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Invariant manifolds of parabolic fixed points (I). Existence and dependence on parameters

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Abstract

In this paper we study the existence and regularity of stable manifolds associated to fixed points of parabolic type in the differentiable and analytic cases, using the parametrization method.

The parametrization method relies on a suitable approximate solution of a functional equation. In the case of parabolic points, if the manifolds have dimension two or higher, in general this approximation cannot be obtained in the ring of polynomials but as a sum of homogeneous functions and it is given in [4]. Assuming a sufficiently good approximation is found, here we provide an "a posteriori" result which gives a true invariant manifold close to the approximate one. In the differentiable case, in some cases, there is a loss of regularity.

We also consider the case of parabolic periodic orbits of periodic vector fields and the dependence of the manifolds on parameters. Examples are provided.

We apply our method to prove that in several situations, namely, related to the parabolic infinity in the elliptic spatial three body problem, these invariant manifolds exist and do have polynomial approximations.

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Contents

1. Introduction

Parabolic fixed points of maps (or parabolic periodic orbits, in the case of flows) appear in general as bifurcation points but they are also present for all values of the parameters in important problems. For instance, the “parabolic infinity” in several instances of the three body problem. See [29,27,30–32,20,11].

The purpose of this work is, given a map with a parabolic fixed point, that is, a point where the map is tangent to the identity, to provide conditions under which the parabolic point possesses a stable invariant set (which in general will not contain a neighborhood of the fixed point) which can be parametrized as a regular invariant manifold. This is the first part of a two papers work, being [4] the second. In the second one, we study the existence of approximate solutions of the invariance equation that the parabolic invariant manifold should satisfy. Here we are concerned with the existence of the actual manifold.

The existence of invariant manifolds of parabolic fixed points and their regularity has been considered in [28,13,30], when the dynamical system is analytic and the stable manifold set is one dimensional. Invariant manifolds of parabolic fixed points with nilpotent linear part were studied in [9,10,18]. In [26] the authors use the manifolds of a parabolic point as pieces of the boundary of regions with regular and ergodic behavior respectively for a specially chosen family of two dimensional symplectic maps. The case of stable manifolds of higher dimension, but still in the analytic category, was considered in [2]. All these works share the use of the graph transform method to obtain the parabolic invariant manifold.

The problem of parabolic fixed points in the context of holomorphic maps has also been studied in a completely different approach by [22,14]. See also the survey [1].

When the map is not analytic, but $\mathcal{C}^k$, 1-dimensional stable manifolds of parabolic points have been studied in [3]. In this work, unlike the previously cited ones, the parametrization method is used [6–8,24,25]. See also [23].

The procedure here is as follows. Let $F : U \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ be a map and assume $(0, 0) \in U$ is a parabolic point, i.e., $F(0, 0) = (0, 0)$ and $DF(0, 0) = \mathrm{Id}$. Assume furthermore certain conditions on the first non-vanishing nonlinear terms which imply some "weak contraction" in the $(x, 0)$-directions and some "weak expansion" in the $(0, y)$-directions, to be specified later. Even if our conditions are more general and in fact do not always imply "weak expansion" in the $(0, y)$-directions, for the sake of simplicity of this introduction, let us assume that there is this expansion. Then one looks for an invariant stable manifold W^s of the origin as an immersion $K : V \subset \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, which we call parametrization of the manifold, with $K(0) = (0, 0)$, $DK(0) = (\mathrm{Id}\,, 0)^\top$, $\mathrm{range}(K) = W^s$ and satisfying the *invariance equation*

$$F \circ K = K \circ R, \tag{1.1}$$

where $R : V \to V$ is a reparametrization of the dynamics of F on W^s. In general, V is a domain which contains 0 on its boundary. The procedure to find such K and R has two steps. First, find functions $K^\leq$ and R solving approximately the invariance equation, that is, satisfying

$$F \circ K^\leq(x) - K^\leq \circ R(x) = \mathcal{O}(\|x\|^\ell), \tag{1.2}$$

for some ℓ large enough, depending on the degree of the first non-vanishing nonlinear terms of F at $(0, 0)$. Once these functions are obtained, the invariance equation can be rewritten as a fixed point equation for a perturbation of $K^\leq$ and solved in appropriate Banach spaces.

Of course, if the invariance equation does have solutions K and R, they will not be unique, since for any diffeomorphism $T : V \to V$, the functions $K \circ T$ and $T^{-1} \circ R \circ T$ also satisfy the same equation. The same claim holds for the approximate invariant equation (1.2) if, for instance, $T(x) = x + o(\|x\|)$. The parametrization method aims to obtain the "simplest" parametrization (or the parametrization that provides the "simplest" R).

