1}\chi_Q\|_{p'(\cdot)} < \infty.$$

Further, we have the following result, which was discovered by the authors and Neugebauer [21] and independently by Diening and Hästö [31]. (Another proof was given in [16].)

THEOREM 4.22. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}^n)$, then $w \in A_{p(\cdot)}$ if and only if*

$$\|(Mf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

4.5. Rubio de Francia extrapolation

The theory of extrapolation is an extremely powerful tool in the study of weighted norm inequalities. Our treatment in this section is derived from [24] which gives a comprehensive development of the theory. To put our main result in context, we first state the classical result, albeit in a recent formulation. While a tool for proving weighted norm inequalities for operators, a surprising feature of the proof is that the properties of the operator play no role. Therefore, we will work with pairs (F, G) of non-negative, measurable functions. This may seem a superfluous generalization, but it allows the theory of extrapolation to be extended to prove a much wider class of results.

Hereafter, let $\mathcal{F}$ denote a family of pairs of non-negative, measurable functions; given p, q, $1 \leq p, q < \infty$, if for some $w \in A_q$ we write

$$\int_{\mathbb{R}^n} F(x)^p w(x)\, dx \leq C_0 \int_{\mathbb{R}^n} G(x)^p w(x)\, dx, \qquad (F,G) \in \mathcal{F}, \tag{4.11}$$

then we mean that this inequality holds for all pairs $(F,G) \in \mathcal{F}$ such that the left-hand side is finite, and that the constant may depend on n, p, and $[w]_{A_q}$ but not on w.

Theorem 4.23. *Suppose that for some p_0, $1 \leq p_0 < \infty$, the family $\mathcal{F}$ is such that for all $w \in A_{p_0}$,*

$$\int_{\mathbb{R}^n} F(x)^{p_0} w(x)\, dx \leq C_{p_0} \int_{\mathbb{R}^n} G(x)^{p_0} w(x)\, dx, \qquad (F,G) \in \mathcal{F}. \tag{4.12}$$

Then for every p, $1 < p < \infty$, and every $w \in A_p$,

$$\int_{\mathbb{R}^n} F(x)^p w(x)\, dx \leq C_p \int_{\mathbb{R}^n} G(x)^p w(x)\, dx, \qquad (F,G) \in \mathcal{F}. \tag{4.13}$$

The utility of the more abstract approach comes from ability to choose the family of pairs. To apply the theorem to prove norm inequalities for an operator T, we would consider a family of pairs of the form $(|Tf|, |f|)$, where f ranges over some appropriate collection of functions. We can also use extrapolation to prove Coifman-Fefferman type inequalities of the form

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} Sf(x)^p w(x)\, dx,$$

where T is (usually) some more singular operator and S is some positive operator such as a maximal operator or square function. Here we would apply extrapolation to a family of pairs $(|Tf|, Sf)$. We can also use extrapolation to prove weak-type inequalities of the form

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx;$$

in this case we would apply extrapolation to the family of pairs

$$(t\chi_{\{x\in\mathbb{R}^n : |Tf(x)|>t\}}, |f|);$$

see Corollary 4.28 below. In all of these applications, some care must be exercised in constructing the family $\mathcal{F}$ so that the left-hand sides of the inequalities (4.12) and (4.13) are finite, and so that the desired norm inequality can be shown to hold for all functions in the space. We will consider this question further in the next section when we discuss applications of extrapolation.

To state our version of Rubio de Francia extrapolation for variable Lebesgue spaces, we extend our convention for the family $\mathcal{F}$ as follows: if we write

$$\|F\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_{p(\cdot)} \|G\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \qquad (F,G) \in \mathcal{F}, \tag{4.14}$$

then we mean that this inequality holds for all pairs such that the left-hand side is finite and the constant may depend on n and $p(\cdot)$.

Theorem 4.24. *Suppose that for some $p_0 > 0$ the family $\mathcal{F}$ is such that for all $w \in A_1$,*

$$\int_{\mathbb{R}^n} F(x)^{p_0} w(x)\, dx \leq C_0 \int_{\mathbb{R}^n} G(x)^{p_0} w(x)\, dx, \qquad (F,G) \in \mathcal{F}. \tag{4.15}$$

Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $p_0 \leq p_- \leq p_+ < \infty$ and the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$, then

$$\|F\|_{p(\cdot)} \leq C_{p(\cdot)} \|G\|_{p(\cdot)}, \qquad (F,G) \in \mathcal{F}.$$

REMARK 4.25. As was the case for Theorem 4.11, the hypothesis $p_+ < \infty$ is redundant: if $p_+ = \infty$, then $((p(\cdot)/p_0)')_- = 1$ and the maximal operator cannot be bounded on $L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$. We include it for clarity.

REMARK 4.26. If $p_0 \leq 1$, then the hypothesis $p_0 \leq p_-$ automatically holds. However, this result can be extended to variable Lebesgue spaces defined for exponents $p(\cdot)$ such that $p_- < 1$; these are quasi-Banach function spaces. For details, see [20].

