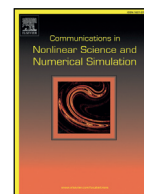




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Research paper

Hopf-Zero singularities truly unfold chaos

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ABSTRACT

We provide conditions to guarantee the occurrence of Shilnikov bifurcations in analytic unfoldings of some Hopf-Zero singularities through a beyond all order phenomenon: the exponentially small breakdown of invariant manifolds which coincide at any order of the normal form procedure. The conditions are computable and satisfied by generic singularities and generic unfoldings.

The existence of Shilnikov bifurcations in the C^r case was already argued by Guckenheimer in the 80's. About the same time, endowing the space of C^∞ unfoldings with a convenient topology, persistence and density of the Shilnikov phenomenon was proved by Broer and Vegter in 1984. However, since the proof involves the use of flat perturbations, this approach is not valid in the analytic context. What is more, none of the mentioned approaches provides a computable criteria to decide whether a given unfolding exhibits Shilnikov bifurcations or not.

Many people appeals to the appearance of Hopf-Zero singularities to explain the emergence of chaos in a huge number of applications. However, no one can refer to a specific theorem establishing the conditions that a given unfolding should satisfy to ensure that chaotic dynamics are exhibited. We fill this gap by providing an ultimate result about the appearance of Shilnikov bifurcations in analytic unfoldings of a certain class of Hopf-Zero singularities. These conditions are computable and satisfied by generic families. One of these conditions depends on the full jet of the singularity and comes from a beyond all order phenomenon. It can be related with Stokes constants. The other conditions only depend on the 2-jet of the family.

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1. Introduction

Results guaranteeing the existence of chaotic dynamics are hardly available in the literature. Although there is a plenty of examples based on numerical evidences, analytical proofs are rare. It is only recently that simple criteria are available for certain sets of dynamical systems as, for instance, unfoldings of singularities of vector fields; the general framework of this paper.

The route through dynamical complexity starts with Poincaré. In his seminal essay [33], he discovered that homoclinic intersections between the invariant manifolds of a hyperbolic fixed point were sources of very complicated behaviours. Later [7], Birkhoff showed that, for planar diffeomorphisms, near a transverse homoclinic intersection there exists an extremely intricate set of periodic orbits. By the mid 60's, Smale [38] conceived his celebrated horseshoe to explain, via

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conjugations to Bernoulli shifts, the Birkhoff result and also additional features of the complicated dynamics arising near a homoclinic intersection. Mora and Viana [31] proved the appearance of strange attractors when tangent homoclinic intersections were unfolded in families of planar diffeomorphisms. These attractors are like those in [6] for the Hénon family, that is, they are non hyperbolic and persistent in the sense of measure.

On the other hand, Shilnikov [36] proved the counterpart of the Birkhoff results in the context of smooth vector fields on \mathbb{R}^3 . Namely, he proved the existence of a countable set of periodic orbits in every neighbourhood of a homoclinic orbit to a hyperbolic equilibrium point with eigenvalues λ and $-\rho \pm \omega i$, with $0 < \rho < \lambda$. Because of the resemblance with the analogous statement for transverse homoclinic points in diffeomorphisms, one would expect to find the Smale horseshoes playing a key role. And so it was that Tresser [40] showed that in every neighbourhood of a Shilnikov homoclinic orbit, an infinite number of linked horseshoes can be defined in such a way that the dynamics is conjugated to a subshift of finite type on an infinite number of symbols. Moreover, when the homoclinic connection is generically unfolded, disappearance of horseshoes is accompanied by unfoldings of homoclinic tangencies to periodic orbits, leading to persistent non hyperbolic strange attractors like those in [31]. In [25,34,35], it was proven that infinitely many of these attractors can coexist for non generic families of vector fields. For an extensive study of the phenomena accompanying homoclinic bifurcations see [8,32]. Shilnikov homoclinic orbits are the simplest global configurations which, as just argued, can explain the existence of chaotic dynamics in families of vector fields, but, unfortunately, their existence is again not easy to prove in a given system.

Finally, in this searching of friendly criteria, we come into the world of singularities. They are, needless to say, manageable objects, much more manageable than global structures such as homoclinic intersections. It has been proven that Shilnikov saddle-focus homoclinic orbits appear in generic smooth unfoldings of certain singularities. Indeed, in [26] it was proven that these configurations were unfolded in generic unfoldings of three-dimensional nilpotent singularities of codimension four. Degeneracy was reduced in [27], where Shilnikov homoclinic bifurcations were proven to exist in any generic unfolding of a three-dimensional nilpotent singularity of codimension three. Therefore, suspended Hénon like strange attractors, appear in generic unfoldings of such singularities. See [5] for additional technical details and [18,19] for complementary results regarding the rich unfolding of the three-dimensional nilpotent singularity. In higher dimensions there also exists singularities of low codimension which can play the role of organizing centers of chaotic dynamics. For instance, [16] provides numerical evidences of the existence of strange attractors in the unfolding of the codimension-three Hopf-Bogdanov-Takens bifurcation. This bifurcation has also been studied in the context of Hamiltonian systems (see [28]).

One of the main motivations for obtaining simple criteria for the existence of strange attractors in given families of vector fields is their applicability. Results in [27] have been successfully used to prove the existence of chaotic dynamics in a model of coupled oscillators [17], in a general class of delay differential equations [13] and also in a three-dimensional predator-prey model [15].

Once it has been proven that chaos is unfolded by low codimension singularities, a natural question arises: which is the lowest codimension level where there exist singularities unfolding strange attractors or, in other words, what is the simplest local bifurcation displaying strange attractors. Obviously, one is not such lowest codimension because the only codimension one bifurcations are the saddle-node bifurcation of equilibrium points and the Hopf bifurcation, none of them including chaos. So, if strange attractors are unfolded from codimension three singularities, the question is: do there exist codimension-two local bifurcations unfolding generically strange attractors?

In this paper we consider Hopf-Zero (HZ in the sequel) singularities, that is, three-dimensional vector fields X^* such that $X^*(0, 0, 0) = 0$ and $DX^*(0, 0, 0)$ has eigenvalues $\pm i\alpha^*$ and 0, with $\alpha^* > 0$. In fact, without loss of generality, we can assume that:

$$DX^*(0, 0, 0) = \begin{pmatrix} 0 & \alpha^* & 0 \\ -\alpha^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

Classification of HZ singularities was first done by Takens [39]. Up to an analytic local change of coordinates the 2-jet of a HZ singularity becomes:

$$\begin{cases} x' = y - axz \\ y' = -x - ayz \\ z' = cz^2 + b(x^2 + y^2). \end{cases} \quad (2)$$

At the 1-jet level we need to impose two degeneracy condition which, joined to the open conditions $abc \neq 0$, define a stratum of codimension two in the space of germs of singularities of vector fields on \mathbb{R}^3 . Because $c \neq 0$, we can assume that this coefficient has been normalized by means of an appropriate scaling of coordinates. Takens proved that there are six topological types, see Fig. 1, but we are only interested in one of them, that type characterized by the conditions

$$a > 0, \quad b > 0. \quad (3)$$

HZ singularities satisfying (3) are denoted by HZ^* in the sequel.

Generic unfoldings of HZ singularities of codimension two were first studied by Guckenheimer [23] and Gavrilov [22]. The reader can find the bifurcation diagrams of suitable truncated normal forms for each of the six topological types in [24,29].

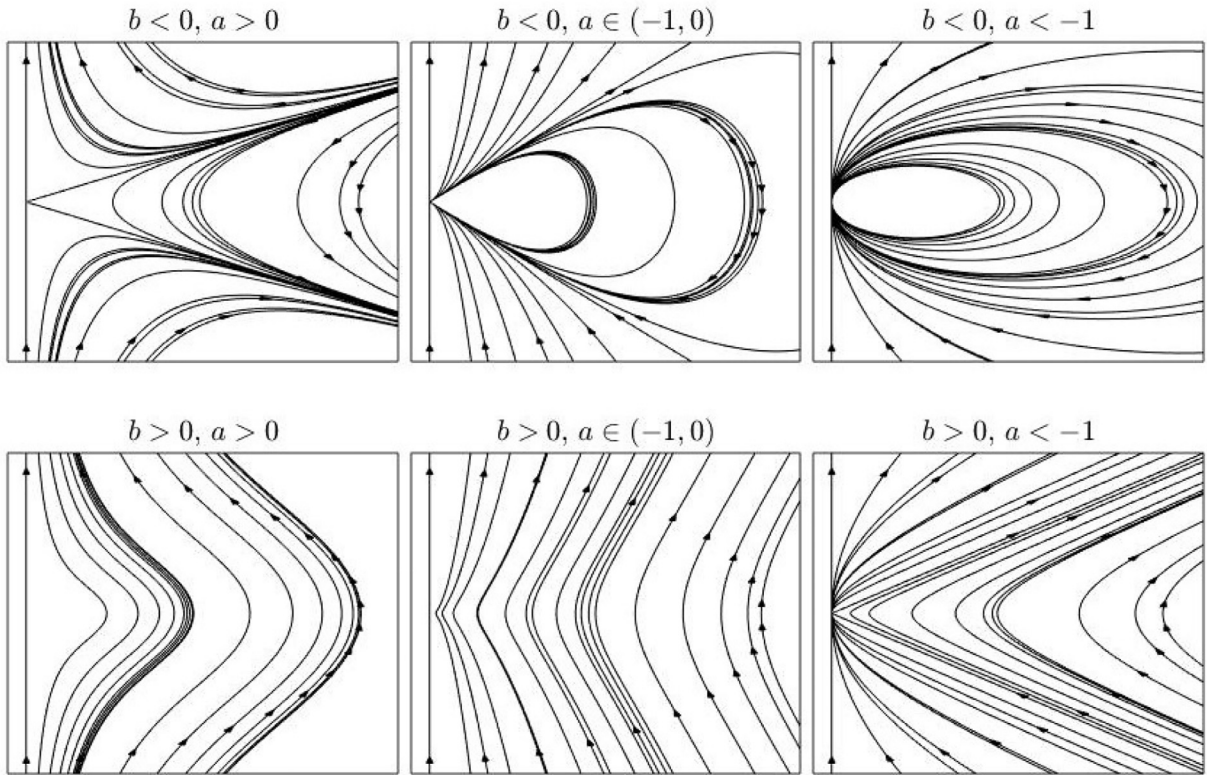


Fig. 1. Different topological types of Hopf-Zero singularities depending on a and b .

Truncating at second order, any generic unfolding is written as:

$$\begin{cases} x' = y + vx - axz \\ y' = -x + vy - ayz \\ z' = -\mu + z^2 + b(x^2 + y^2) \end{cases} \tag{4}$$

being μ, v the parameters of the unfolding, or, in cylindrical coordinates, as

$$\begin{cases} r' = vr - arz \\ z' = -\mu + z^2 + br^2 \\ \theta' = -1. \end{cases} \tag{5}$$

The characterization of HZ^* singularities by means of their normal form does not allow to detect if a given vector field belongs to HZ^* without performing changes of variables to reduce it to its normal form. For that reason, in Lemma 2.1, we prove that the intrinsic conditions:

$$\left[\partial_{z^2}^2 \pi^z X^* \cdot (\partial_{x^2}^2 \pi^z X^* + \partial_{y^2}^2 \pi^z X^*) \right] (\mathbf{0}) > 0, \quad \left[\partial_{z^2}^2 \pi^z X^* \cdot (\partial_{x,z}^2 \pi^x X^* + \partial_{y,z}^2 \pi^y X^*) \right] (\mathbf{0}) < 0, \tag{6}$$

where (π^x, π^y, π^z) denote the (x, y, z) components of the vector field X^* , define the vector fields $X^* \in HZ^*$ in terms of the 2-jet of X^* . Furthermore, we prove that any unfolding $X_{\mu,v}$ satisfying the generic condition:

$$\left[\left(\partial_{x,v}^2 \pi^x X_{0,0} + \partial_{y,v}^2 \pi^y X_{0,0} \right) \partial_{\mu} \pi^z X_{0,0} - \left(\partial_{x,\mu}^2 \pi^x X_{0,0} + \partial_{y,\mu}^2 \pi^y X_{0,0} \right) \partial_v \pi^z X_{0,0} \right] (\mathbf{0}) \neq 0 \tag{7}$$

has (4) as truncated normal form of order two.