There are two important reasons to use the parametrization method to obtain the invariant manifolds of a parabolic fixed point. The first one, from the theoretical point of view, is that is better suited to deal with cases of finite differentiability than the graph transform method since the operators involved are more regular. The second one is related to the computation of the approximate solutions of the invariance equation. From a computational point of view, it provides a way to explicitly obtain such approximations. And reciprocally, if one is able to produce functions $K^{\leq}$ and R that are approximate solutions of the invariance equation, then there exists a true solution close to the given approximation. This is a type of *a posteriori* argument (see [12,24,25,19,15–17]).

The parametrization method is used in [6,7] to find nonresonant manifolds of fixed points of maps in Banach spaces. In such setting, the approximations $K^{\leq}$ and R can be taken as polynomials. The degrees of $K^{\leq}$ and R depend on the spectrum of $DF(0,0)$. The homogeneous terms of these polynomials are found recursively. The homogeneous term of degree j must satisfy a linear equation which depends on the terms of degree i, for $1 \leq i \leq j-1$. In solving these equations, $K^{\leq}$ and R play different roles and are not unique, even in the class of polynomials. A possible criterium to determine them is to look for the "simplest" polynomial R, in the sense that the majority of its coefficients vanish. This fact only depends on the spectrum of $DF(0,0)$.

In the case when the origin is parabolic and $n=1$, in [3] it is shown that it is also possible to find polynomials $K^{\leq}$ and R which are approximate solutions of the invariance equation. Again, these polynomials are not determined uniquely, but there is a choice in which R is the "simplest". Its degree only depends on the degree of the first non-vanishing term of the contracting part. A related result was obtained in [5] where the Gevrey character of the manifolds is established for analytic maps.

The situation changes drastically when one considers invariant manifolds of parabolic points of dimension two or more. Although these cases were successfully dealt in the analytical context [2] by means of the graph transform method, a simple computation shows that generically there are no polynomial approximate solutions of the invariance equation. In the spirit of the parametrization method, if it is not possible to find approximate solutions, the fixed point part of the argument cannot be carried on. We remark that this fact implies that, generically, the invariant manifolds obtained in [2], which are analytic outside the origin, do not have a polynomial approximation.

In the present paper, we deal with the actual existence of the invariant manifold and we study its regularity and dependence on parameters, assuming that a suitable approximate solution of the invariance equation is known. In the companion paper [4], we derive a method to find such approximations and their regularity. However, since, in general, these approximations are not polynomials but sums of homogeneous functions of increasing degree, we reproduce in Section 3 the algorithm derived in [4] to obtain them. It should be remarked that, in general, these homogeneous functions need not be rational functions. We also remark that the conditions under which these approximations can be found allow several characteristic directions in the domain under consideration (see [22,1]).

When considering parabolic points, one has to look at the first non-vanishing homogeneous terms of the Taylor expansion of the map at the parabolic point. One looks for "contracting" and "expanding" directions (in certain subsets) in the dynamics generated by the polynomial map obtained by truncating the Taylor expansion of the map at the parabolic point at the lowest non vanishing order in each component. We will assume that the degree of all the "contracting" directions is N, the degree of all the "expanding" directions is M, without assuming that $N = M$. The fact that $N \neq M$ has consequences both at a formal level, when solving the approximate invariance equations, and at the analytical level, when considering the fixed point equation that provides the manifold. In particular, the behavior and regularity at the origin of the formal approximation and the invariant manifold depend on the relation between N and M.

We remark that, as it is often the case, the hypotheses to carry out the fixed point procedure are milder than the ones required for solving the approximate invariance equation. The reason is that to solve the fixed point equation it is enough to start with an approximate solution having an error of prescribed high enough order depending of the first non-vanishing nonlinear terms. Of course, some care is required to deal with the regularity of the involved objects.

We include in our study the dependence on parameters of the invariant manifolds, which is rather cumbersome but useful for the applications. In particular, it allows to derive the analogous statement for flows from the one for maps. This is performed separately for the actual manifold, in the present paper, and for the approximate solutions of the invariance equation, in the companion paper. The dependence on parameters of the invariant manifolds in the case that they are one dimensional and the map is analytic is already done in [21], where it was used to find regular foliations of the invariant manifolds of some parabolic cylinders.

As a side application of our method, we prove that, in several instances of the three body problem, namely in perturbations of the restricted spatial elliptic three body problem, the “parabolic infinity” is foliated with parabolic fixed points with stable manifolds of dimension two *that have polynomial approximation at the origin*. This fact is rather surprising, since to be able to solve the approximate invariance equations in the ring of polynomials, one obtains a larger number of obstructions than coefficients at each order. Then, the fixed point machinery works at any order and as a result one obtains the invariant manifolds of the “parabolic infinity” and their expansion at the origin.