To motivate the proof of Theorem 4.24, we first reconsider the proof of Theorem 4.11. Given a potential-type approximate identity $\{\phi_t\}$, the heart of the proof is a duality argument that yields

$$\int_{\mathbb{R}^n} (\Phi_t * |f|)(x)h(x)\,dx \leq C\|\Phi\|_1 \int_{\mathbb{R}^n} |f(x)|Mh(x)\,dx. \tag{4.16}$$

Suppose for the moment that $h \in A_1$. Then we would have that $Mh(x) \leq [h]_{A_1} h(x)$, and so we could rewrite (4.16) as

$$\int_{\mathbb{R}^n} (\Phi_t * |f|)(x)h(x)\,dx \leq C\|\Phi\|_1 [h]_{A_1} \int_{\mathbb{R}^n} |f(x)|h(x)\,dx. \tag{4.17}$$

At this point, the proof would continue as before. In other words: the weighted norm inequality (4.17) would imply that the convolution operators $\Phi_t * f$ are uniformly bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

The problem with this argument is obvious: in general h is not an A_1 weight. In the actual proof we overcame this by keeping Mh and using the norm inequalities for M after we applied Hölder's inequality. A more flexible approach is to use the iteration algorithm of Rubio de Francia and replace h by $\mathcal{R}h$. In this case we have that $\mathcal{R}h$ is an A_1 weight, and we can use the theory of weighted norm inequalities.

For the proof we need one lemma on the variable Lebesgue space norm.

LEMMA 4.27. *Given $\mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $|\mathbb{R}^n_\infty| = 0$, then for all s, $1/p_- \leq s < \infty$,*

$$\||f|^s\|_{p(\cdot)} = \|f\|^s_{sp(\cdot)}.$$

PROOF. This follows at once from the definition of the norm: since $|\mathbb{R}^n_\infty| = 0$, if we let $\mu = \lambda^{1/s}$,

$$\begin{aligned}\||f|^s\|_{p(\cdot)} &= \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|^s}{\lambda}\right)^{p(x)} dx \leq 1\right\} \\ &= \inf\left\{\mu^s > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\mu}\right)^{sp(x)} dx \leq 1\right\} = \|f\|^s_{sp(\cdot)}.\end{aligned}$$ □

PROOF OF THEOREM 4.24. Fix $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ as in the hypotheses, and let $\bar{p}(x) = p(x)/p_0$. By assumption the maximal operator is bounded on $L^{\bar{p}'(\cdot)}(\mathbb{R}^n)$. As in Lemma 4.19, define the iteration algorithm $\mathcal{R}$ on $L^{\bar{p}'(\cdot)}(\mathbb{R}^n)$ by

$$\mathcal{R}h(x) = \sum_{k=0}^\infty \frac{M^k h(x)}{2^k \|M\|^k_{\mathcal{B}(L^{\bar{p}'(\cdot)}(\mathbb{R}^n))}}.$$

Then we have that

(a) for all $x \in \mathbb{R}^n$, $|h(x)| \leq \mathcal{R}h(x)$;
(b) $\mathcal{R}$ is bounded on $L^{\bar{p}'(\cdot)}(\mathbb{R}^n)$ and $\|\mathcal{R}h\|_{\bar{p}'(\cdot)} \leq 2\|h\|_{\bar{p}'(\cdot)}$;
(c) $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{\mathcal{B}(L^{\bar{p}'(\cdot)}(\mathbb{R}^n))}$.

Fix a pair $(F, G) \in \mathcal{F}$ such that $F \in L^{p(\cdot)}(\mathbb{R}^n)$ (i.e., so that the left-hand side of (4.33) is finite). By Proposition 4.27 and Theorem 2.36,

$$\|F\|_{p(\cdot)}^{p_0} = \|F^{p_0}\|_{\bar{p}(\cdot)} \le C \sup \int_{\mathbb{R}^n} F(x)^{p_0} h(x)\, dx,$$

where the supremum is taken over all non-negative $h \in L^{\bar{p}'(\cdot)}(\mathbb{R}^n)$ with $\|h\|_{\bar{p}'(\cdot)} = 1$. Fix any such function h; we will show that

$$\int_{\mathbb{R}^n} F(x)^{p_0} h(x)\, dx \le C\|G\|_{p(\cdot)}^{p_0}$$

with the constant C independent of h. First note that by Property (a) we have that

$$\int_{\mathbb{R}^n} F(x)^{p_0}\, h(x)\, dx \le \int_{\mathbb{R}^n} F(x)^{p_0}\, \mathcal{R}h(x)\, dx. \tag{4.18}$$

We want to apply our hypothesis (4.31) to the right-hand term in (4.18). To do so we have to show that it is finite: by the generalized Hölder's inequality (Theorem 2.32), Property (b) and Proposition 4.27,

$$\begin{aligned}\int_{\mathbb{R}^n} F(x)^{p_0} \mathcal{R}h(x)\, dx &\le K_{p(\cdot)}\|F^{p_0}\|_{\bar{p}(\cdot)}\|\mathcal{R}h\|_{\bar{p}'(\cdot)} \\ &\le 2K_{p(\cdot)}\|F\|_{p(\cdot)}^{p_0}\|h\|_{\bar{p}'(\cdot)} < \infty.\end{aligned}$$