When $v = 0$, family (5) has a first integral

$$H(r, z) = r^{\frac{2}{a}} \left(-\mu + z^2 + \frac{b}{1+a} r^2 \right).$$

Regarding the case of HZ^* singularities, it easily follows from the existence of H that, when $\mu > 0$, the two-dimensional invariant manifolds of the equilibrium points $p_{\pm} = (\pm\sqrt{\mu}, 0, 0)$ form an invariant globe and, moreover, the branches of the one-dimensional invariant manifolds contained inside the globe are also coincident (see Fig. 2).

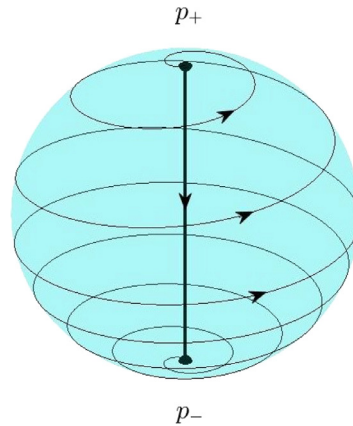


Fig. 2. Sketch of the invariant globe in the phase portrait of (4) when $\nu = 0$. There are two saddle-focus points p_+ and p_- such that $\dim(W^u(p_+)) = \dim(W^s(p_-)) = 1$ and $\dim(W^s(p_+)) = \dim(W^u(p_-)) = 2$. One of the branches of $W^u(p_+) \setminus \{p_+\}$ coincides with one of the branches of $W^s(p_-) \setminus \{p_-\}$. Moreover $W^u(p_-) \setminus \{p_-\} = W^s(p_+) \setminus \{p_+\}$.

Adding higher order terms, these invariant structures can be destroyed (even if the rotational symmetry is preserved). One can guess that the splitting of the invariant manifolds could lead to saddle-focus homoclinic connections. The complexity to decide if such invariant manifolds whether split or not, strongly depends on the regularity of the considered unfoldings, being the analytic case (the one considered in this paper) the more intricate one.

Simple geometrical arguments were supplied in [23,24] to support the possibility of the existence of Shilnikov homoclinic bifurcations in C^k unfoldings of HZ^* singularities. For the specific case of families $X + \varepsilon Y$, where X is the normal form (4) and $\varepsilon > 0$ is a perturbation parameter, Gaspard [21] proves the occurrence of Shilnikov homoclinic bifurcations near the codimension two point. The tricky point here is that the generic unfoldings of a HZ^* singularity can not be written in the form $X + \varepsilon Y$ (see also [14]).

The question is considered with a different perspective by Broer and Vegter in [12]. They prove that for any generic C^∞ unfolding of a HZ^* singularity there exists a C^∞ flat perturbation providing a family exhibiting Shilnikov homoclinic bifurcations. Moreover, abundance of unfoldings displaying the Shilnikov scenario is discussed, but using suitable topologies. Because of the “flat” nature of the techniques, the remarkable results in [12] neither include any usable criterion for the existence of Shilnikov homoclinic bifurcations in a given unfolding of HZ^* singularities nor can be applied to analytic families unfolding these singularities. Therefore the analytic case remains open since [12]. As it is made clear in that paper, Shilnikov phenomenon is beyond all orders. The results in [12] were extended to time reversible unfoldings in [30]. In this paper we end with the whole discussion by showing that under generic hypotheses, any analytic unfolding of a HZ^* singularity contains Shilnikov homoclinic orbits.

General unfoldings of the HZ^* singularity were considered in [20]. As argued there, introducing a scaling parameter $\varepsilon = \sqrt{\mu}$ and scaling variables and time, one obtains either a singular perturbation problem with a pure rotation when $\varepsilon = 0$ or a family with rotation speed tending to ∞ as $\varepsilon \rightarrow 0$. In any of the two approaches there is no clear limit for the invariant manifolds of the two equilibrium points corresponding to the poles of the invariant globe already discussed for (5). Nevertheless, one can apply the results in [9,10] to prove that, when the scaling parameter tends to 0, the invariant manifolds have a limit position, given by the invariant manifolds of the equilibrium points at the 2-jet level, at least when one considers restrictions to $z \geq 0$ or $z \leq 0$. Therefore, for any generic unfolding of the HZ^* singularity, splitting distance functions are well defined for both the one-dimensional and the two-dimensional invariant manifolds. Using conjectured formulas for the splitting functions and some extra conditions, existence of Shilnikov homoclinic bifurcation points is proven for general unfoldings.

On the other hand, in [1–3] explicit formulas for the splitting functions are obtained. The splitting function for the one-dimensional invariant manifolds was achieved in [1]. It follows that the distance between the one-dimensional invariant manifolds is exponentially small with respect to ε . Moreover the coefficient in front of the dominant term depends on the full jet of the singularity. The splitting function for the two-dimensional invariant manifolds was derived in [2,3]. The mean free terms in the asymptotic formula are exponentially small with respect to ε and, again, constants involved in the expression for those terms, which now depend on an angular variable, depend on the full jet of the singularity. Because constants involved in the dominant terms depend on the full jet of the singularity, their computation can only be done by means of numerical techniques (see [20] for several examples). In any case, such constants are the essential pieces to establish general criteria for the existence of Shilnikov homoclinic orbits. Namely, depending on the accuracy of the result one deals with, one needs to assume that either one or two of these constants do not vanish.

The point is that, putting together [1–3,20] as well as additional results, we are able to provide general results for the existence of Shilnikov homoclinic bifurcations in generic analytic unfoldings of the HZ^* singularity.

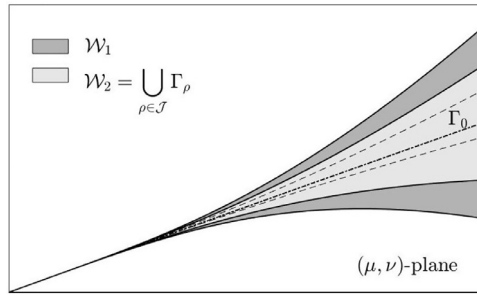


Fig. 3. Wedge-shaped domains $\mathcal{W}_2 \subset \mathcal{W}_1$ and the curve Γ_0 . \mathcal{W}_1 corresponds to the union of dark and light grey regions. The discontinuous lines in \mathcal{W}_2 are two curves Γ_ρ .

At this point it should be noticed that HZ singularities can be considered in two different contexts. At the 1-jet level any HZ singularity has divergence zero (note that the trace of (1) is zero). If one restricts to the set of volume-preserving systems, HZ singularities are of codimension one, that is, they occur generically in one parameter families of volume preserving vector fields. In such a case we should consider $\nu = 0$ in (2) and (5) and refer to unfoldings X_μ . We will require in this case, replacing (7) the generic condition:

$$\partial_\mu \pi^z X_0(0, 0, 0) \neq 0. \tag{8}$$

In the general context, as already explained, HZ singularities have codimension two, that is, they arise generically in two parameter families $X_{\mu,\nu}$ of vector fields. Note that according to [11], when working with families of vector fields with divergence zero, the change of coordinates to reduce to normal form may be chosen volume preserving.

Now we are able to provide a qualitative version of our main results. Later, after introducing some technical details in Section 2, we will give a more formal statement. We separate the results in the general case, Theorem 1.1 and the volume preserving setting, Theorem 1.2.

Theorem 1.1 (General case). *Let X^* be a HZ singularity satisfying the open conditions in (6), the additional open condition*

$$\left| (\partial_{x,z}^2 \pi^x X^* + \partial_{y,z}^2 \pi^y X^*)(\mathbf{0}) \right| < 2 \left| \partial_{z,z}^2 \pi^z X^*(\mathbf{0}) \right|, \tag{9}$$

and some generic conditions (that will be given explicitly in Theorem 2.7). Assume that $X_{\mu,\nu}$ is any analytic unfolding satisfying the generic condition (7).

Then, in the (μ, ν) plane, there exists an analytic curve $\Gamma_0 = \{(\mu, \nu) : \nu = \nu_0(\mu)\}$, with $\nu_0(0) = 0$, and domains $\mathcal{W}_2 \subset \mathcal{W}_1$ contained in a wedge-shaped neighbourhood of Γ_0 of a width that is exponentially small in $\sqrt{\mu}$ (see Fig. 3) such that:

1. There exists an immersed curve $\Lambda \subset \mathcal{W}_1$ (maybe with more than one component) such that the vector field $X_{\mu,\nu}$ has a Shilnikov homoclinic orbit, for $(\mu, \nu) \in \Lambda$.
2. For $(\mu, \nu) \in \mathcal{W}_2$, the vector field $X_{\mu,\nu}$ has, at least, two heteroclinic orbits, formed by the intersection between the two-dimensional invariant manifolds of the equilibrium points.
3. There exists an open neighbourhood of the origin $\mathcal{J} \subset \mathbb{R}^3$ such that $\mathcal{W}_2 = \bigcup_{\rho \in \mathcal{J}} \Gamma_\rho$, where for any $\rho \in \mathcal{J}$, $\Gamma_\rho = \{(\mu, \nu) : \nu = \nu_\rho(\mu)\}$ is a curve exponentially close to Γ_0 . In addition, for any $\rho \in \mathcal{J}$ each curve Γ_ρ possesses a sequence $(\mu_n, \nu_\rho(\mu_n)) \rightarrow \mathbf{0}$ as $n \rightarrow \infty$ such that each vector field $X_{\mu_n, \nu_\rho(\mu_n)}$ has a Shilnikov homoclinic orbit.

Theorem 1.2 (Volume preserving). *Let X^* be a HZ singularity satisfying the open conditions in (6) and some generic conditions (that will be given explicitly in Theorem 2.6).*

In the volume preserving case, any divergence free generic unfolding X_μ satisfying the generic condition (8) has the following properties:

1. For μ small enough, the vector field X_μ has, at least, two heteroclinic orbits, formed by the intersection between the two-dimensional invariant manifolds of the equilibrium points.
2. There exists a sequence of parameter values $\{\mu_n\}$ with $\mu_n \rightarrow 0$ as $n \rightarrow \infty$ such that the vector fields X_{μ_n} have a Shilnikov homoclinic orbit.

Remark 1.3. The generic conditions on the singularity X^* in Theorems 1.1 and 1.2 depend on the full jet of the singularity. In fact they involve that some Stokes constants have to be different from zero.

Remark 1.4. The additional open condition (9) is not necessary to prove the existence of homoclinic orbits to saddle-focus equilibrium points, but it appears in the main Theorem 1.1 to get the expansivity condition which is required to have Shilnikov homoclinic orbits. Namely, denoting the eigenvalues at the saddle-focus as $-\varrho \pm \omega i$ and λ , with $\varrho > 0$ and $\lambda > 0$, the expansivity condition is $\lambda > \varrho$ (see [36,37]). If (9) holds, the eigenvalues at p_+ satisfy such condition for ν and μ small enough. At p_- the expansivity condition is satisfied for the backward flow.

Remark 1.5. It should be noticed that the occurrence of chaotic dynamics is also expectable in the unfolding of certain codimension-two Hopf-Hopf singularities, that is, singularities with two pairs of pure imaginary eigenvalues without resonances. The classification was obtained in [39] and, together with the study of truncated unfoldings, it can be found in [24]. Similarly to the case of the Hopf-Zero singularities, the truncation of the normal form leads to planar reductions exhibiting, under appropriate assumptions, heteroclinic cycles involving equilibrium points and, bearing in mind the four-dimensional system, periodic orbits. The treatment required to get results of existence of homoclinic orbits is likely to be not far from the techniques described and used in this paper, where we deal with the Hopf-Zero case.