The structure of the paper is as follows. In Section 2.1 we present the setting and hypotheses as well as two theorems of existence of invariant manifolds for maps. In Section 2.2, we present the result concerning the regularity with respect to parameters and in Section 2.3 we deal with the results for flows. In Section 3 we describe the algorithm from [4] developed to compute the approximate solutions of the invariance equation. In Section 4 we apply the algorithm to the elliptic spatial restricted three body problem to obtain the invariant manifolds of the “parabolic infinity”. In Section 5 we provide two examples that show that our hypotheses are indeed necessary and that the loss of differentiability can take place. We remark the differences between one-dimensional and multidimensional parabolic manifolds. The rest of the paper is devoted to the actual proofs of the results.

2. Main results

This section is devoted to present all the results of this work related to the existence and regularity of parametrizations of invariant sets. There are three settings we consider: the map case in Section 2.1, the dependence on parameters in the map case in Section 2.2 and the periodic flow case in Section 2.3.

2.1. The map case

The first result is a posteriori type theorem which assures the existence of an invariant manifold close to a sufficiently good approximate solution of the invariance equation (1.1). Then we provide sufficient conditions to ensure the existence of an invariant manifold by means of the results in [4] about approximate solutions of the invariance equation, that is, solutions of (1.2).

2.1.1. Set up

Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set such that $0 \in U$. We consider $\mathcal{C}^r$ maps $F : U \to \mathbb{R}^{n+m}$, with r to be specified later, of the form

$$F(x, y) = \begin{pmatrix} x + p(x, y) + f(x, y) \\ y + q(x, y) + g(x, y) \end{pmatrix}, \qquad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m, \tag{2.1}$$

where p and q are homogeneous polynomials of degrees $N \geq 2$ and $M \geq 2$ respectively, $f(x, y) = \mathcal{O}(\|(x, y)\|^{N+1})$ and $g(x, y) = \mathcal{O}(\|(x, y)\|^{M+1})$. With these conditions, the origin is a parabolic fixed point of F.

We introduce the constants

$$L = \min\{N, M\}, \qquad \eta = 1 + N - L. \tag{2.2}$$

We denote the projection onto a variable as a subscript, i.e. X_x, and by B_ϱ the open ball centered at the origin of radius $\varrho > 0$.

Given $V \subset \mathbb{R}^n$ such that $0 \in \partial V$ and $\varrho > 0$, we introduce the set

$$V_\varrho = V \cap B_\varrho.$$

We will consider sets V *star-shaped with respect to* 0, *i.e.*, $0 \in \partial V$ and, for all $x \in V$ and $\lambda \in (0, 1)$, $\lambda x \in V$.

We define the stable set of F over V associated to the origin 0 as:

$$W^{\mathrm{s}}_V = \{(x, y) \in U : F^k_x(x, y) \in V,\ k \geq 0,\ F^k(x, y) \to 0 \text{ as } k \to \infty\}$$

and its local version, when we restrict V to the set V_ϱ:

$$W^{\mathrm{s}}_{V,\varrho} = \{(x, y) \in U : F^k_x(x, y) \in V_\varrho,\ k \geq 0,\ F^k(x, y) \to 0 \text{ as } k \to \infty\}. \tag{2.3}$$

Let $V \subset \mathbb{R}^n$ be an open star-shaped with respect to 0 set. Take $\varrho > 0$, some norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ and consider the following constants:

$$\begin{aligned}
&a_p = -\sup_{x \in V_\varrho} \frac{\|x + p(x, 0)\| - \|x\|}{\|x\|^N}, && b_p = \sup_{x \in V_\varrho} \frac{\|p(x, 0)\|}{\|x\|^N}, \\
&A_p = -\sup_{x \in V_\varrho} \frac{\|\mathrm{Id} + D_x p(x, 0)\| - 1}{\|x\|^{N-1}}, && B_p = \sup_{x \in V_\varrho} \frac{\|\mathrm{Id} - D_x p(x, 0)\| - 1}{\|x\|^{N-1}}, \\
&B_q = -\sup_{x \in V_\varrho} \frac{\|\mathrm{Id} - D_y q(x, 0)\| - 1}{\|x\|^{M-1}}, && \\
&c_p = \begin{cases} a_p, & \text{if } B_q \leq 0, \\ b_p, & \text{otherwise,} \end{cases} && d_p = \begin{cases} a_p, & \text{if } A_p \leq 0, \\ b_p, & \text{otherwise,} \end{cases}
\end{aligned} \tag{2.4}$$

where the norms of linear maps are the corresponding operator norms. We emphasize that all the previous constants depend on ϱ. Nevertheless there are some straightforward relations among them.

Lemma 2.1. *The constants A_p, B_q, B_p, a_p and b_p are finite. They satisfy $|a_p| \leq b_p$, $B_p \geq A_p$, $a_p \geq A_p/N$ and $B_p \geq Na_p > 0$.*

In addition, if $0 < \overline{\varrho} \leq \varrho$ and denoting by $\overline{A_p}, \overline{B_p}, \overline{B_q}, \overline{a_p}, \overline{b_p}$ the corresponding constants for $\overline{\varrho}$, we have that

$$\overline{A_p} \geq A_p, \quad \overline{B_p} \leq B_p, \quad \overline{B_q} \geq B_q, \quad \overline{a_p} \geq a_p, \quad \overline{b_p} = b_p.$$

This lemma is proven in Section 3.1 of [4] (in a slightly different set up).