Therefore, by Property (c), (4.31) holds with $w = \mathcal{R}h$. Further, the constant C_0 only depends on $[\mathcal{R}h]_{A_1}$ and so is independent of h. Hence, by (4.31) and again by Theorem 2.32 and Proposition 4.27 we get

$$\begin{aligned}\int_{\mathbb{R}^n} F(x)^{p_0} \mathcal{R}h(x)\, dx &\le C_0 \int_{\mathbb{R}^n} G(x)^{p_0} \mathcal{R}h(x)\, dx \\ &\le C_0\|G^{p_0}\|_{\bar{p}(\cdot)}\|\mathcal{R}h\|_{\bar{p}'(\cdot)} \\ &= C_0\|G\|_{p(\cdot)}^{p_0}\|\mathcal{R}h\|_{\bar{p}'(\cdot)}\end{aligned}$$

Finally, we need to show that $\|(\mathcal{R}h)\|_{\bar{p}'(\cdot)}$ is bounded by a constant independent of h. But by Property (b),

$$\|\mathcal{R}h\|_{\bar{p}'(\cdot)} \le 2\|h\|_{\bar{p}'(\cdot)} = 2.$$

This completes our proof. □

Theorem 4.41 has two corollaries, both of which further illustrate the value of defining extrapolation for arbitrary pairs of functions. The first yields weak type inequalities and the second vector-valued inequalities.

Corollary 4.28. *Given $\mathbb{R}^n$, suppose that for some $p_0 \ge 1$, the family $\mathcal{F}$ is such that for all $w \in A_1$,*

$$w(\{x \in \mathbb{R}^n : F(x) > t\}) \le C_0 \frac{1}{t^{p_0}} \int_{\mathbb{R}^n} G(x)^{p_0} w(x)\, dx \qquad (F, G) \in \mathcal{F}. \tag{4.19}$$

Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p_0 \le p_- \le p_+ < \infty$, if the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$, then for all $t > 0$,

$$\|t\chi_{\{x\in\mathbb{R}^n : F(x)>t\}}\|_{p(\cdot)} \le C_{p(\cdot)}\|G\|_{p(\cdot)}, \qquad (F, G) \in \mathcal{F}. \tag{4.20}$$

Proof. Define a new family $\tilde{\mathcal{F}}$ consisting of the pairs

$$(F_t, G) = (t\chi_{\{x\in\mathbb{R}^n : F(x)>t\}}, G), \qquad (F, G) \in \mathcal{F},\ t > 0.$$

Then we can restate (4.19) as follows: for every $w \in A_1$,

$$\|F_t\|_{L^{p_0}(w)} = tw(\{x \in \mathbb{R}^n : F(x) > t\})^{1/p_0} \le C_0^{1/p_0}\|G\|_{L^{p_0}(w)}, \qquad (F_t, G) \in \tilde{\mathcal{F}}.$$

Therefore, we can apply Theorem 4.41 to the family $\tilde{\mathcal{F}}$ to conclude that (4.33) holds for the pairs $(F_t, G) \in \tilde{\mathcal{F}}$, which in turn immediately implies (4.20). □

COROLLARY 4.29. *Given $\mathbb{R}^n$, suppose that for some $p_0 \geq 1$ the family $\mathcal{F}$ is such that for all $w \in A_{p_0}$,*

$$\int_{\mathbb{R}^n} F(x)^{p_0} w(x)\,dx \leq C_0 \int_{\mathbb{R}^n} G(x)^{p_0} w(x)\,dx, \qquad (F,G) \in \mathcal{F}. \tag{4.21}$$

Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if $p_0 \leq p_- \leq p_+ < \infty$ and the maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$, then for every r, $1 < r < \infty$, and sequence $\{(F_i, G_i)\} \subset \mathcal{F}$,

$$\Big\| \Big(\sum_i F_i^r \Big)^{1/r} \Big\|_{p(\cdot)} \leq C_{p(\cdot)} \Big\| \Big(\sum_i G_i^r \Big)^{1/r} \Big\|_{p(\cdot)}. \tag{4.22}$$

Corollary 4.29 requires a more restrictive hypothesis than Theorem 4.41 or Corollary 4.28 since it requires (4.21) to hold for a larger class of weights. In practice, however, this restriction is only a problem if the operator T is very "rough" or "singular". Most of the classical operators in harmonic analysis satisfy weighted L^p norm inequalities with weights in A_p.