In Section 2, to understand the role of all the open conditions stated in Theorems 1.1 and 1.2, we explain the derivation of the normal form of the unfoldings of a HZ singularity up to second order. Using such normal form, after appropriate scalings, we will write our unfoldings in the form to be used in the rest of the paper. We will discuss the existence of equilibrium points, their invariant manifolds and their possible intersections. At the end of Section 2 we will give a more quantitative version of our main theorems also split into two results, Theorem 2.6 in the general case and Theorem 2.7 in the volume preserving case. Proofs are given in Section 3.

2. Preliminaries and a more quantitative result

This section is mainly devoted to recall some previous results about the dynamics of the unfoldings $X_{\mu,\nu}$ as well as to write a more quantitative version of Theorems 1.1 and 1.2. In addition, we also prove Lemma 2.1 which gives the correspondence between the intrinsic conditions on X^* and $X_{\mu,\nu}$ stated in Theorems 1.1 and 1.2 and the truncated normal form (4).

In the sequel, we will deal with both cases (the volume preserving and the general case) at the same time because the volume preserving case is contained in the general one by putting $\nu = 0$ and assuming that $\text{tr}DX_{\mu,0}(0,0,0) = 0$.

2.1. Normal form and previous results

Even if the proof of the result below (Lemma 2.1) is elementary following the procedure indicated for instance in [23,24,39], we will include it in Appendix A, in order to follow the relation between the original vector field and the corresponding normal form.

Lemma 2.1. *Let X^* be a HZ singularity satisfying the open condition (6). Assume the generic condition (7) on the unfolding $X_{\mu,\nu}$ in the general case and (8) in the volume preserving case.*

Then there exists a real analytic change of variables and parameters such that the unfolding $X_{\mu,\nu}$ is given by

$$\begin{cases} x' = y + \nu x - axz + f(x, y, z, \nu, \mu) \\ y' = -x + \nu y - ayz + g(x, y, z, \nu, \mu) \\ z' = -\mu + z^2 + b(x^2 + y^2) + h(x, y, z, \nu, \mu) \end{cases} \quad (10)$$

where $a, b > 0$, the functions f, g, h are real analytic and $\mathcal{O}(\|(x, y, z, \nu, \mu)\|^3)$. In addition, if X^* also satisfies (9), then $0 < a < 2$.

We also introduce the new parameters ε, σ as:

$$(\mu, \nu) = (\varepsilon^2, \varepsilon\sigma), \quad \varepsilon > 0$$

and the blow up

$$x = \varepsilon\bar{x}, \quad y = \varepsilon\bar{y}, \quad z = \varepsilon\bar{z}.$$

Then, System (10) becomes (dropping bars of the notation):

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon x(\sigma x - az) + y + \varepsilon^{-1}f(\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon^2, \varepsilon\sigma) \\ \frac{dy}{dt} &= -x + \varepsilon y(\sigma - az) + \varepsilon^{-1}g(\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon^2, \varepsilon\sigma) \\ \frac{dz}{dt} &= -\varepsilon + \varepsilon z^2 + \varepsilon b(x^2 + y^2) + \varepsilon^{-1}h(\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon^2, \varepsilon\sigma). \end{aligned} \quad (11)$$

Scaling time, $s = \varepsilon t$ we obtain the vector field below, that we will call, abusing notation, $X_{\varepsilon,\sigma}$ in the general case and X_ε in the volume preserving case ($\sigma = 0$):

$$\begin{aligned} \frac{dx}{ds} &= x(\sigma - az) + \frac{y}{\varepsilon} + \varepsilon^{-2}f(\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon^2, \varepsilon\sigma) \\ \frac{dy}{ds} &= -\frac{x}{\varepsilon} + y(\sigma - az) + \varepsilon^{-2}g(\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon^2, \varepsilon\sigma) \\ \frac{dz}{ds} &= -1 + z^2 + b(x^2 + y^2) + \varepsilon^{-2}h(\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon^2, \varepsilon\sigma). \end{aligned} \quad (12)$$

From now on we will work with the generic unfoldings already in the form (12).

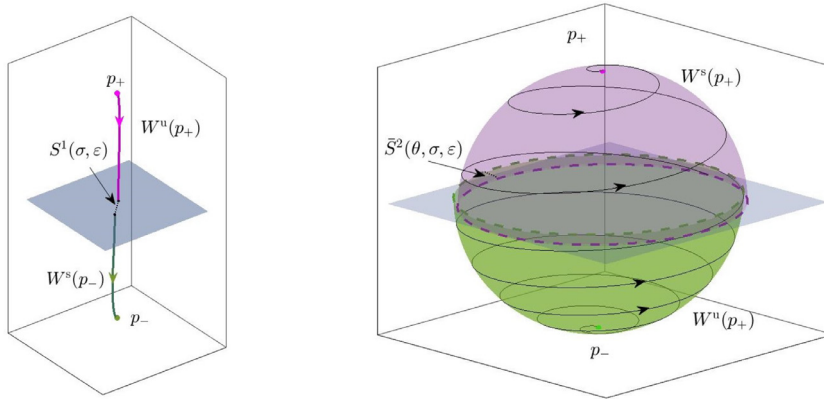


Fig. 4. On the left, the splitting of the one dimensional heteroclinic connection W^1 . On the right, the corresponding breakdown of the two dimensional heteroclinic connection.

Remark 2.2. Families (11) and (12) were also used in [1–3,20]. Note that for family (11), the limit when $\varepsilon \rightarrow 0$ is merely a rotation around the vertical axis. On the contrary, in the case of (12) there is no regular limit when $\varepsilon \rightarrow 0$ because the rotation speed tends to ∞ .

We will state and prove the quantitative version of Theorems 1.1 and 1.2 in terms of System (12). For that reason, the first result we use is the following lemma, whose proof can be found in [4], which assures the existence of saddle-focus equilibrium points:

Lemma 2.3. Consider System (12) with $b > 0$, $a > 0$ and $|\sigma| < a$. There exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the vector field has two equilibrium points p_{\pm} (depending on ε) of saddle-focus type such that p_+ (p_-) has a one-dimensional unstable (stable) manifold and a two-dimensional stable (unstable) one.

Observe that, when $f = g = h = 0$ in (12), the points p_{\pm} are $(0, 0, \pm 1)$ and they are connected by the heteroclinic orbit:

$$W_1 = \{x = y = 0, |z| < 1\} \tag{13}$$

which consists on a branch of the 1-dimensional unstable manifold of $p_+ = (0, 0, 1)$ that coincides with a branch of the 1-dimensional stable manifold of $p_- = (0, 0, -1)$.

However, one expects that for generic f, g, h , the one dimensional heteroclinic connection (13) breaks. Next theorem, which corresponds to Theorem 1 in [1] just adapting the notation, gives the distance $S^1(\sigma, \varepsilon)$ between the 1-dimensional invariant manifolds of p_+ and p_- of System (12), which is different from zero under generic conditions on the singularity X^* , see Fig. 4.

Theorem 2.4 [See [1]]. Consider System (12) with $a, b > 0$ and $|\sigma| < a$. Then there exist $\varepsilon_0 > 0$ and a real constant C^* , such that, for $0 < \varepsilon \leq \varepsilon_0$, the distance $S^1(\varepsilon, \sigma)$ between the one-dimensional stable manifold of p_- and the one-dimensional unstable manifold of p_+ when they meet the plane $z = 0$ is given by

$$S^1(\varepsilon, \sigma) = \varepsilon^{-1+a} e^{-\frac{\pi}{2\varepsilon}} e^{-\frac{\pi c_0}{2}} (C^* + \mathcal{O}(|\log \varepsilon|^{-1}))$$

with $c_0 = \lim_{z \rightarrow 0} z^{-3} h(0, 0, z, 0, 0)$.

The constant C^* depends on the full jet of the singularity X^* and is different from zero for generic singularities.

The study of the 2-dimensional invariant manifolds of the equilibrium points p_{\pm} of System (12) is more involved, see [2,3] for a detailed study of the relative position of these manifolds. We give some details below.

When $f = g = h = 0$ and $\sigma = 0$ one can see that the 2-dimensional manifolds coincide forming an ellipsoid given by:

$$z^2 + \frac{b}{a+1}(x^2 + y^2) = 1, \tag{14}$$

but in the general case ($\sigma \neq 0$) it is possible that the 2-dimensional unstable manifold of p_- and the 2-dimensional stable manifold of p_+ do not intersect. Indeed, when the parameter σ is not of order ε , the position of the 2-dimensional manifolds is already known; they do not intersect and the distance between them is of order σ . This fact is easily obtained by doing another step of the normal form procedure to System (12) and studying the position of these manifolds in the normal form of order three. Moreover, in this case, the existence of Shilnikov homoclinic bifurcations is not possible. Indeed, when the 2-dimensional invariant manifolds do not intersect, there exist either forward or backward trapping regions which prevent the existence of homoclinic connections (see details in [20]).

For this reason, from now on, we restrict our parameters and we will take $\sigma = \mathcal{O}(\varepsilon)$. The result that we will use in our case is already done in [3] (and also in [2]). In that work, symplectic polar coordinates were considered:

$$x = \sqrt{2r} \cos \theta, \quad y = \sqrt{2r} \sin \theta,$$

and it was proven that the 2-dimensional invariant manifolds of p_{\pm} can be parameterized by θ and $z = \tanh(au)$ as

$$r = r^{u,s}(u, \theta) = \frac{a + 1}{2b \cosh^2(au)} + r_1^{u,s}(u, \theta), \quad r_1^{u,s}(u, \theta) = \mathcal{O}(\varepsilon), \quad u \in [-T, T].$$

In particular, the intersections of the two dimensional invariant manifolds of p_{\pm} with the plane $z = 0$ are two closed curves $\mathbf{C}^{u,s}$ that can be parameterized as:

$$\mathbf{C}^{u,s} = \{(x, y, 0), x = \sqrt{2r^{u,s}(0, \theta)} \cos \theta, y = \sqrt{2r^{u,s}(0, \theta)} \sin \theta, \theta \in \mathbb{T}\}. \tag{15}$$

Let us call

$$\bar{S}^2(\theta, \varepsilon, \sigma) = r^u(0, \theta) - r^s(0, \theta), \tag{16}$$

the radial symplectic distance between the manifolds when they meet the plane $z = 0$, see Fig. 4. From Theorem 2.16 of Baldomá et al. [3] one has:

Theorem 2.5 [See [3]]. Consider the radial symplectic distance $\bar{S}^2(\theta, \varepsilon, \sigma)$ defined in (16). There exist $\varepsilon_0 > 0$, $\sigma_0 > 0$ and a complex constant $C_1^* + iC_2^*$, such that for $|\sigma| \leq \sigma_0 \varepsilon$ and $0 < \varepsilon \leq \varepsilon_0$, one has

$$\bar{S}^2(\theta, \varepsilon, \sigma) = \Upsilon^{[0]}(1 + \mathcal{O}(\varepsilon)) + \varepsilon^{-2-\frac{2}{a}} e^{-\frac{\pi}{2a\varepsilon}} [C_1^* \cos(\theta - a^{-1}L_0 \log \varepsilon) + C_2^* \sin(\theta - a^{-1}L_0 \log \varepsilon) + \mathcal{O}(|\log \varepsilon|^{-1})]$$

where L_0 is a constant depending on the terms of degree three of the singularity X^* (see Remark 5.17 in [2] for details) and

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\varepsilon, \sigma) = \sigma I + \varepsilon J + \mathcal{O}_2(\varepsilon, \sigma)$$

with $I \neq 0$ and J computable constants (for details, see formulas (90) and (91) in [2]).

In addition, there exists a curve

$$\Gamma_0^* = \{(\varepsilon, \sigma), \sigma = \sigma_0^*(\varepsilon) = -\frac{J}{I}\varepsilon + \mathcal{O}(\varepsilon^2)\}$$

such that for all $0 \leq \varepsilon \leq \varepsilon_0$ one has:

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\varepsilon, \sigma_0^*(\varepsilon)) = 0.$$

Moreover, given any constants c_1, c_2 and $c_3 > 0$, there exists a curve

$$\Gamma_{\rho}^* = \{(\varepsilon, \sigma), \sigma = \sigma_{\rho}^*(\varepsilon) = \sigma_0^*(\varepsilon) + \mathcal{O}(\varepsilon^{c_2} e^{-\frac{c_3 \pi}{2a\varepsilon}})\}$$

with $\rho = (c_1, c_2, c_3)$, such that for all $0 \leq \varepsilon \leq \varepsilon_0$ one has:

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\varepsilon, \sigma_{\rho}^*(\varepsilon)) = c_1 \varepsilon^{c_2} e^{-\frac{c_3 \pi}{2a\varepsilon}}.$$

In the volume preserving case $\Upsilon^{[0]} = 0$.