As usual for parabolic points, their invariant manifolds are defined over a subset V such that $0 \notin \operatorname{int} V$. For this reason, in order to study the regularity of the invariant manifold at the origin, we define the following natural concept:

Definition 2.2. Let $V \subset \mathbb{R}^l$ be an open set with $x_0 \in \overline{V}$ and $f : V \cup \{x_0\} \subset \mathbb{R}^l \to \mathbb{R}^k$. We say that f is C^1 at x_0 if f is C^1 in $V \cap (B_\varepsilon(x_0) \setminus \{x_0\})$, for some $\varepsilon > 0$ and $\lim_{x \to x_0,\, x \in V} Df(x)$ exists.

We finally introduce a quantity related with the minimum differentiability degree we require to F:

$$\ell_0 := N - 1 + \frac{B_p}{a_p} + \max\left\{\eta - \frac{A_p}{d_p}, 0\right\}. \tag{2.5}$$

Note that $\ell_0 \geq 2N - 1 \geq N + 1$.

2.1.2. A posteriori result

Let $V \subset \mathbb{R}^n$ be open, star-shaped with respect to 0. Assume that there exist appropriate norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ and $\varrho > 0$ small enough such that

H1 The homogeneous polynomial p satisfies that $a_p > 0$,
H2 $q(x, 0) = 0$, for $x \in V_\varrho$ and

$$\begin{aligned} B_q &> 0, && \text{if } M < N, \\ B_q &> -Na_p, && \text{if } M = N, \end{aligned}$$

H3 There exists a constant $a_V > 0$ such that, for all $x \in V_\varrho$,

$$\operatorname{dist}(x + p(x, 0), (V_\varrho)^c) \geq a_V \|x\|^N.$$

Remark 2.3. It is easily checked that if hypotheses H1, H2 and H3 hold true for $\varrho > 0$, they also hold for any $0 < \overline{\varrho} \leq \varrho$, so that we will take ϱ as small as we need.

Theorem 2.4. *Let $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ be a C^r map (the case $r = \infty$ is also included), of the form (2.1) with U an open set such that $0 \in U$.*

Assume that, there exists an open set V and $\varrho_0 > 0$ such that:

(a) Hypotheses H1, H2 and H3 hold for $\varrho_0 > 0$.
(b) The degree of differentiability satisfies $r > \ell_0$ with ℓ_0 defined in (2.5).

(c) There exist $K^{\leq} : V_{\varrho_0} \to U$ and $R : V_{\varrho_0} \to V_{\varrho_0}$, $\mathcal{C}^{r^{\leq}}$ functions, for some $r^{\leq} \geq 1$, of the form

$$\Delta K^{\leq}(x) := K^{\leq}(x) - (x,0) = \mathcal{O}(\|x\|^2), \qquad D^j \Delta K^{\leq}(x) = \mathcal{O}(\|x\|^{2-j}),$$
$$\Delta R(x) := R(x) - x - p(x,0) = \mathcal{O}(\|x\|^{N+1}), \quad D^j \Delta R(x) = \mathcal{O}(\|x\|^{N+1-j}),$$

for $0 \leq j \leq r^{\leq}$, satisfying the invariance equation up to order ℓ for $\ell_0 < \ell \leq r$, i.e.:

$$F \circ K^{\leq} - K^{\leq} \circ R = \mathcal{O}(\|x\|^{\ell}).$$

Then, there exists $\varrho > 0$ small enough and a unique function $K^{>} : V_{\varrho} \to U$ such that $K^{>}(x) = \mathcal{O}(\|x\|^{\ell-N+1})$ and $K = K^{\leq} + K^{>}$ satisfies the invariance equation

$$F \circ K = K \circ R. \tag{2.6}$$

Moreover, $R^k(x) \to 0$ as $k \to \infty$, K_x is invertible and, as a consequence,

$$\{K(x)\}_{x \in (K_x)^{-1}(V_\varrho)} \subset W^{\mathrm{s}}_{V,\varrho}. \tag{2.7}$$

Concerning regularity, the parametrization K and the reparametrization R on $W^{\mathrm{s}}_{V,\varrho}$ are $\mathcal{C}^1$ functions at the origin in the sense of Definition 2.2. Moreover, they are $\mathcal{C}^{r^{>}}$ functions on V_ϱ according to the cases

(1) If $A_p \geq \eta d_p$, $r^{>} = \min\{r, r^{\leq}\}$.
(2) If $A_p < \eta d_p$, $r^{>} = \min\{r, r_0, r^{\leq}\}$ with r_0 defined by

$$r_0 = \max\left\{k \in \mathbb{N} : \left(\eta - \frac{A_p}{d_p}\right)k < r - \frac{B_p}{a_p} - N + 1\right\}. \tag{2.8}$$

(3) If $F \in \mathcal{C}^\infty$, then $r^{>} = r^{\leq}$, where the case $r^{\leq} = \infty$ is also included.