PROOF. Fix r, $1 < r < \infty$. We first reduce the proof to the special case of finite sums. For if this case holds, given any sequence $\{(F_i, G_i)\} \subset \mathcal{F}$, by Fatou's lemma for variable Lebesgue spaces (Theorem 2.19),

$$\begin{aligned} \Big\| \Big(\sum_i F_i^r \Big)^{1/r} \Big\|_{p(\cdot)} &\leq \liminf_{N\to\infty} \Big\| \Big(\sum_{i=1}^N F_i^r \Big)^{1/r} \Big\|_{p(\cdot)} \\ &\leq C_{p(\cdot)} \liminf_{N\to\infty} \Big\| \Big(\sum_{i=1}^N G_i^r \Big)^{1/r} \Big\|_{p(\cdot)} \leq C_{p(\cdot)} \Big\| \Big(\sum_i G_i^r \Big)^{1/r} \Big\|_{p(\cdot)}. \end{aligned}$$

Now form a new family $\mathcal{F}_r$ that consists of the pairs of functions $(F_{r,N}, G_{r,N})$ defined by

$$F_{r,N}(x) = \left(\sum_{i=1}^N F_i(x)^r \right)^{1/r}, \quad G_{r,N}(x) = \left(\sum_{i=1}^N G_i(x)^r \right)^{1/r},$$

where $N > 1$ and $\{(F_i, G_i)\}_{i=1}^N \subset \mathcal{F}$. We first apply the classical extrapolation theorem: given (4.21), by Theorem 4.23 applied to the family $\mathcal{F}$ we have that for all $w \in A_r$,

$$\int_{\mathbb{R}^n} F(x)^r w(x)\,dx \leq C_0 \int_{\mathbb{R}^n} G(x)^r w(x)\,dx, \qquad (F,G) \in \mathcal{F}.$$

Hence, for any $w \in A_1 \subset A_r$ and $(F_{r,N}, G_{r,N}) \in \mathcal{F}_r$,

$$\begin{aligned} \int_{\mathbb{R}^n} F_{r,N}(x)^r w(x)\,dx &= \sum_{i=1}^N \int_{\mathbb{R}^n} F_i(x)^r w(x)\,dx \\ &\leq C_0 \sum_{i=1}^N \int_{\mathbb{R}^n} G(x)^r w(x)\,dx = C_0 \int_{\mathbb{R}^n} F_{r,N}(x)^r w(x)\,dx. \end{aligned}$$

Therefore, we can apply Theorem 4.24 to the family $\mathcal{F}_r$ and get

$$\|F_{r,N}\|_{p(\cdot)} \leq C_{p(\cdot)} \|G_{r,N}\|_{p(\cdot)}, \qquad (F_{r,N}, G_{r,N}) \in \mathcal{F}_r.$$

But this is (4.22) for all finite sums, which is what we needed to prove. □

4.6. Applications of extrapolation

In this section we apply Theorem 4.24 to prove $L^{p(\cdot)}$ estimates for three operators. We will concentrate on singular integrals, as these provide a good illustration of the technicalities involved in using extrapolation. Following this we will discuss more briefly the sharp maximal operator and the Riesz potentials.

Singular integrals. We begin with a definition.

DEFINITION 4.30. Given a tempered distribution K suppose that the Fourier transform $\widehat{K} \in L^\infty$ and on $L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, K coincides with a locally integral function that satisfies

$$|K(x)| \leq \frac{C_0}{|x|^n}$$

and

$$|K(x+h) - K(x)| \leq C_0 \frac{|h|}{|x|^{n+1}}, \quad |x| > 2|h| > 0.$$

Define the singular integral operator $Tf = K * f$, where f is a Schwartz function.

The basic properties of singular integrals are recorded in the following result.

THEOREM 4.31. *Given a singular integral with kernel K, if $f \in L^1(\mathbb{R}^n)$, then for all $t > 0$,*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > t\}| \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then

$$\|Tf\|_p \leq C\|f\|_p.$$

Furthermore, for $f \in L^p$, $1 \leq p < \infty$, Tf is defined pointwise a.e. by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x-y)f(y)\, dy = \lim_{\epsilon \to 0} \int_{\{|x-y|>\epsilon\}} K(x-y)f(y)\, dy. \tag{4.23}$$

The classical examples of singular integral operators are the Hilbert transform on the real line,

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\{|x-y|>\epsilon\}} \frac{f(y)}{x-y}\, dy,$$

and in higher dimensions the Riesz transforms R_j, $1 \leq j \leq n$,

$$R_j f(x) = \lim_{\epsilon \to 0} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\{|x-y|>\epsilon\}} \frac{x^j - y^j}{|x-y|^{n+1}} f(y)\, dy.$$

Theorem 4.31 holds for a more general class of operators, referred to as Calderón-Zygmund operators, that are not (singular) convolution operators. With appropriate assumptions, everything we say below extends to this larger class, but we restrict ourselves to singular integrals for simplicity. See [**33**, **46**] for more information.

We can extend Theorem 4.31 to the variable Lebesgue spaces. This result was proved by another method by Diening and Růžička [**32**]; the extrapolation proof is from [**20**].