In addition, the constant $C_1^* + iC_2^* \neq 0$ for generic singularities X^* and depends on the full jet of X^* .

2.2. More quantitative results

We emphasize that, since the unfoldings are analytic, the distance between the invariant manifolds is exponentially small for adequate values of the parameters. This fact is intrinsic to the analytic case and lies in the field of singular perturbation theory. Conversely, when we are in the finitely many differentiable case, the classical perturbation theory can be applied to compute the distances S^1 and \bar{S}^2 . They both will be, generically, $\mathcal{O}(\varepsilon^k)$ for some $k > 0$, for values of σ in an adequate curve.

Using the notation we have already introduced, we can write a the announced quantitative version of Theorem 1.1 for the general case:

Theorem 2.6 [General case]. Consider System (12) and assume the open conditions $0 < a < 2$ and $b > 0$ and the generic condition $C^* \neq 0$, where C^* is given in Theorem 2.4, on the initial singularity X^* .

- In the (ε, σ) plane, with ε small enough, there exists an immerse curve Λ^* , which lies in a wedge-shaped neighbourhood \mathcal{W}_1^* of the curve Γ_0^* given in Theorem 2.5 of a width that is at most $2c_1 \varepsilon^{-2-\frac{2}{a}} e^{-\frac{\pi}{2a\varepsilon}}$, with either $c_1 = \sqrt{(C_1^*)^2 + (C_2^*)^2}$ if $C_1^* + iC_2^* \neq 0$ or $c_1 > 0$ if $C_1^* + iC_2^* = 0$, such that for $(\varepsilon, \sigma) \in \Lambda^*$, any analytic unfolding $X_{\varepsilon, \sigma}$ has a Shilnikov homoclinic orbit to the equilibrium point p_+ .
- Assume moreover the generic condition on the singularity X^* : $C_1^* \neq 0$ or $C_2^* \neq 0$. Let $0 < \kappa < 1$ be any constant. Take any curve Γ_{ρ}^* , $\rho = (c_1, c_2, c_3)$ as given in Theorem 2.5 with

$$|c_1| \leq \kappa \sqrt{(C_1^*)^2 + (C_2^*)^2}, \quad c_2 \geq -2 - \frac{2}{a}, \quad c_3 \geq 1$$

and ε small enough. Let $\mathcal{W}_2^* = \bigcup_{\rho} \Gamma_{\rho}^*$. Then

1. For any $(\varepsilon, \sigma) \in \mathcal{W}_2^*$ there are, at least, two heteroclinic orbits from p_- to p_+ .
2. For any curve $\Gamma_\rho^* \subset \mathcal{W}_2^*$, there exists a sequence of parameter values $\{\varepsilon_n\}$, with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $(\varepsilon_n, \sigma_\rho^*(\varepsilon_n)) \in \mathcal{W}_2^*$, such that the vector field $X_{\varepsilon_n, \sigma_\rho^*(\varepsilon_n)}$ has a Shilnikov homoclinic orbit to the equilibrium point p_+ .

The precise results in the volume preserving case are collected in the following result:

Theorem 2.7 [Volume preserving case]. Consider System (12) and assume that it is volume preserving. In particular, $a = 1$ and $\nu = 0$. Assume moreover the open condition $b > 0$ and the generic conditions $C^* \neq 0$, $(C_1^*)^2 + (C_2^*)^2 \neq 0$ on the initial singularity X^* .

1. Then, for ε small enough there are, at least, two heteroclinic orbits from p_- to p_+ .
2. There exists a sequence of parameter values $\{\varepsilon_n\}$, with $\varepsilon_n > 0$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that the vector field X_{ε_n} has a Shilnikov homoclinic orbit to the equilibrium point p_+ .

Observe that Theorems 1.1 and 1.2 are just a corollary of Theorems 2.6 and 2.7 undoing the changes of variables from System (12) to System (10).

Remark 2.8. Recall that $0 < a < 2$ is equivalent to (9). As we pointed out in Remark 1.4, to prove the existence of homoclinic orbits to saddle-focus equilibrium points we only need to assume $a > 0$. However, the condition $0 < a < 2$ is necessary to guarantee that such homoclinic orbits are of Shilnikov type.

3. Proof of Theorems 2.6 and 2.7

The proof of these two theorems will have some common setting: to prove the existence of homoclinic orbits to the north point p_+ we need to control its global 1-dimensional unstable manifold $W^u(p_+)$ and more concretely its intersections with the plane $\{z = 0\}$.

Theorem 2.4 gives us information about the “first” intersection of $W^u(p_+)$ with the plane $\{z = 0\}$ and the distance $S^1(\varepsilon, \sigma)$, between $W^u(p_+)$ and the 1-dimensional stable manifold of the south point p_- ; $W^s(p_-)$ (see Fig. 4). Using the precise results in Theorems 2.4 and 2.5 we will prove that $W^u(p_+)$ intersects again the plane $z = 0$ in a point Q_0 which is “close enough” to the 2-dimensional manifold $W^u(p_-)$. The goal is to get a sharp bound for the distance between Q_0 and the curve $C^u = W^u(p_-) \cap \{z = 0\}$ (see Fig. 5).

We rewrite the curve $C^{u,s}$ (in (15)) in polar (non symplectic) coordinates as:

$$C^{u,s} = \{(x, y, 0), x = \bar{r}^{u,s}(\theta, \varepsilon, \sigma) \cos \theta, y = \bar{r}^{u,s}(\theta, \varepsilon, \sigma) \sin \theta, \theta \in \mathbb{T}\}. \tag{17}$$

Here we have written the dependence on the parameters ε, σ and we will do this whenever it is convenient.

More explicitly, we will prove the following key result:

Proposition 3.1. Consider System (12). Then, the unstable invariant manifold, $W^u(p_+)$, crosses the plane $z = 0$ at least twice.

Let us call q_0 and Q_0 the first and the second intersection points, $\varphi(t; q)$ the flow and $\tau > 0$ such that $Q_0 = \varphi(\tau; q_0)$. Consider the C^∞ function $\vartheta(t)$ giving the argument of $\varphi(t; q_0)$. We denote by $\theta_0(\varepsilon, \sigma) = \vartheta(\tau)$ and by $r_0(\varepsilon, \sigma)$ the radius of Q_0 . Then we have:

1. The radial distance between Q_0 and the unstable curve C^u given in (17) is bounded from above by

$$0 < r_0(\varepsilon, \sigma) - \bar{r}^u(\theta_0(\varepsilon, \sigma), \varepsilon, \sigma) \leq \bar{C}_1 \varepsilon^{2-\frac{2}{a}} e^{-\frac{\tau}{a\varepsilon}},$$

for some constant $\bar{C}_1 > 0$.

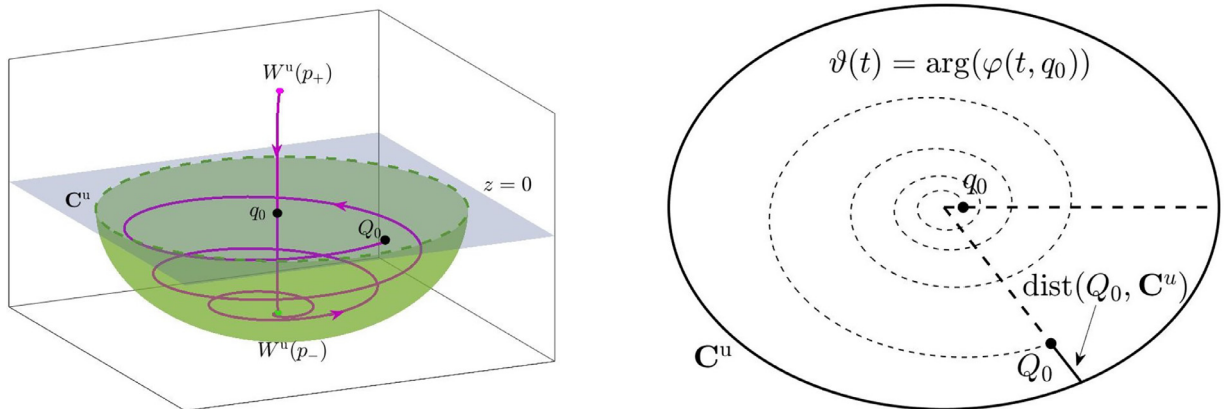


Fig. 5. The points q_0 and Q_0 correspond to the first and second intersections of $W^u(p_+)$ with $z = 0$. The curve C^u is given by the first intersection of $W^u(p_-)$ with $z = 0$. We need to get appropriate bounds for the angle $\theta_0 = \vartheta(t)$ and for the distance between Q_0 and C^u . On the left the three dimensional figure, on the right the plane $\{z = 0\}$.

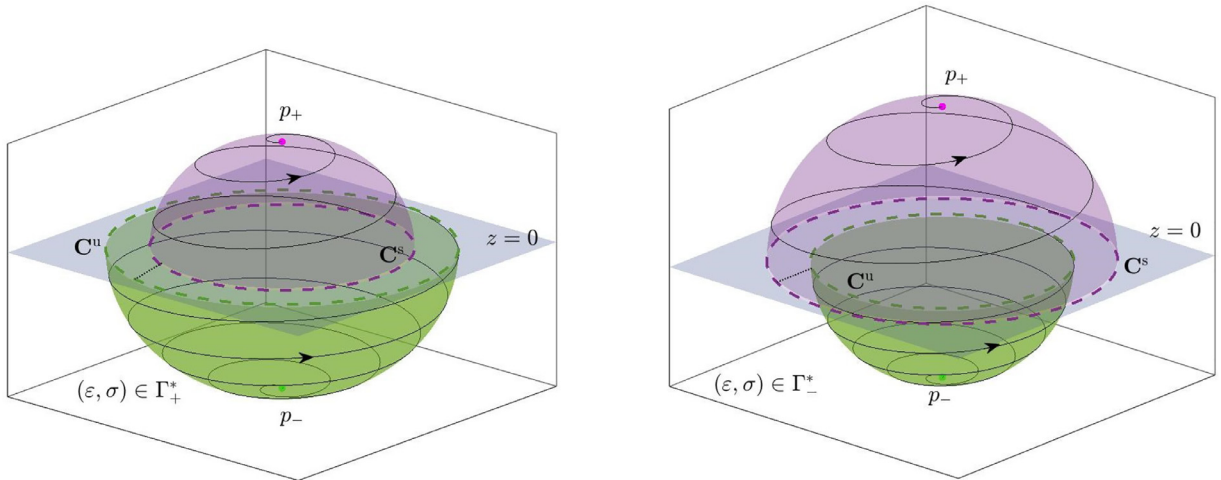


Fig. 6. On the left, the parameters $(\varepsilon, \sigma) \in \Gamma_+^*$ and C^u is outside of C^s . On the right, the parameters belongs to Γ_-^* and C^u is inside of C^s . In both cases there are no intersections.

2. There exists a constant $d > 0$ such that $\theta_0(\varepsilon, \sigma) \geq \frac{d}{\varepsilon^2}$.

The arguments to check the existence of Q_0 and to get the estimation of the distance complete the results in [20], where the required quantitative estimates were not proven. The proof of Proposition 3.1 is deferred to Section 3.2.

3.1. Proof of the existence of heteroclinic and homoclinic connections

In order to prove that System (12) undergoes a Shilnikov bifurcation (first item in Theorem 2.6) we are going to use a classical argument, similar to the one used in [20] (see also [12]) to obtain the curve Λ^* . But since we are in the analytic context, we have to use the accurate information we have proven about the splitting of the stable and unstable manifolds of p_{\pm} and about the distance between Q_0 and the unstable manifold of p_- .