In addition, if F, $K^{\leq}$ and R are real analytic, $A_p > b_p$ and item (c) is true for $j = 0$, then K is also real analytic.

2.1.3. Existence results of invariant manifolds

As a corollary of Theorem 2.4 and the work [4] we can prove an existence result. We first formulate the new set of hypotheses which are (as usual) slightly stronger than the previous ones. They coincide with the ones assumed in [4] for the existence of approximate solutions of the invariance equation (1.1). We include them here for completeness. We summarize the algorithm to find these approximate solutions in Section 3.

Let $V \subset \mathbb{R}^n$ be an open set which is star-shaped with respect to 0. Assume that, with the appropriate norms in $\mathbb{R}^n$ and $\mathbb{R}^m$, there exists ϱ small enough such that Hypothesis H3 is satisfied and

H1' The homogeneous polynomial p satisfies that

$$a_p > 0.$$

If $M > N$, we further ask $A_p/d_p > -1$.

H2' The homogeneous polynomial q satisfies $q(x,0) = 0$, for $x \in V_\varrho$ and

$$\begin{aligned} &B_q > 0, && \text{if } M < N, \\ &2 + \frac{B_q}{c_p} > \max\left\{1 - \frac{A_p}{d_p}, 0\right\}, && \text{if } M = N. \end{aligned}$$

Unlike the hyperbolic case, as we claimed in Theorem 2.4, here we can lose differentiability in the case $A_p < \eta d_p$ even at points $x \in V_\varrho$ with $x \neq 0$. In fact, the formal approximation is only $\mathcal{C}^{r_*}$ when $A_p < d_p$ and $M \geq N$, r_* being:

$$r_* = \begin{cases} \max\left\{k \in \mathbb{N} : \left(1 - \dfrac{A_p}{d_p}\right)k < 2 + \dfrac{B_q}{c_p}\right\}, & \text{if } M = N, \\ \max\left\{k \in \mathbb{N} : \left(1 - \dfrac{A_p}{d_p}\right)k < 2\right\}, & \text{if } M > N. \end{cases}$$

See [4].

The existence result is as follows:

Corollary 2.5. *Let $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ be a $\mathcal{C}^r$ map, of the form (2.1). Assume that, for some $\varrho_0 > 0$, $r > \ell_0$, hypotheses H1', H2' and H3 are satisfied in an open star-shaped with respect to 0 set V.*

Then, there exist $\varrho > 0$ small enough and maps $K : V_\varrho \to U$ and $R : V_\varrho \to V_\varrho$ solutions of the invariance equation (2.6) satisfying (2.7).

In addition, $K = K^{\leq} + K^{>}$ with $K^{\leq}$ and R provided by Theorem 2.2 in [4].

The parametrization K and the reparametrization R on $W^{\mathrm{s}}_{V,\varrho}$ are only $\mathcal{C}^1$ functions at the origin restricting them to the set V_ϱ and they are $\mathcal{C}^{r^>}$ functions on V_ϱ and $r^>$ takes the values:

(1) If $A_p \geq \eta d_p$, $r^> = r$.
(2) If either $d_p \leq A_p < \eta d_p$ or $M < N$, $r^> = \min\{r, r_0\}$ with r_0 defined in (2.8).
(3) If $A_p < d_p$ and $M \geq N$, $r^> = \min\{r, r_0, r_*\}$.
(4) If $F \in \mathcal{C}^\infty$ and $A_p \geq d_p$, then $K^> \in \mathcal{C}^\infty$.

Moreover, if F is real analytic and $A_p > b_p$, K is also real analytic.

Finally, substituting H1' and H2' by the new conditions $A_p > 0$ and $B_q > 0$, we have that $W^{\mathrm{s}}_{V,\varrho} = \{K(x)\}_{x \in (K_x)^{-1}(V_\varrho)}$.

Proof. Obviously H1' implies H1. It remains to check that when $M = N$, the condition in H2' implies that $B_q > -Na_p$. This is immediate if $B_q \geq 0$. When $B_q < 0$, H2' implies $2a_p + B_q > 0$ and hence $B_q > -2a_p \geq -Na_p$.

Now we set a good enough initial approximation of the invariant manifold K by means of Theorem 2.2 in [4].