THEOREM 4.32. *Let T be a singular integral operator with kernel K. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \infty$, if M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then*

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{4.24}$$

If $p_- = 1$ and M is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$, then for all $t > 0$,

$$\|t\chi_{\{x:|Tf(x)|>t\|}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{4.25}$$

To use extrapolation, we need a weighted norm inequality for singular integrals. For a proof of the following result, see [**33**, **44**, **46**].

THEOREM 4.33. *Given a singular integral T with kernel K, if $w \in A_1$, then for all $t > 0$,*

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| w(x)\, dx. \tag{4.26}$$

Further, if $1 < p < \infty$ and $w \in A_p$, then

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx. \tag{4.27}$$

REMARK 4.34. Part of the proof of Theorem 4.33 is showing that if $f \in L^p(w)$, then Tf is well-defined, since Definition 4.30 only defines Tf for f in the unweighted spaces $L^p(\mathbb{R}^n)$. However, if f is a bounded function of compact support it is in L^p for all $p < \infty$, and if $w \in A_p$, then $f \in L^p(w)$. Since such functions are dense in $L^p(w)$, Tf can be defined on the whole space by an approximation argument.

PROOF OF THEOREM 4.32. We will prove the strong-type inequality when $p_- > 1$; the weak-type inequality is proved in essentially the same way. Since the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, by Theorem 3.35 there exists $p_0 > 1$ such that M is bounded on $L^{p(\cdot)/p_0}(\mathbb{R}^n)$ and so on $L^{(p(\cdot)/p_0)'}(\mathbb{R}^n)$. Define the family $\mathcal{F}$ to be all pairs $(|Tf|, |f|)$ with f a bounded function of compact support. By Lemma 4.17, if $w \in A_1$, then $w \in A_{p_0}$. Hence, by Theorem 4.33, T is bounded on $L^{p_0}(w)$. In particular, for all such f, $\|Tf\|_{L^{p_0}(w)} < \infty$. Therefore, by Theorem 4.24,

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$$

for every bounded function of compact support such that the left-hand side is finite. But this is the case for every such f. Fix f and let B be a ball centered at the origin such that $\operatorname{supp}(f) \subset B$, and let $2B$ be the ball with the same center and twice the radius. Then for $x \in \mathbb{R}^n \setminus 2B$ and $y \in B$, we have that $|x-y| \geq |x| - |y| \geq \frac{1}{2}|x|$, and so

$$\begin{aligned} |Tf(x)| &= \left| \int_B K(x-y) f(y)\, dy \right| \\ &\leq C \int_B \frac{|f(y)|}{|x-y|^n}\, dy \leq C(n) \fint_{B_{|x|}(0)} |f(y)|\, dy \leq C(n) Mf(x). \end{aligned}$$

Since M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|Tf\|_{L^{p(\cdot)}(\mathbb{R}^n \setminus 2B)} \leq C\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)} < \infty,$$

and by Theorems 2.26 and 4.31,

$$\|Tf\|_{L^{p(\cdot)}(2B)} \leq (1 + |2B|)\|Tf\|_{L^{p_+}(2B)} \leq C\|f\|_{L^{p_+}(B)} < \infty.$$

This proves inequality (4.24) for bounded functions of compact support.

To complete the proof, fix $f \in L^{p(\cdot)}(\mathbb{R}^n)$; we need to define Tf and show that (4.24) holds. By Theorem 2.29 we can write $f = f_1 + f_2$, where $f_1 \in L^{p_-}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n)$ and $f_2 \in L^{p_+}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n)$. Since $p_+ < \infty$, by Theorem 4.31 we can define

$$Tf(x) = Tf_1(x) + Tf_2(x).$$

Again since $p_+ < \infty$, by Theorem 2.30 there exist sequences $\{f_i^j\}_{j=1}^\infty$, $i = 1, 2$, of bounded functions of compact support that converge to f_i in $L^{p_\pm}$ and in $L^{p(\cdot)}$. (This simultaneous convergences follows from the construction.) Therefore, since T is bounded on $L^{p_\pm}$, the sequence Tf_i^j converges to Tf_i in $L^{p_\pm}$ norm; by passing to a subsequence we may assume it also converges pointwise a.e. But then by Theorem 2.19,

$$\begin{aligned}
\|Tf\|_{p(\cdot)} &\leq \|Tf_1\|_{p(\cdot)} + \|Tf_2\|_{p(\cdot)} \\
&\leq \liminf_{j\to\infty} \big(\|Tf_1^j\|_{p(\cdot)} + \|Tf_2^j\|_{p(\cdot)}\big) \\
&\leq C \liminf_{j\to\infty} \big(\|f_1^j\|_{p(\cdot)} + \|f_2^j\|_{p(\cdot)}\big) \\
&= C(\|f_1\|_{p(\cdot)} + \|f_2\|_{p(\cdot)}) \\
&\leq C\|f\|_{p(\cdot)}.
\end{aligned}$$

The final inequality follows by the definition of the f_i. This completes the proof. □

By Corollary 4.29 we also have vector-valued inequalities.