Denote by Γ_{\pm}^* the curves Γ_{ρ}^* given in Theorem 2.5, with $\rho_{\pm} = (\pm c_1, c_2, c_3)$ corresponding to the constants: $c_2 = -2 - \frac{2}{a}$, $c_3 = 1$ and $c_1 = 2\sqrt{(C_1^*)^2 + (C_2^*)^2}$ when $C_1^* + iC_2^* \neq 0$ and $c_1 > 0$ if $C_1^* + iC_2^* = 0$.

When the parameters are in these curves, by Theorem 2.5, the two dimensional stable and unstable manifolds do not intersect. Moreover, we know that when $(\varepsilon, \sigma) \in \Gamma_+^*$ the curve C^u is outside C^s and the contrary in Γ_-^* , and the distance between these curves is of order $\mathcal{O}(c_1 \varepsilon^{-2-\frac{2}{a}} e^{-\frac{\pi}{2a\varepsilon}})$, see Fig. 6.

We introduce the radial distance between the 2-dimensional invariant manifolds denoted by $S^2(\theta, \varepsilon, \sigma(\varepsilon))$. It is related with the symplectic radial distance by (see (15) and (16)):

$$S^2(\theta, \varepsilon, \sigma(\varepsilon)) = \sqrt{\frac{b}{a+1}} \bar{S}^2(\theta, \varepsilon, \sigma(\varepsilon))(1 + \mathcal{O}(\varepsilon)).$$

Fix $0 < \varepsilon < \varepsilon_0$ and consider $\sigma \in [\sigma_{\rho_-}^*(\varepsilon), \sigma_{\rho_+}^*(\varepsilon)]$. By Proposition 3.1 the radial distance between Q_0 and C^u is less than the radial distance S^2 between C^u and C^s . Therefore we can ensure that when $\sigma = \sigma_{\rho_-}^*(\varepsilon)$ the point Q_0 is inside C^s , and for $\sigma = \sigma_{\rho_+}^*(\varepsilon)$ the point Q_0 is outside C^s , therefore, we know that it exists at least one value $\sigma = \sigma^s(\varepsilon) \in [\sigma_{\rho_-}^*(\varepsilon), \sigma_{\rho_+}^*(\varepsilon)]$ where $Q_0 \in C^s$ and, consequently, $W^u(p_+) \subset W^s(p_+)$ giving rise to a homoclinic orbit to p_+ .

Remark 3.2. The previous argumentation resembles the proof of the existence of Shilnikov homoclinic orbits argued in [20] based in the notion of trapping region. However, using our results we have a sharp accurate domain on the parameters where the Shilnikov homoclinic orbits take place.

This reasoning gives the existence of a homoclinic orbit to the point p_+ and therefore the existence of, at least, one homoclinic bifurcation for any $0 < \varepsilon < \varepsilon_0$. Then the existence of the curve Λ^* in the parameter plane where System (12) has Shilnikov bifurcations. Note that

$$\Lambda^* \subset \mathcal{W}_1^* = \{(\varepsilon, \sigma) : \sigma_{\rho_-}^*(\varepsilon) < \sigma < \sigma_{\rho_+}^*(\varepsilon)\}$$

With respect to the heteroclinic connections, by Theorem 2.5, there is a heteroclinic connection if and only if the symplectic distance $\bar{S}^2(\theta, \varepsilon, \sigma) = 0$ for some θ . In the general case, let θ_* be the argument of $C_1^* + iC_2^*$, take any constant $\kappa \in (0, 1)$ and choose $|c_1| \leq \kappa \sqrt{(C_1^*)^2 + (C_2^*)^2}$, $c_2 = -2 - \frac{2}{a}$ and $c_3 = 1$. Notice that with this choice of $\rho = (c_1, c_2, c_3)$, the curve

$\Gamma_\rho^* \subset \mathcal{W}_2^*$. Therefore, if $(\varepsilon, \sigma) \in \Gamma_\rho^*$, we need θ satisfying

$$\cos(\theta_* - \theta + a^{-1}L_0 \log \varepsilon) = -\frac{c_1}{\sqrt{(C_1^*)^2 + (C_2^*)^2}}(1 + \mathcal{O}(\varepsilon)) + \mathcal{O}(|\log \varepsilon|^{-1}).$$

Obviously, this equation has two solutions when ε is small enough provided $|c_1| \leq \kappa \sqrt{(C_1^*)^2 + (C_2^*)^2}$ with $0 < \kappa < 1$. Analogously we prove the first item in [Theorem 2.7](#).

Now we prove the rest of the items in [Theorems 2.6](#) and [2.7](#). In the volume preserving case, take $\sigma \equiv 0$. In the general case, as for the heteroclinic orbits, take any curve $\Gamma_\rho^* \subset \mathcal{W}_2^*$ defined in [Theorem 2.6](#). Let Q_0 be the second crossing with $\{z = 0\}$ defined in [Proposition 3.1](#). The polar coordinates of Q_0 are $r_0(\varepsilon, \sigma)$, $\theta_0(\varepsilon, \sigma)$ being θ_0 a C^∞ function for $\varepsilon > 0$ small enough. Then Q_0 belongs to \mathbb{C}^s , and therefore there is a Shilnikov orbit at the curve $\sigma = \sigma_\rho^*(\varepsilon)$, if and only if the function defined by

$$\mathcal{H}(\varepsilon) = r_0(\varepsilon, \sigma_\rho^*(\varepsilon)) - \bar{r}^u(\theta_0(\varepsilon, \sigma_\rho^*(\varepsilon)), \varepsilon, \sigma_\rho^*(\varepsilon)) + S^2(\theta_0(\varepsilon, \sigma_\rho^*(\varepsilon)), \varepsilon, \sigma_\rho^*(\varepsilon))$$

is zero for some values of ε .

Let $\theta_1^*(\varepsilon) = \theta_* - a^{-1}L_0 \log \varepsilon$ and $\theta_2^*(\varepsilon) = \theta_* - \pi - a^{-1}L_0 \log \varepsilon$, where, as before, θ_* is the argument of $C_1^* + iC_2^*$. We have that:

$$S^2(\theta_1^*(\varepsilon), \varepsilon, \sigma_\rho^*(\varepsilon)) = \sqrt{\frac{b}{a+1}} \varepsilon^{-2-\frac{2}{a}} e^{-\frac{\pi}{2a\varepsilon}} \left(c_1 + \sqrt{(C_1^*)^2 + (C_2^*)^2} + \mathcal{O}(|\log \varepsilon|^{-1}) \right)$$

$$S^2(\theta_2^*(\varepsilon), \varepsilon, \sigma_\rho^*(\varepsilon)) = \sqrt{\frac{b}{a+1}} \varepsilon^{-2-\frac{2}{a}} e^{-\frac{\pi}{2a\varepsilon}} \left(c_1 - \sqrt{(C_1^*)^2 + (C_2^*)^2} + \mathcal{O}(|\log \varepsilon|^{-1}) \right)$$

and it is clear that

$$S^2(\theta_1^*(\varepsilon), \varepsilon, \sigma_\rho^*(\varepsilon)) > 0, \quad S^2(\theta_2^*(\varepsilon), \varepsilon, \sigma_\rho^*(\varepsilon)) < 0. \tag{18}$$

By the second item of [Proposition 3.1](#), for any $n \in \mathbb{N}$, there exists $\varepsilon_n^1, \varepsilon_n^2 \rightarrow 0$ as $n \rightarrow \infty$ satisfying

$$\theta_0(\varepsilon, \sigma_\rho^*(\varepsilon)) = \theta_1^*(\varepsilon) + 2\pi n, \quad \theta_0(\varepsilon, \sigma_\rho^*(\varepsilon)) = \theta_2^*(\varepsilon) + 2\pi n.$$

Using [\(18\)](#) and the first item in [Proposition 3.1](#),

$$\mathcal{H}(\varepsilon_n^1) > 0, \quad \mathcal{H}(\varepsilon_n^2) < 0.$$

Therefore we conclude that there exists $\varepsilon_n \in \overline{\varepsilon_n^1 \varepsilon_n^2}$ $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ satisfying that $\mathcal{H}(\varepsilon_n) = 0$ and the proof is complete.

Condition $0 < a < 2$ ensures that the homoclinic orbit is of Shilnikov type.

Remark 3.3. Our meticulous argumentation is close to the qualitative and geometrical approaches used in [\[20\]](#) and, in some sense, in [\[12\]](#).

3.2. Proof of Proposition 3.1

As [Theorem 2.4](#) proves that the 1-dimensional unstable manifold of the north pole, $W^u(p^-)$, intersects $z = 0$ very close to the 1-dimensional stable manifold of the south pole $W^s(p^+)$, the first step is to study the solutions of System [\(12\)](#) near $W^s(p_-)$ in $z \leq 0$. More concretely, we want to see that orbits entering the plane $z = 0$ near the 1-dimensional stable manifold $W^s(p_-)$, leave the plane again near the 2-dimensional unstable manifold $W^u(p_-)$. This is done in [Section 3.2.1](#). After that, in [Sections 3.2.2](#) and [3.2.3](#) we indeed prove the results in [Proposition 3.1](#).

3.2.1. Transition through $z \leq 0$

In order to control the behavior of the solutions which enter $z \leq 0$ near the 1-dimensional stable manifold $W^s(p_-)$, in particular $W^u(p_+)$, we perform an analytic change of coordinates such that the point p_- becomes $(0, 0, -1)$ and its stable manifold $W^s(p_-)$ becomes the z -axis. As this change does not affect the terms in normal form, we obtain the following system, that we write keeping the notation (x, y, z) :

$$\begin{aligned} \frac{dx}{dt} &= x(\sigma - az) + \frac{y}{\varepsilon} + \varepsilon \tilde{f}(x, y, z, \varepsilon, \sigma), \\ \frac{dy}{dt} &= -\frac{x}{\varepsilon} + y(\sigma - az) + \varepsilon \tilde{g}(x, y, z, \varepsilon, \sigma), \\ \frac{dz}{dt} &= -1 + b(x^2 + y^2) + z^2 + \varepsilon \tilde{h}(x, y, z, \varepsilon, \sigma), \end{aligned} \tag{19}$$

where $\tilde{f}(x, y, z, \varepsilon, \sigma)$, $\tilde{g}(x, y, z, \varepsilon, \sigma) = \mathcal{O}(\|(x, y)\|)$ and the function \tilde{h} satisfies $\tilde{h}(x, y, z, \varepsilon, \sigma) = \mathcal{O}(\|(x, y, z + 1)\|)$. To finish we perform a suitable linear change of variables, $\mathcal{O}(\varepsilon^2)$ close to the identity, explained with detail in [\[2, Lemma 4.1\]](#), to put \tilde{h} of the form

$$\tilde{h}(x, y, z, \varepsilon, \sigma) = (z + 1)\tilde{h}_1(x, y, z, \varepsilon, \sigma) + \tilde{h}_2(x, y, z, \varepsilon, \sigma),$$

with $\tilde{h}_2(x, y, z, \varepsilon, \sigma) = \mathcal{O}(\|x, y, z + 1\|^2)$ and \tilde{h}_1 bounded. In addition, the corresponding \tilde{f}, \tilde{g} keep the same properties.

From the results in [10] and [2, Proposition 4.4], we easily deduce that

$$W^u(p_-) \cap \{z \leq 0\} = \left\{ \frac{b}{a+1}(x^2 + y^2) + z^2 = 1 + \varepsilon(1 - z^2)\psi(z, \theta) \right\}, \tag{20}$$

where $\psi(z, \theta)$ is an analytic periodic function in θ satisfying that there exists a constant $M > 0$ such that for $z \leq 0, \theta \in [0, 2\pi)$:

$$|\psi(\theta, z)| \leq M, \quad \left| \frac{\partial \psi}{\partial z}(\theta, z) \right| \leq M, \quad \left| \frac{\partial \psi}{\partial \theta}(\theta, z) \right| \leq \varepsilon M.$$

That is, the 2-dimensional unstable manifold of the point $p_- = (0, 0, -1)$ in the domain $z \leq 0$ is ε -close to the ellipsoid (14).