We take $\ell \in \mathbb{N}$ such that $\ell_0 < \ell \leq r$ with ℓ_0 introduced in (2.5) and we decompose our map F into

$$F(x, y) = P(x, y) + G_\ell(x, y), \tag{2.9}$$

where P is the Taylor expansion of F up to degree $\ell - 1$ and $G_\ell(x) = o(\|x\|^{\ell-1})$. In fact, since $\ell \le r$, we actually have $G_\ell(x) = \mathcal{O}(\|x\|^\ell)$. We apply Theorem 2.2 in [4] to P to obtain $K^\le$ and R such that

$$P \circ K^\le - K^\le \circ R = T^\ell, \qquad T^\ell(x) = o(\|x\|^{\ell-1}). \tag{2.10}$$

Moreover, both $K^\le$ and R are sums of homogeneous functions satisfying that $\Delta K^\le := K^\le(x) - (x, 0) = \mathcal{O}(\|x\|)^2)$ and $\Delta R(x) : R(x) - x - p(x, 0) = \mathcal{O}(\|x\|^{N+1})$. By Theorem 2.2 in [4], $K^\le$ and R are analytic functions if $A_p > b_p$, $\mathcal{C}^\infty$ functions if $A_p = b_p$ and $\mathcal{C}^{r_*}$ if $A_p < b_p$, therefore, F, $K^\le$ and R are $\mathcal{C}^r$ functions if $A_p \ge b_p$ and $\mathcal{C}^{\min\{r,r_*\}}$ functions otherwise. We use the symbol $r^\le$ to denote the degree of differentiability in each case.

Since P is a polynomial, the remainder $T^\ell(x) = \mathcal{O}(\|x\|^\ell)$ is also a finite sum of homogeneous functions. Therefore, using that the derivative of a homogeneous function of degree j is also a homogeneous function of degree $j - 1$, we have that, for any $0 \le j \le r^\le$,

$$D^j \Delta K^\le(x) = \mathcal{O}(\|x\|^{2-j}), \quad D^j \Delta R(x) = \mathcal{O}(\|x\|^{N+1-j}), \quad D^j T^\ell(x) = \mathcal{O}(\|x\|^{\ell-j}).$$

Therefore, we are under the conditions of Theorem 2.4 which implies the stated existence and the regularity in the present results.

The last statement follows from Theorem 3.1 in [2] which states that the stable set $W^{\mathrm{s}}_{V,\varrho}$ defined in (2.3) is the graph of a Lipschitz function. Since $K_x(x) = x + O(\|x\|^2)$ is invertible, the result follows immediately since the new conditions $A_p, B_q > 0$ imply the hypotheses of the results in [2]. □

Now we state a corollary from Theorem 2.4 and Theorem 2.7 in [4].

Corollary 2.6. *Assume the conditions in Corollary 2.5 and take ℓ such that $\ell_0 < \ell \le r$. For $j = 2, \cdots, \ell - N$, let $K_x^j : V_\varrho \to \mathbb{R}^n$ be $\mathcal{C}^{r^>}$ homogeneous functions of degree j. Denote*

$$K_x^*(x) = x + \sum_{j=2}^{\ell-N} K_x^j(x).$$

Then there exists $R^* : V_\varrho \to \mathbb{R}^n$, a finite sum of $\mathcal{C}^{r^>}$ homogeneous functions of order less than $\ell - 1$, of the form $R^*(x) - x - p(x, 0) = \mathcal{O}(\|x\|^{N+1})$ such that for any $\mathcal{C}^{r^>}$ function $\Delta R : V_\varrho \to \mathbb{R}^n$ with $\Delta R(x) = \mathcal{O}(\|x\|^\ell)$ there exists a $\mathcal{C}^{r^>}$ function K satisfying the invariance equation (2.6) with $R = R^* + \Delta R$ and $K_x(x) - K_x^*(x) = \mathcal{O}(\|x\|^{\ell-N+1})$.

Proof. We proceed as in the proof of Corollary 2.5 decomposing F as in (2.9) and applying Theorem 2.7 in [4] instead of Theorem 2.2 in [4] which assures the existence of $K^\le$ and R^* satisfying the invariance equation (2.10) up to order ℓ. Moreover, $K_x^\le(x) - K_x^*(x) = \mathcal{O}(\|x\|^{\ell-N+1})$. Since $\Delta R(x) = \mathcal{O}(\|x\|^\ell)$, we have that $K^\le(R^*(x) + \Delta R(x)) = K^\le \circ R^*(x) + \mathcal{O}(\|x\|^\ell)$ and, consequently, writing $R = R^* + \Delta R$,

$$F \circ K^\le(x) - K^\le \circ R(x) = \mathcal{O}(\|x\|^\ell).$$

Applying Theorem 2.4, we get the result. □

2.2. Dependence on parameters

In this section we deal with the dependence on parameters of the parametrization K and the reparametrization R provided by Theorem 2.4 and Corollary 2.5.