THEOREM 4.35. *Let T be a singular integral operator with kernel K. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \infty$, if M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then T satisfies a vector-valued inequality on $L^{p(\cdot)}(\mathbb{R}^n)$: for each r, $1 < r < \infty$,*

$$\left\| \left(\sum_{i=1}^{\infty} |Tf_i|^r \right)^{1/r} \right\|_{p(\cdot)} \leq C \left\| \left(\sum_{i=1}^{\infty} |f_i|^r \right)^{1/r} \right\|_{p(\cdot)}. \tag{4.28}$$

PROOF. Fix r and define the family $\mathcal{F}$ to consist of all pairs of functions (F, G) such that

$$F(x) = \left(\sum_{i=1}^{N} |Tf_i(x)|^r \right)^{1/r}, \qquad G(x) = \left(\sum_{i=1}^{N} |f_i(x)|^r \right)^{1/r},$$

where N is any positive integer and each f_i is a bounded function of compact support. If we now argue as in the proof of Theorem 4.32, we get (4.28) for all pairs in $\mathcal{F}$. By Theorem 2.19, inequality (4.28) extends to any sequence of functions in $L^{p(\cdot)}(\mathbb{R}^n)$. □

We conclude with two results that suggest that our hypotheses are in fact necessary. For a proof of both, see [19].

THEOREM 4.36. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, if the Riesz transforms R_j, $1 \leq j \leq n$, are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then $1 < p_- \leq p_+ < \infty$.*

We conjecture that if the Riesz transforms are all bounded on $L^{p(\cdot)}$, then the maximal function is as well. We cannot prove this, but we have the following slightly weaker result.

THEOREM 4.37. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose that all the Riesz transforms satisfy the weak type inequality (4.25). Then $p(\cdot) \in K_0(\mathbb{R}^n)$.*

4.6.1. Sharp maximal function estimates. Given a locally integrable function f, define the sharp maximal function by

$$M^{\#}f(x) = \sup_Q \fint_Q |f(y) - f_Q|\, dy,$$

where $f_Q = \fint_Q f(y)\, dy$ and the supremum is taken over all cubes Q with sides parallel to the coordinate axes. The sharp maximal function was introduced by C. Fefferman and Stein [41] and can be used (for instance) to define BMO. Its importance comes from the fact that it controls the oscillation of functions by the

Hardy-Littlewood maximal. For example, if T is a singular integral operator and $0 < \delta < 1$, then

$$M_\delta^\#(Tf)(x) = M^\#(|Tf|^\delta)(x)^{1/\delta} \leq C_\delta Mf(x).$$

(This is due to Álvarez and Pérez [6]; see also [24].)

Given a function f, f and $M^\# f$ are comparable in L^p norm: for all p, $0 < p < \infty$,

$$\|f\|_p \leq \|Mf\|_p \leq C\|M^\# f\|_p;$$

indeed, this is true if the L^p norm is replaced by the $L^p(w)$ norm for any $w \in A_\infty$ (Again see [24, 33].) The same inequality also holds in the variable Lebesgue spaces.

Theorem 4.38. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < p_- \leq p_+ < \infty$, if the maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|Mf\|_{p(\cdot)} \leq C\|M^\# f\|_{p(\cdot)}.$$

This follows from extrapolation applied to the family $\mathcal{F}$ of pairs $(Mf, M^\# f)$, where f is runs over all bounded functions of compact support. Note that in this case the left-hand expressions $\|Mf\|_{L^p(w)}$ and $\|Mf\|_{p(\cdot)}$ are automatically finite.

A weaker version of Theorem 4.38, with $\|f\|_{p(\cdot)}$ on the left-hand side, was proved by Diening and Růžička [32] (see also [30]). The full result was proved via extrapolation in [20].

Theorem 4.38 is an example of a Coifman-Fefferman type inequality; many such inequalities can be proved using extrapolation: see [24] for details.

Riesz potentials. The Riesz potentials, sometimes referred to as fractional integrals, play an important role in PDEs and Sobolev space theory.

Definition 4.39. Given α, $0 < \alpha < n$, define the Riesz potential I_α, also referred to as the fractional integral operator with index α, to be the convolution operator

$$I_\alpha f(x) = \gamma(\alpha, n) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

where

$$\gamma(\alpha, n) = \frac{\Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)}{\pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}.$$

The constant $\gamma(\alpha, n)$ is chosen so that if f is a Schwartz function, then the Fourier transform of the Riesz potential is

$$\widehat{I_\alpha f}(\xi) = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi).$$

(See Stein [99].) The Riesz potentials are not bounded on $L^p(\mathbb{R}^n)$, but satisfy off-diagonal inequalities: $\|I_\alpha f\|_q \leq C\|f\|_p$ provided $1 < p < n/\alpha$, and q satisfies $1/p - 1/q = \alpha/n$.