Lemma 3.4. *Let $D = \bar{D}$ be the closed region $D \subset \{(x, y, z) \in \mathbb{R}^3 : z \leq 0\}$ with boundaries:*

$$\partial D = \{z = 0\} \cup W^u(p_-).$$

Take $\zeta_1(t), \zeta_2(t)$ two solutions of System (19) such that

$$\zeta_1(0), \zeta_2(0) \in \partial D_0 := \{z = 0\} \cap \left\{ b(x^2 + y^2) \leq \frac{1}{2} \right\}.$$

Then, there exist $\tau, C > 0$ such that $\zeta_1(t), \zeta_2(t) \in D$ for all $t \in [0, \tau]$ and

$$\|\zeta_1(t) - \zeta_2(t)\| \leq \|\zeta_1(0) - \zeta_2(0)\| e^{Ct}, \quad t \in [0, \tau].$$

Proof. First we note that $\partial D_0 \subset \partial D \cap \{z = 0\}$, because the maximum radius in ∂D is $\sqrt{\frac{a+1}{b}} + \mathcal{O}(\varepsilon)$ which is greater than $\sqrt{\frac{1}{2b}}$ the maximum radius in ∂D_0 . The fact that $\tau > 0$ is a direct consequence from the fact that, on ∂D_0 , if ε is small enough, we have that $\dot{z} < -\frac{1}{2} + \mathcal{O}(\varepsilon) < 0$ and therefore the solutions $\zeta_1(t)$ and $\zeta_2(t)$ enter in $\text{int } D$.

We consider now $\Delta \zeta = \zeta_1 - \zeta_2$. As both are solutions of System (19) which is an analytical vector field, and we know that $\zeta_1(t), \zeta_2(t) \in D$ for $0 \leq t \leq \tau$ and D is a bounded region, it is clear, by the mean value Theorem, that $\Delta \zeta$ is a solution of the homogeneous linear equation

$$\frac{d\Delta \zeta}{dt} = A\Delta \zeta + \mathcal{A}(t)\Delta \zeta, \quad \max_{t \in [0, \tau]} \|\mathcal{A}(t)\| \leq K,$$

being A the matrix

$$A = \begin{pmatrix} \sigma + a & \frac{1}{\varepsilon} & 0 \\ -\frac{1}{\varepsilon} & \sigma + a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, since, $\|e^{At}\| \leq e^{(a+\sigma)t}$, where $\|\cdot\|$ is the euclidian norm, we have that

$$\|\Delta \zeta(t) e^{-(a+\sigma)t}\| \leq \|\Delta \zeta(0)\| + K \int_0^t e^{-(a+\sigma)s} \|\Delta \zeta(s)\| ds$$

and by Gronwall's lemma one deduces

$$\|\Delta \zeta(t) e^{-(\sigma+a)t}\| \leq \|\Delta \zeta(0)\| e^{Kt}$$

which gives the result taking $C = K + \sigma + a$. \square

Next two lemmas are devoted to show how the flow of (19) behaves on suitable surfaces.

Lemma 3.5. *For any $R_0 > 0$, there exists $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_0)$, $|\sigma| \leq \sigma_0 \varepsilon$ (see Theorem 2.5) and $0 < R \leq R_0$ if one considers the cylinder*

$$C_R = \{x^2 + y^2 = R^2, -1 \leq z \leq -\varepsilon |\log \varepsilon|\}.$$

the flow of System (19) is pointing outwards the side boundary of C_R .

Proof. The normal vector to ∂C_R is $(x, y, 0)$. Therefore we need to check that $x\dot{x} + y\dot{y} > 0$ for $(x, y, z) \in \partial C_R$:

$$x\dot{x} + y\dot{y} = (\sigma - az)R^2 + \varepsilon \mathcal{O}(R^2) \geq a\varepsilon R^2 (|\log \varepsilon| + \mathcal{O}(1)) > 0$$

if ε is small enough. \square

Lemma 3.6. *Take $\nu_1 > 0$ and the ellipsoid*

$$S_{\nu_1} = \left\{ z^2 + \frac{b}{a+1}(x^2 + y^2) = 1 - \nu_1 \right\}.$$

Then there exists $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_0)$, $|\sigma| \leq \sigma_0 \varepsilon$ the surface defined by $S_{\nu_1} \cap \{-1 \leq z \leq -\varepsilon |\log \varepsilon|\}$ is contained in the region D defined in Lemma 3.4 and the flow of System (19) points outwards.

Proof. As the unstable manifold of p_- , $W^u(p_-)$, is ε -close to the ellipsoid (14) (see (20)) in the region $z \leq 0$ and ν_1 is small but independent of ε , it is clear that if ε is small enough $S_{\nu_1} \cap \{-1 \leq z \leq -\varepsilon |\log \varepsilon|\}$ is contained in the region D . Moreover:

$$\begin{aligned} z\dot{z} + \frac{b}{a+1}(x\dot{x} + y\dot{y}) &= z(-1 + z^2 + b(x^2 + y^2) + \varepsilon \mathcal{O}(\|x, y, z + 1\|)) \\ &\quad + \frac{b}{a+1}(x^2 + y^2)(\sigma - az + \mathcal{O}(\varepsilon)) \\ &= z(a - \nu_1(a + 1) - az^2 + \varepsilon \mathcal{O}(\|x, y, z + 1\|)) \\ &\quad + (1 - \nu_1 - z^2)(-az + \mathcal{O}(\varepsilon)) \\ &= -\nu_1 z + \mathcal{O}(\varepsilon) \geq \varepsilon(\nu_1 |\log \varepsilon| + \mathcal{O}(1)) > 0 \end{aligned}$$

if ε is small enough. \square

Recall that in the coordinates which give System (19), the “north” equilibrium point is of the form $p_+ = (0, 0, 1) + \mathcal{O}(\varepsilon)$.

3.2.2. Proof of second item in Proposition 3.1

By Theorem 2.4 we know that the points

$$\zeta^u = W^u(p_+) \cap \{z = 0\}, \quad \zeta^s = W^s(p_-) \cap \{z = 0\},$$

satisfy that, for some constant \tilde{C}^* ,

$$\|\zeta^u - \zeta^s\| = \varepsilon^{(-1+a)} e^{-\frac{\pi}{2\varepsilon}} (\tilde{C}^* + \mathcal{O}(\varepsilon)). \tag{21}$$

Let us consider $\zeta_u(t)$ and $\zeta_s(t)$ the solutions of System (19) such that $\zeta_{u,s}(0) = \zeta^{u,s}$. It is clear that, ζ_s is defined for all $t \geq 0$ and $\lim_{t \rightarrow +\infty} \zeta_s(t) = p_- = (0, 0, -1)$, therefore it belongs to the domain D defined in Lemma 3.4. In fact $\zeta_s(t) \in \{x = y = 0\}$ for all $t \geq 0$. Let $\tau > 0$, given by this lemma, be such that $\zeta_u(t) \in D$ for $t \in [0, \tau]$. Then, applying Lemma 3.4, there exist a positive constant C , such that, for $0 \leq t \leq \tau$:

$$\|\zeta_u(t) - \zeta_s(t)\| \leq \frac{\tilde{C}^*}{\varepsilon^{1-a}} e^{-\frac{\pi}{2\varepsilon}} e^{Ct}. \tag{22}$$

As $\zeta_s(t) \rightarrow p_- = (0, 0, -1)$ as $t \rightarrow +\infty$, one concludes that, in order to leave the domain D , τ has to satisfy

$$\frac{1}{\varepsilon^{1-a}} e^{-\frac{\pi}{2\varepsilon}} e^{C\tau} = \mathcal{O}(1)$$

and therefore, we can assure that, at least

$$\tau \geq \frac{\pi}{4C\varepsilon}. \tag{23}$$

Observe that the lower bound for τ is not sharp but it will be enough for our purposes.

Now we prove that, indeed, $\zeta_u(t)$ leaves the domain D , that is, it crosses the plane $\{z = 0\}$ for some $t \geq \tau$. With this result we prove the second item in Proposition 3.1. We proceed by assuming the contrary, that is, that $\zeta_u(t) \in \text{int}D$ for all $t > 0$. We do the proof in three steps:

Step 1: Preliminary considerations

Take $\nu_1 > 0$ and consider the closed region V_{ν_1} with boundary

$$\partial V_{\nu_1} = W^u(p_-) \cup \{z = -\varepsilon |\log \varepsilon|\} \cup S_{\nu_1}$$

where the ellipsoid S_{ν_1} is defined in Lemma 3.6. Recall that, as it is indicated in (20), up to order ε the unstable manifold $W^u(p_-)$ is well approximated by the ellipsoid

$$z^2 + \frac{b}{a+1}(x^2 + y^2) = 1.$$

Consequently, the region V_{ν_1} is $\mathcal{O}(\varepsilon)$ close to the region

$$\{z \leq -\varepsilon |\log \varepsilon|\} \cap \left\{ -\sqrt{1 - \frac{b}{a+1}(x^2 + y^2)} \leq z \leq -\sqrt{1 - \nu_1 - \frac{b}{a+1}(x^2 + y^2)} \right\}.$$

Step 2: Leaving V_{ν_1} by $\{z = -\varepsilon |\log \varepsilon|\}$

We claim that there exists $\tau_0 > 0$ independent of ε such that

$$\zeta_u(\tau_0) \in S_{\nu_1}.$$

Indeed, in [1] was proven that $\zeta_s(t) = (0, 0, \tanh(-t) + \mathcal{O}(\varepsilon))$, among other results. Then we can go down along $\zeta_u(t)$ as close to $z = -1$ as we want. Since ν_1 is independent of ε we can ensure that there exists $\bar{\tau}_0$ independent of ε such that $\pi^z \zeta_s(\bar{\tau}_0) = -1 + \nu_1/2$, consequently, as (22) assures that $\zeta_u(t)$ will be exponentially close to $\zeta_s(t)$, there exists τ_0 independent of ε such that $\zeta_u(\tau_0) \in S_{\nu_1}$ and enters in V_{ν_1} .

Let τ_1 be such that $\zeta_u(t) \in V_{\nu_1}$ if $t \in [\tau_0, \tau_1]$. By Lemma 3.6, the only way to leave the region V_{ν_1} is to cross the section $\{z = -\varepsilon|\log \varepsilon|\}$. In addition, Lemma 3.5 assures that $\zeta_u(t)$ leaves every cylinder $\{x^2 + y^2 = R^2\}$. Therefore, we conclude that $\zeta_u(\tau_1) \in \{z = -\varepsilon|\log \varepsilon|\}$.

Step 3: Final conclusion

Since we are assuming that $\zeta_u(t) \in \text{int}D$ for all $t > 0$, we have now that if $t \geq \tau_1$, then $\pi^z \zeta_u(t) \in [-\varepsilon|\log \varepsilon|, 0)$. We write $\zeta_u(t) = (x_u(t), y_u(t), z_u(t))$. It is now clear that $b(x_u^2(\tau_1) + y_u^2(\tau_1)) \geq (a + 1)(1 - \nu_1 + \mathcal{O}(\varepsilon^2|\log \varepsilon|^2))$. As, by Taylor's formula, there exists $\xi_t \in [\tau_1, t]$ such that

$$\begin{aligned} z_u(t) &= z_u(\tau_1) + \dot{z}_u(\tau_1)(t - \tau_1) + \frac{1}{2} \ddot{z}_u(\xi_t)(t - \tau_1)^2 \\ &= -\varepsilon|\log \varepsilon| + (-1 + \varepsilon^2 \log^2 \varepsilon + b(x_u^2(\tau_1) + y_u^2(\tau_1)) + \mathcal{O}(\varepsilon))(t - \tau_1) \\ &\quad + \frac{1}{2} \ddot{z}_u(\xi_t)(t - \tau_1)^2 \\ &\geq -\varepsilon|\log \varepsilon| + (-1 + (a + 1)(1 - \nu_1) + \mathcal{O}(\varepsilon))(t - \tau_1) \\ &\quad + \frac{1}{2} \ddot{z}_u(\xi_t)(t - \tau_1)^2. \end{aligned}$$

Take now $t_m = \tau_1 + m\varepsilon|\log \varepsilon|$ with $m > 0$. Since the solution $\zeta_u(s)$ remains bounded for $s \geq \tau_1$,

$$z_u(t_m) \geq \varepsilon|\log \varepsilon|(-1 - m + m(a + 1)(1 - \nu_1)) + \mathcal{O}(\varepsilon^2|\log \varepsilon|^2).$$

Take $\nu_1 > 0$ such that $(a + 1)\nu_1 < a$, then $(a + 1)(1 - \nu_1) - 1 > 0$ and hence for

$$m > \frac{2}{(a + 1)(1 - \nu_1) - 1}$$

we have that $z_u(t_m) \geq \varepsilon|\log \varepsilon| + \mathcal{O}(\varepsilon^2|\log \varepsilon|^2) > 0$ which is a contradiction since we have assumed that $z_u(t) < 0$ if $t > 0$.