2.2.1. Set up

Let $\Lambda \subset \mathbb{R}^{n'}$ be an open set of parameters and $U \subset \mathbb{R}^n$ be an open set. We consider $\mathcal{C}^r$ maps $F : U \times \Lambda \to \mathbb{R}^{n+m}$ having the form (2.1) for any $\lambda \in \Lambda$, namely:

$$F(x, y, \lambda) = \begin{pmatrix} x + p(x, y, \lambda) + f(x, y, \lambda) \\ y + q(x, y, \lambda) + g(x, y, \lambda) \end{pmatrix}, \qquad (x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n'}, \tag{2.11}$$

where p, q are homogeneous polynomials for any fixed λ of degree $N, M \geq 2$ respectively and $f(x, y, \lambda) = \mathcal{O}(\|(x, y)\|^{N+1})$, $g(x, y, \lambda) = \mathcal{O}(\|(x, y)\|^{M+1})$ uniformly in λ.

In this context, the constants introduced in (2.4), (2.5) and Hypothesis H3, depend on λ. We denote this dependence by a superindex, for instance A_p^λ, ℓ_0^λ, etc. We redefine the constants (independent of λ) $A_p, B_p, a_p, b_p, B_q, a_V, c_p, d_p, \ell_0$ by

$$\begin{aligned} &A_p = \inf_{\lambda\in\Lambda} A_p^\lambda, \qquad a_p = \inf_{\lambda\in\Lambda} a_p^\lambda, \qquad B_q = \inf_{\lambda\in\Lambda} B_q^\lambda, \\ &B_p = \sup_{\lambda\in\Lambda} B_p^\lambda, \qquad b_p = \inf_{\lambda\in\Lambda} a_p^\lambda, \qquad a_V = \inf_{\lambda\in\Lambda} a_V^\lambda, \\ &c_p = \begin{cases} a_p, & \text{if } B_q \leq 0, \\ b_p, & \text{otherwise,} \end{cases} \qquad d_p = \begin{cases} a_p, & \text{if } A_p \leq 0, \\ b_p, & \text{otherwise,} \end{cases} \\ &\ell_0 = N - 1 + \frac{B_p}{a_p} + \max\left\{\eta - \frac{A_p}{d_p}, 0\right\}. \end{aligned} \tag{2.12}$$

Lemma 2.7. *If the conditions in H1, H2, H1', H2' and H3 hold true for the constants $A_p, B_p, a_p, b_p, B_q, c_p, d_p, a_V$, they are also true for $A_p^\lambda, B_p^\lambda, a_p^\lambda, b_p^\lambda, B_q^\lambda, c_p^\lambda, d_p^\lambda, a_V^\lambda$ for any $\lambda \in \Lambda$.*

In addition $\ell_0^\lambda \leq \ell_0$.

The proof of this lemma is straightforward from the definitions.

The differentiability class we work in was used in [7] and is the one considered in [4] for the approximate solutions. For any $s, r \in (\mathbb{Z}^+)^2$, we define the set

$$\Sigma_{s,r} = \left\{(i, j) \in (\mathbb{Z}^+)^2 : i + j \leq r + s,\ i \leq s\right\}$$

and for an open set $\mathcal{U} \subset \mathbb{R}^l \times \mathbb{R}^{n'}$, the function space

$$\mathcal{C}^{\Sigma_{s,r}} = \left\{f : \mathcal{U} \to \mathbb{R}^k\ :\ \forall (i, j) \in \Sigma_{s,r},\ D_\mu^i D_z^j f \text{ exists, is continuous and bounded}\right\}. \tag{2.13}$$

Here D_μ and D_z means the derivative with respect to μ and z respectively. We also denote by

$$\mathcal{C}^{\Sigma_{s,\omega}} = \{f : \mathcal{U} \to \mathbb{R}^k\ :\ \text{for all } \mu,\ f(\cdot, \mu) \text{ is analytic and } f \in \mathcal{C}^s\}.$$

We note that $\mathcal{C}^r \subset \mathcal{C}^{\Sigma_{r,r}}$.

2.2.2. Dependence on parameters results

Note that assuming that the conditions in both Theorem 2.4 and Corollary 2.5 are satisfied for any $\lambda \in \Lambda$, we obtain the existence of K, R solutions of the invariance equation

$$F(K(x,\lambda),\lambda) = K(R(x,\lambda),\lambda). \tag{2.14}$$

To have regularity with respect to λ we need to impose some uniformity conditions.

Let V be an open set as in Section 2.1.2 and $\varrho > 0$. We rewrite H1, H2 and H3 to become uniform with respect to $\lambda \in \Lambda$ and we add an extra condition:

- Hλ The constants $a_p, a_V > 0$. Moreover $q(x,0,\lambda) = 0$ for $(x,\lambda) \in V_\varrho \times \Lambda$ and either $B_q > 0$ if $M > N$ or $B_q > -Na_p$ if $M = N$.
- HP $D_z^j f(x,y,\lambda) = \mathcal{O}(\|(x,y)\|^{N+1-j})$ and $D_z^j g(x,y,\lambda) = \mathcal{O}(\|(x,y)\|^{M+1-j})$ uniformly in Λ with $z = (x,y)$ and $j = 0, 1$.