The Riesz potentials are well defined on the variable Lebesgue spaces. If $p_+ < n/\alpha$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$, then $I_\alpha f(x)$ converges for every x. To see this, apply Theorem 2.29 and let $f = f_1 + f_2$, where $f_1 \in L^{p_-}(\mathbb{R}^n)$ and $f_2 \in L^{p_+}(\mathbb{R}^n)$. Then it is straightforward to show that $I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x)$ converges absolutely.

We can use extrapolation to prove norm inequalities for the Riesz potentials on the variable Lebesgue spaces. This result was proved independently by Diening [26] and the authors and Capone [12] using other means. Our proof using extrapolation is from [20].

THEOREM 4.40. *Fix α, $0 < \alpha < n$. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < n/\alpha$, define $q(\cdot)$ by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}.$$

If there exists $q_0 > \frac{n}{n-\alpha}$ such that M is bounded on $L^{(q(\cdot)/q_0)'}(\mathbb{R}^n)$, then

$$\|I_\alpha f\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{4.29}$$

If $p_- = 1$ and if M is bounded on $L^{(q(\cdot)/q_0)'}(\mathbb{R}^n)$ when $q_0 = \frac{n}{n-\alpha}$, then for every $t > 0$,

$$\|t\chi_{\{x\in\mathbb{R}^n : |I_\alpha f(x)|>t\}}\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}. \tag{4.30}$$

The proof of Theorem 4.40 is very similar to the proof of Theorem 4.32 and requires two ingredients: an extended version of Theorem 4.24 that yields off-diagonal inequalities and the appropriate weighted norm inequalities. We will state these results but omit the details of the proof itself.

The necessary extrapolation result was proved in [20] (see also [19]); the proof is a straightforward modification of the proof of Theorem 4.24.

THEOREM 4.41. *Suppose that for some p_0, q_0, $1 \leq p_0 \leq q_0$, the family $\mathcal{F}$ is such that for all $w \in A_1$,*

(4.31)
$$\left(\int_{\mathbb{R}^n} F(x)^{q_0} w(x)\,dx\right)^{1/q_0} \leq C_0 \left(\int_{\mathbb{R}^n} G(x)^{p_0} w(x)^{p_0/q_0}\,dx\right)^{1/p_0}, \qquad (F,G) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p_0 \leq p_- \leq p_+ < \frac{p_0 q_0}{q_0 - p_0}$, define $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}. \tag{4.32}$$

If the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(\mathbb{R}^n)$, then

$$\|F\|_{q(\cdot)} \leq C_{p(\cdot)}\|G\|_{p(\cdot)}, \qquad (F,G) \in \mathcal{F}. \tag{4.33}$$

The requisite weighted norm inequalities are due to Muckenhoupt and Wheeden [75]. Our version of the necessary inequalities is non-standard, but can be easily derived from their results.

DEFINITION 4.42. Given α, $0 < \alpha < n$, and p, $1 < p < n/\alpha$, define q by

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Then a weight w satisfies the $A_{p,q}$ condition (denoted by $w \in A_{p,q}$) if

$$[w]_{A_{p,q}} = \sup_Q \left(⨍_Q w(x)\,dx\right)\left(⨍_Q w(x)^{-p'/q}\,dx\right)^{q/p'} < \infty.$$

When $p = 1$, let $A_{1,q} = A_1$.

The connection between $A_{p,q}$ and the Muckenhoupt A_p classes is an immediate consequence of Definition 4.42.

LEMMA 4.43. *Given α, $0 < \alpha < n$, and p, $1 < p < n/\alpha$, a weight $w \in A_{p,q}$ if and only if $w \in A_r$, $r = 1 + q/p'$.*

THEOREM 4.44. *Given α, $0 < \alpha < n$, and p, $1 \le p < n/\alpha$, define q by $1/p - 1/q = \alpha/n$ and let $w \in A_{p,q}$. If $p = 1$, then for every $t > 0$,*

$$w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > t\}) \le C\left(\frac{1}{t}\int_{\mathbb{R}^n} |f(x)|w(x)^{1/q}\,dx\right)^q.$$

If $p > 1$, then

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q w(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^{p/q}\,dx\right)^{1/p}.$$

We conclude this section with an application of Theorem 4.41 to variable Sobolev spaces: we prove the Sobolev embedding theorem. When $p_- > 1$ this result follows from the strong type norm inequalities for the Riesz potential: see [12] for details and references. When $p_- = 1$ the proof is more difficult. It was proved by Harjulehto and Hästö [50] for bounded domains and extended to all of $\mathbb{R}^n$ by Hästö [52]. The proof we give here is from [24]; see also [19].

The variable Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^n)$ is the set functions f such that f, $\nabla f \in L^{p(\cdot)}(\mathbb{R}^n)$, where ∇f is the distributional gradient. The norm is $\|f\|_{W^{1,p(\cdot)}} = \|f\|_{p(\cdot)} + \|\nabla f\|_{p(\cdot)}$. If $p_+ < n$, define the Sobolev exponent $p^*(\cdot)$ by

$$\frac{1}{p(\cdot)} - \frac{1}{p^*(\cdot)} = \frac{1}{n}.$$

THEOREM 4.45. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 \le p_- \le p_+ < n$ and $p(\cdot) \in LH(\mathbb{R}^n)$, then $W^{1,p(\cdot)}(\mathbb{R}^n) \subset L^{p^*(\cdot)}(\mathbb{R}^n)$; in fact,*

$$\|f\|_{p^*(\cdot)} \le C\|\nabla f\|_{p(\cdot)}.$$

REMARK 4.46. They hypothesis that $p(\cdot)$ is log-Hölder continuous can be weakened; see [19]. For simplicity we consider this simpler case here.