If we call $Q_0 = (r_0 \cos \theta_0, r_0 \sin \theta_0, 0) = \zeta_u(\tau)$ the point where $z_u(\tau) = 0$, and write the Eq. (19) in cylindrical coordinates we have that

$$\frac{d\theta}{dt} = \frac{-1}{\varepsilon} + \mathcal{O}(\varepsilon)$$

therefore, the estimate for θ_0 in Proposition 3.1 comes from (23).

3.2.3. Proof of first item in Proposition 3.1

It remains to estimate the distance from Q_0 to the unstable circle C^u . Let us introduce $r^2 = x^2 + y^2$. It is a well known fact that, as $\sigma = \mathcal{O}(\varepsilon)$, System (19) written in cylindrical coordinates is $\mathcal{O}(\varepsilon)$ -close to an integrable system with first integral

$$H(r, z) = r^{2/a} \left[1 - z^2 - \frac{b}{a + 1} r^2 \right].$$

We introduce the new variable

$$\rho = r(1 + \varepsilon\psi(z, \theta))^{-1/2}, \tag{24}$$

with ψ defined by (20). Then the two dimensional unstable manifold of $p_- = (0, 0, -1)$ is

$$W^u(p_-) = \left\{ z^2 + \frac{b}{a + 1} \rho^2 = 1 \right\}.$$

The variation with respect to t of (ρ, z) is

$$\begin{aligned} \frac{d\rho}{dt} &= (\sigma - az)\rho + \varepsilon\rho\widehat{f}(\rho, z, \theta, \varepsilon, \sigma) \\ \frac{dz}{dt} &= -1 + z^2 + b\rho^2(1 + \varepsilon\psi) + \varepsilon(z + 1)\widehat{h}_1(\rho, z, \theta, \varepsilon, \sigma) + \varepsilon\widehat{h}_2(\rho, z, \theta, \varepsilon, \sigma) \end{aligned} \tag{25}$$

with $\widehat{f}, \widehat{h}_1$ bounded and \widehat{h}_2 of $\mathcal{O}(\|\rho, z + 1\|^2)$. That is, the properties of \tilde{f} and \tilde{h}_1, \tilde{h}_2 are satisfied also after the change of variables.

Step 1. A bound of $H(\rho(t), z(t))$.

We first prove that, if $(\rho(t), z(t)) \in \{H(\rho, z) \geq 0\} \cap \{z \leq 0\}$ for $t \geq t_0$, then

$$H(\rho(t), z(t)) \leq H(\rho(t_0), z(t_0))e^{K\varepsilon(t-t_0)} \tag{26}$$

for some constant K independent of the initial condition.

It is not difficult to see that

$$\begin{aligned} \frac{dH}{dt}(\rho, z) &= \frac{2}{a}H(\rho, z)(\sigma + \varepsilon\widehat{f}(\rho, z, \theta, \varepsilon, \sigma)) \\ &\quad - 2z\rho^{2/a}(\varepsilon(z+1)\widehat{h}_1(\rho, z, \theta, \varepsilon, \sigma) + \varepsilon\widehat{h}_2(\rho, z, \theta, \varepsilon, \sigma)) \\ &\quad - \rho^{2+\frac{2}{a}}\frac{2b}{a+1}(\sigma + \varepsilon\widehat{f}(\rho, z, \theta, \varepsilon, \sigma)). \end{aligned}$$

Let us denote $\bar{\sigma} = \sigma\varepsilon^{-1} = \mathcal{O}(1)$ and

$$\bar{h}_2(\rho, z, \theta, \varepsilon, \sigma) = 2z\widehat{h}_2(\rho, z, \theta, \varepsilon, \sigma) + \rho^2\frac{2b}{a+1}(\bar{\sigma} + \widehat{f}(\rho, z, \theta, \varepsilon, \sigma)).$$

The unstable manifold of p_- is contained in $H(\rho, z) = 0$ and therefore is given by $\rho = \varphi(z) := \sqrt{\frac{a+1}{b}(1-z^2)^{1/2}}$. Consequently

$$\frac{dH}{dt}(\varphi(z), z) \equiv 0.$$

and using that $H(\varphi(z), z) \equiv 0$, we obtain

$$(z+1)z\widehat{h}_1(\varphi(z), z, \theta, \varepsilon, \sigma) + \bar{h}_2(\varphi(z), z, \theta, \varepsilon, \sigma) \equiv 0.$$

Notice that $D_\rho\widehat{h}_1$ is bounded and $D_\rho\bar{h}_2 = \mathcal{O}(\|\rho, z + 1\|)$. Then, using the mean's value theorem we have that

$$\begin{aligned} |\widehat{h}_1(\rho, z, \theta, \varepsilon, \sigma) - \widehat{h}_1(\varphi(z), z, \theta, \varepsilon, \sigma)| &= \left| \int_0^1 D_\rho\widehat{h}_1(\rho + \lambda(\varphi(z) - \rho), z, \theta, \varepsilon, \sigma) d\lambda(\varphi(z) - \rho) \right| \\ &\leq K|\varphi(z) - \rho| \end{aligned}$$

and

$$\begin{aligned} |\widehat{h}_2(\rho, z, \theta, \varepsilon, \sigma) - \widehat{h}_2(\varphi(z), z, \theta, \varepsilon, \sigma)| &= \left| \int_0^1 D_\rho\bar{h}_2(\rho + \lambda(\varphi(z) - \rho), z, \theta, \varepsilon, \sigma) d\lambda(\varphi(z) - \rho) \right| \\ &\leq K(|z+1| + \rho + |\varphi(z) - \rho|)|\varphi(z) - \rho|. \end{aligned}$$

As a consequence

$$\left| \frac{dH}{dt}(\rho, z) \right| \leq \varepsilon KH(\rho, z) + K\varepsilon\rho^{2/a}(|z+1| + \rho + |\varphi(z) - \rho|)|\varphi(z) - \rho|. \tag{27}$$

Recall that $\varphi(z) = \sqrt{\frac{a+1}{b}}\sqrt{1-z^2}$. Notice on the one hand that

$$H(\rho, z) = \rho^{2/a}\frac{b}{a+1}\left(\frac{a+1}{b}(1-z^2) - \rho^2\right) = \rho^{2/a}\frac{b}{a+1}(\varphi(z) + \rho)(\varphi(z) - \rho)$$

which implies

$$\rho^{2/a}|\varphi(z) - \rho| = \sqrt{\frac{a+1}{b}}H(\rho, z)\left(\sqrt{1-z^2} + \rho\sqrt{\frac{b}{a+1}}\right)^{-1}$$

On the other hand, using that $\varphi(z), \rho > 0$, we have that $|\varphi(z) - \rho| \leq |\varphi(z) + \rho|$. Therefore, using that $z \leq 0$, from (27) and changing K if necessary, we obtain:

$$\begin{aligned} \left| \frac{dH}{dt}(\rho, z) \right| &\leq \varepsilon KH(\rho, z)\left(1 + \frac{|z+1| + \rho + |\varphi(z) - \rho|}{\sqrt{1-z^2} + \sqrt{\frac{b}{a+1}}\rho}\right) \\ &\leq \varepsilon KH(\rho, z)\left(1 + \sqrt{\frac{|z+1|}{|1-z|}} + \frac{b}{a+1} + 1\right) \\ &\leq \varepsilon KH(\rho, z) \end{aligned}$$

This inequality implies, by Gronwall's lemma, bound (26), provided $z(t_0) \leq 0$.

Step 2. A bound for τ .

Now we provide a bound from above of τ , the time that the solution $\zeta_u(t)$ needs for crossing again $\{z = 0\}$. Recall that $\zeta_{u,s}(t)$ are the solutions such that $\zeta_{u,s}(0) = \zeta^{u,s} \in \{z = 0\}$. We call (r_u, θ_u, z_u) the corresponding cylindrical coordinates and $\rho_u(t)$ the corresponding radius defined in (24). We will prove that,

$$\rho_u(\tau) = \mathcal{O}(1), \quad \tau \leq \frac{d_0}{\varepsilon}. \tag{28}$$

for some constant d_0 .

First we consider $t_1 > 0$ such that $\zeta_u(t)$ crosses by the first time the plane $z = -\eta$ for some $\eta > 0$. We have that $z_u(t_1) = -\eta$. As argued before, t_1 is independent of ε and the corresponding radius $\rho^u(t_1) \sim K\varepsilon^{-1+a}e^{-\frac{\pi}{2\varepsilon}}$, that is, it is of the same order as $\|\zeta^u - \zeta^s\|$ which is written in (21).

Now we consider the minimum value $t_2 > t_1$ such that $\zeta_u(t_2) \in \{z = -\eta\}$. We have that for $t \in [t_1, t_2]$, $z_u(t) \in [-1, -\eta]$. Then, by (25) we have that

$$\dot{\rho}_u = -az_u\rho_u + \rho_u\mathcal{O}(\varepsilon) \geq c_1\rho_u$$

with $c_1 = a\eta + \mathcal{O}(\varepsilon)$. This implies that

$$\rho_u(t) \geq \rho_u(t_1)e^{c_1(t-t_1)}, \quad t \in [t_1, t_2]. \tag{29}$$

Assume η small enough to take $\nu_1 < 1 - \eta^2$, with $\nu_1 > 0$. Note that

$$z_u^2(t_1) + \frac{b}{a+1}\rho_u^2(t_1) = \eta^2 + \mathcal{O}(\varepsilon^{-1+a}e^{-\frac{\pi}{2\varepsilon}}) < 1 - \nu_1,$$

and hence, $(\rho_u(t_1), z_u(t_1)) \notin V_{\nu_1}$. Then, reasoning as before, there exists $t'_1 > t_1$ independent of ε such that $\zeta_u(t) \in V_{\nu_1}$ for $t \in [t'_1, t_2]$. Then

$$\frac{b}{a+1}\rho_u^2(t_2) \geq 1 - \nu_1 - z_u^2(t_2) = 1 - \nu_1 - \eta^2 > 0$$

which is an independent of ε constant. Consequently, when ζ_u crosses the plane $\{z = -\eta\}$ the radius $\rho_u(t_2) = \mathcal{O}(1)$. We recall that $\rho_u(t_1) \sim K\varepsilon^{-1+a}e^{-\frac{\pi}{2\varepsilon}}$. Then by (29) t_2 has to satisfy

$$t_2 - t_1 \leq \frac{1}{c_1} \left[\log \rho_u(t_2) + \frac{\pi}{2\varepsilon} + (a-1) \log \varepsilon \right] \leq \frac{c_2}{\varepsilon}$$

for some positive constant c_2 .

The time τ that ζ_u needs to meet the plane $z = 0$ satisfies $\tau - t_2 = \mathcal{O}(1)$. Indeed, on the one hand, meanwhile the solution is in V_{ν_1} , that is if $z_u(t) \leq -\varepsilon|\log \varepsilon|$, we have that

$$\dot{z}_u = -1 + b\rho_u^2 + z_u^2 + \mathcal{O}(\varepsilon) \geq -1 + (a+1)(1 - \nu_1 - \eta^2) + \mathcal{O}(\varepsilon) \geq \frac{a}{2}$$

if we take ν_1, η and ε small enough. Then, in this case we get $z = -\varepsilon|\log \varepsilon|$ in a finite time. On the other hand the transition from $z = -\varepsilon|\log \varepsilon|$ has been studied before, obtaining also a finite (in fact of $\mathcal{O}(\varepsilon|\log \varepsilon|)$) time.