To prove Theorem 4.45 using extrapolation we need the corresponding weighted norm inequality. This estimate was implicit in [42] and its proof is based on an argument due to Long and Nie [72] which in turn uses an idea from Maz'ja [73].

LEMMA 4.47. *For all p, $1 \le p < n$, $w \in A_1$, and $f \in C_c^\infty$,*

$$\left(\int_{\mathbb{R}^n} |f(x)|^{p^*} w(x)\,dx\right)^{1/p^*} \le C\left(\int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)^{p/p_*}\,dx\right)^{1/p}.$$

PROOF. Fix $f \in C_c^\infty$. For each $j \in \mathbb{Z}$, define

$$\Omega_j = \{x \in \mathbb{R}^n : 2^j < |f(x)| \le 2^{j+1}\},$$

and the function f_j by

$$f_j(x) = \begin{cases} |f(x)| - 2^j & x \in \Omega_j, \\ 2^j & x \in \Omega_i,\ i > j, \\ 0 & \text{otherwise.} \end{cases}$$

It follows immediately that $|\nabla f_j(x)| = |\nabla f(x)|\chi_{\Omega_j}$. Further, by a standard inequality (see [100]) we have that if $x \in \Omega_j$, then

$$c_n I_1(|\nabla f_{j-1}|)(x) \ge |f_{j-1}(x)| \ge 2^{j-1}, \tag{4.34}$$

where I_1 is the Riesz potential. Since $w \in A_1 \subset A_{1+p^*/p'}$, $w^{1/p^*} \in A_{p,p^*}$. Therefore, the Riesz potential satisfies the weak-type inequality in Theorem 4.44, so we can estimate as follows:

$$\int_{\mathbb{R}^n} |f(x)|^{p^*} w(x)\,dx = \sum_j \int_{\Omega_j} |f(x)|^{p^*} w(x)\,dx$$

$$\begin{aligned}
&\leq \sum_j \int_{\Omega_j} 2^{(j+1)p^*} w(x)\,dx \\
&= 4^{p^*} c_n^{p^*} \sum_j \int_{\Omega_j} \left[c_n^{-1} 2^{j-1}\right]^{p^*} w(x)\,dx \\
d &\leq C \sum_j \int_{\{x\in\mathbb{R}^n : I_1(|\nabla f_{j-1}|)(x) > c_n^{-1}2^{j-1}\}} \left[c_n^{-1} 2^{j-1}\right]^{p^*} w(x)\,dx \\
&\leq C \sum_j \left(\int_{\mathbb{R}^n} |\nabla f_{j-1}(x)|^p w(x)^{p/p_*}\,dx \right)^{p^*/p} \\
&\leq C \left(\sum_j \int_{\Omega_{j-1}} |\nabla f(x)|^p w(x)^{p/p_*}\,dx \right)^{p^*/p} \\
&\leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p w(x)^{p/p_*}\,dx \right)^{p^*/p}. \qquad \square
\end{aligned}$$

Proof of Theorem 4.45. Let $p_0 = 1$ and $q_0 = 1^* = n/(n-1)$. Then $p_+ < n = p_0q_0/(q_0-p_0)$, and $1/p(x) - 1/p^*(x) = 1/p_0 - 1/q_0$. Define the family $\mathcal{F}$ to consist of the pairs $(|f|, |\nabla f|)$, where $f \in C_c^\infty$. Since $p(\cdot) \in LH(\mathbb{R}^n)$ we have that the maximal operator satisfies the necessary norm inequalities. Therefore, by Theorem 4.41, for all $f \in C_c^\infty$, $\|f\|_{p^*(\cdot)} \leq C\|\nabla f\|_{p(\cdot)}$. (Note that the left-hand side is automatically finite.)

Now fix $f \in W^{1,p(\cdot)}$. Since $p(\cdot) \in LH(\mathbb{R}^n)$, C_c^∞ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ (see [17, 19]), so there exists a sequence $\{f_k\} \subset C_c^\infty$ such that $f_k \to f$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$. By Theorem 2.22, if we pass to a subsequence, we may assume that $f_k \to f$ pointwise a.e. Hence, by Theorem 2.19 and the above estimate,

$$\|f\|_{p^*(\cdot)} \leq \liminf_{k\to\infty} \|f_k\|_{p^*(\cdot)} \leq C \liminf_{k\to\infty} \|\nabla f_k\|_{p(\cdot)} \leq C\|\nabla f\|_{p(\cdot)}.$$

$\square$

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