Step 3. End of the proof.

Using (26) with $t_0 = 0$ and $t = \tau$ and (28):

$$H(\rho_u(\tau), z_u(\tau)) \leq KH(\rho_u(0), z_u(0)).$$

Again by (28), we deduce from the above bound that

$$1 - \frac{b}{a+1}\rho_u^2(\tau) \leq K\rho_u(0)^{2/a}(1 + \mathcal{O}(\rho_u^2(0))).$$

It follows that, there exists a constant $L > 0$ such that

$$\sqrt{\frac{a+1}{b}} - \rho_u(\tau) \leq L\rho_u(0)^{2/a}.$$

The required bound for the distance between Q_0 and the unstable curve \mathbf{C}^u follows from the above inequality taking into account (21) and that \mathbf{C}^u is the circumference of radius $\sqrt{\frac{a+1}{b}}$.

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Appendix A. Proof of Lemma 2.1

We prove the result in the general case, being the divergence free case a straightforward consequence by performing the obvious changes. The unfolding $X_{\mu,v}$ can be written as

$$X_{\mu,v}(x, y, z) = (\alpha^*y, -\alpha^*x, 0)^\top + c\mu + d\nu + \mathcal{O}_2,$$

with $c = (c_1, c_2, c_3)^\top \in \mathbb{R}^3$, $d = (d_1, d_2, d_3)^\top \in \mathbb{R}^3$ and where \mathcal{O}_2 stands for a vector field with components of order $\mathcal{O}(\|(x, y, z, \mu, \nu)\|^2)$. In general, we will use the notation

$$\mathcal{O}_\ell = \mathcal{O}(\|(x, y, z, \mu, \nu)\|^\ell).$$

By means of the change of variables

$$\bar{x} = x - \frac{(c_2\mu + d_2\nu)}{\alpha^*}, \quad \bar{y} = y + \frac{(c_1\mu + d_1\nu)}{\alpha^*},$$

we transform the unfolding into (keeping the same notation)

$$X_{\mu,v}(x, y, z) = (\alpha^*y, -\alpha^*x, c_3\mu + d_3\nu)^\top + \mathcal{O}_2. \tag{A.1}$$

After a second order normal form procedure applied to $X_{\mu,v}(x, y, z)$, it can be written as (see [23,24,39]):

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{x}(\beta_0^1\nu + \beta_0^2\mu - \beta_1\bar{z}) + \bar{y}(\alpha^* + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \mathcal{O}_3, \\ \frac{d\bar{y}}{dt} &= -\bar{x}(\alpha^* + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \bar{y}(\beta_0^1\nu + \beta_0^2\mu - \beta_1\bar{z}) + \mathcal{O}_3, \\ \frac{d\bar{z}}{dt} &= -\gamma_0^1\mu - \gamma_0^2\nu + \gamma_1\bar{z}^2 + \gamma_2(\bar{x}^2 + \bar{y}^2) + \gamma_3\mu^2 + \gamma_4\nu^2 + \gamma_5\mu\nu + \gamma_1^1\mu\bar{z} + \gamma_1^2\nu\bar{z} + \mathcal{O}_3. \end{aligned}$$

The coefficients $\beta_1, \gamma_1, \gamma_2$ and α_3 depend only on the initial singularity and straightforward computations give:

$$\begin{aligned} \beta_1 &= -\frac{1}{2}(\partial_{x,z}^2\pi^x X^* + \partial_{y,z}^2\pi^y X^*)(\mathbf{0}), \quad \gamma_1 = \frac{1}{2}\partial_{z^2}^2\pi^z X^*(\mathbf{0}), \\ \gamma_2 &= \frac{1}{4}(\partial_{x^2}^2\pi^x X^* + \partial_{y^2}^2\pi^y X^*)(\mathbf{0}). \end{aligned}$$

It follows from conditions in (6) and (9) that

$$\gamma_1\gamma_2 > 0, \quad \gamma_1\beta_1 > 0, \quad |\beta_1| < 2|\gamma_1|. \tag{A.2}$$

The terms $\beta_0^1, \beta_0^2, \gamma_0^1, \gamma_0^2$ depend on the second order derivatives of $X_{\mu,v}$ at zero, that is, on the terms of degree two of the unfolding. Namely, it is straightforward to check that

$$\begin{aligned} \beta_0^1 &= \frac{1}{2}(\partial_{x,v}^2\pi^x X_{\mu,v} + \partial_{y,v}\pi^y X_{\mu,v})(\mathbf{0}), \quad \gamma_0^1 = -c_3, \\ \beta_0^2 &= \frac{1}{2}(\partial_{x,\mu}^2\pi^x X_{\mu,v} + \partial_{y,\mu}\pi^y X_{\mu,v})(\mathbf{0}), \quad \gamma_0^2 = -d_3, \end{aligned}$$

where c_3 and d_3 are introduced in (A.1). Note that

$$\gamma_0^1 = -c_3 = -\partial_\mu\pi^z X_{0,0}(\mathbf{0}) \quad \gamma_0^2 = -d_3 = -\partial_\nu\pi^z X_{0,0}(\mathbf{0}).$$

It follows from the generic condition (7) that $\beta_0^1\gamma_0^1 - \beta_0^2\gamma_0^2 \neq 0$. Hence we can introduce new parameters

$$\tilde{\nu} = \beta_0^1\nu + \beta_0^2\mu, \quad \tilde{\mu} = \gamma_0^1\mu + \gamma_0^2\nu,$$

obtaining

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{x}(\tilde{\nu} - \beta_1\bar{z}) + \bar{y}(\alpha^* + \tilde{\alpha}_1\tilde{\nu} + \tilde{\alpha}_2\tilde{\mu} + \alpha_3\bar{z}) + \mathcal{O}_3, \\ \frac{d\bar{y}}{dt} &= -\bar{x}(\alpha^* + \tilde{\alpha}_1\tilde{\nu} + \tilde{\alpha}_2\tilde{\mu} + \alpha_3\bar{z}) + \bar{y}(\tilde{\nu} - \beta_1\bar{z}) + \mathcal{O}_3, \\ \frac{d\bar{z}}{dt} &= -\tilde{\mu} + \gamma_1\bar{z}^2 + \gamma_2(\bar{x}^2 + \bar{y}^2) + \tilde{\gamma}_3\tilde{\mu}^2 + \tilde{\gamma}_4\tilde{\nu}^2 + \tilde{\gamma}_5\tilde{\mu}\tilde{\nu} + \tilde{\gamma}_1^1\tilde{\mu}\bar{z} + \tilde{\gamma}_1^2\tilde{\nu}\bar{z} + \mathcal{O}_3. \end{aligned}$$

Expressions of $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\gamma}_3, \tilde{\gamma}_4, \tilde{\gamma}_5, \tilde{\gamma}_1^1$ and $\tilde{\gamma}_1^2$ are not provided because they are not relevant in the sequel. Since $\alpha^* > 0$, in a neighbourhood of $(\bar{x}, \bar{y}, \bar{z}, \tilde{\mu}, \tilde{\nu}) = (0, 0, 0, 0, 0)$ we can multiply the above vector field by the function

$$\frac{1}{\alpha^* + \tilde{\alpha}_1\tilde{\nu} + \tilde{\alpha}_2\tilde{\mu} + \alpha_3\bar{z}} = \frac{1}{\alpha^*} - \frac{\tilde{\alpha}_1}{(\alpha^*)^2}\tilde{\nu} - \frac{\tilde{\alpha}_2}{(\alpha^*)^2}\tilde{\mu} - \frac{\alpha_3}{(\alpha^*)^2}\bar{z} + \mathcal{O}(\|(\bar{z}, \tilde{\mu}, \tilde{\nu})\|^2)$$

to get the equivalent family

$$\begin{aligned}\frac{d\bar{x}}{d\tau} &= \bar{x}(\bar{\beta}_0\bar{v} - \bar{\beta}_1\bar{z}) + \bar{y} + \mathcal{O}_3 \\ \frac{d\bar{y}}{d\tau} &= -\bar{x} + \bar{y}(\bar{\beta}_0\bar{v} - \bar{\beta}_1\bar{z}) + \mathcal{O}_3 \\ \frac{d\bar{z}}{d\tau} &= -\bar{\gamma}_0\bar{\mu} + \bar{\gamma}_1\bar{z}^2 + \bar{\gamma}_2(\bar{x}^2 + \bar{y}^2) + \bar{\gamma}_3\bar{\mu}^2 + \bar{\gamma}_4\bar{v}^2 + \bar{\gamma}_5\bar{\mu}\bar{v} + \bar{\gamma}_1^1\bar{\mu}\bar{z} + \bar{\gamma}_1^2\bar{v}\bar{z} + \mathcal{O}_3\end{aligned}$$

with

$$\bar{\beta}_0 = \frac{1}{\alpha^*}, \quad \bar{\beta}_1 = \frac{\beta_1}{\alpha^*}, \quad \bar{\gamma}_0 = \frac{1}{\alpha^*}, \quad \bar{\gamma}_1 = \frac{\gamma_1}{\alpha^*}, \quad \bar{\gamma}_2 = \frac{\gamma_2}{\alpha^*}.$$

Precise expressions for $\bar{\gamma}_3$, $\bar{\gamma}_4$, $\bar{\gamma}_5$, $\bar{\gamma}_1^1$ and $\bar{\gamma}_1^2$ are not relevant for further calculations. Then, since the condition (A.2) is satisfied, $\bar{\gamma}_1 \neq 0$ and we can introduce the change of variables:

$$\hat{x} = \bar{x}, \quad \hat{y} = \bar{y}, \quad \hat{z} = \bar{\gamma}_1\bar{z} + \frac{\bar{\gamma}_1^1\bar{\mu} + \bar{\gamma}_1^2\bar{v}}{2},$$

and parameters:

$$\begin{aligned}\hat{\mu} &= \tilde{\gamma}_0\tilde{\gamma}_1\tilde{\mu} + \frac{(\tilde{\gamma}_1^1\tilde{\mu} + \tilde{\gamma}_1^2\tilde{v})^2}{4} - \tilde{\gamma}_1\tilde{\gamma}_3\tilde{\mu}^2 - \tilde{\gamma}_1\tilde{\gamma}_4\tilde{v}^2 - \tilde{\gamma}_1\tilde{\gamma}_5\tilde{\mu}\tilde{v}, \\ \hat{v} &= \tilde{\beta}_0\tilde{v} + \frac{\tilde{\beta}_1(\tilde{\gamma}_1^1\tilde{\mu} + \tilde{\gamma}_1^2\tilde{v})}{2\tilde{\gamma}_1}\tilde{\mu},\end{aligned}$$

to obtain the normal form (10):

$$\begin{aligned}\frac{d\hat{x}}{d\tau} &= \hat{x}(\hat{v} - a\hat{z}) + \hat{y} + \mathcal{O}_3 \\ \frac{d\hat{y}}{d\tau} &= -\hat{x} + \hat{y}(\hat{v} - a\hat{z}) + \mathcal{O}_3 \\ \frac{d\hat{z}}{d\tau} &= -\hat{\mu} + \hat{z}^2 + b(\hat{x}^2 + \hat{y}^2) + \mathcal{O}_3\end{aligned}$$

with $a = \frac{\tilde{\beta}_1}{\tilde{\gamma}_1}$, $b = \tilde{\gamma}_1\tilde{\gamma}_2$ or equivalently:

$$a = \frac{\beta_1}{\gamma_1}, \quad b = \frac{\gamma_1\gamma_2}{(\alpha^*)^2}.$$

Note that the condition (A.2) is now equivalent to:

$$b > 0, \quad a > 0, \quad a < 2,$$

where the two first conditions correspond to (6) and the third one to (9).

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