We introduce

$$\ell_1 := N - 1 + \frac{B_p}{a_p} + (\eta - 1). \tag{2.15}$$

Theorem 2.8. *Let $F \in \mathcal{C}^{\Sigma_{s,r}}$ be a map of the form (2.11). Let $\varrho_0 > 0$ be such that Hypotheses Hλ, HP hold true and $r > \max\{\ell_0, \ell_1\}$, $s \geq 0$.*

Assume that there exist $K^{\leq} : V_{\varrho_0} \times \Lambda \to U$ and $R : V_{\varrho_0} \times \Lambda \to V_{\varrho_0}$ such that

(a) $K^{\leq}, R \in \mathcal{C}^{\Sigma_{s^{\leq},r^{\leq}}}$.
(b) For $(i,j) \in \Sigma_{s^{\leq},r^{\leq}}$, uniformly over Λ,

$$\Delta K^{\leq}(x,\lambda) := K^{\leq}(x,\lambda) - (x,0) = \mathcal{O}(\|x\|^2), \qquad D_\lambda^i D_x^j \Delta K^{\leq}(x,\lambda) = \mathcal{O}(\|x\|^{2-j}),$$
$$\Delta R(x,\lambda) := R(x,\lambda) - x - p(x,0,\lambda) = \mathcal{O}(\|x\|^{N+1}), \quad D_\lambda^i D_x^j \Delta R(x,\lambda) = \mathcal{O}(\|x\|^{N+1-j}).$$

(c) The invariance equation (2.14) is satisfied up to order $\ell_0 < \ell \leq r$:

$$F(K^{\leq}(x,\lambda),\lambda) - K^{\leq}(R(x,\lambda),\lambda) = \mathcal{O}(\|x\|^{\ell}), \quad \textit{uniformly for } \lambda \in \Lambda.$$

Then the unique function $K^{>} : V_\varrho \times \Lambda \to \mathbb{R}^{n+m}$ found in Theorem 2.4 belongs to $\mathcal{C}^{\Sigma_{s^>,r^>}}$ where $s^>$ and $r^>$ have the following values according to the cases

(1) If $A_p \geq d_p\eta$, $r^> = \min\{r, r^{\leq}\}$ and $s^> \leq \min\{s, s^{\leq}\}$ satisfies

$$s^>(\eta - 1) < r - \frac{B_p}{a_p} - N + 1. \tag{2.16}$$

(2) If $d_p < A_p \leq d_p\eta$, then $r^> \leq \min\{r, r^{\leq}\}$, $s^> \leq \min\{s, s^{\leq}\}$ and

$$r - \frac{B_p}{a_p} - N + 1 - r^> \left(\eta - \frac{A_p}{d_p}\right) > s^>(\eta - 1). \tag{2.17}$$

(3) If $A_p < d_p$, then $r^> \le \min\{r, r^\le\}$, $s^> \le \min\{s, s^\le\}$ and

$$r - \frac{B_p}{a_p} - N + 1 - r^> \left(\eta - \frac{A_p}{d_p}\right) > s^> \left(\eta - \frac{A_p}{d_p}\right). \tag{2.18}$$

(4) If $F \in \mathcal{C}^{\Sigma_{s,\infty}}$, then $r^> = r^\le$ and $s^> = s^\le$.

Finally, if either F, $K^\le$ and R are real analytic or they belong to $\mathcal{C}^{\Sigma_{s,\omega}}$ and $A_p > b_p$, then $K^>$ is either real analytic if item (b) holds true for $i = j = 0$ or $K^> \in \mathcal{C}^{\Sigma_{s,\omega}}$ if item (b) holds true for $j = 0$ respectively.

To finish this section, we formulate an existence result as a corollary of Theorem 2.8 and Theorem 2.7 in [4] which includes the regularity with respect to parameters of the approximate solutions. The following new condition is necessary to ensure the existence of solutions of the invariance equation (2.14) for any value of $\lambda \in \Lambda$:

Hλ' $a_p, a_V > 0$, $q(x, 0, \lambda) \equiv 0$ and the conditions in hypotheses H1', H2' are satisfied for the constants A_p, d_p, B_q, c_p redefined in (2.12).

As we claimed in Lemma 2.7, we have that H1', H2' and H3 are satisfied if Hλ' holds true. Therefore, by the existence Corollary 2.5 there exist K and R satisfying the invariance equation (2.14). Moreover, by construction, $K = K^\le + K^>$ with $K^\le$ provided by Theorem 2.7 in [4].

Corollary 2.9. *Let $F \in \mathcal{C}^{\Sigma_{s,r}}$ be a map of the form (2.11). Assume that there exists $\varrho_0 > 0$ such that Hλ' holds true.*

- *Parametric version of Corollary 2.5: The solutions $K : V_\varrho \times \Lambda \to \mathbb{R}^{n+m}$, $R : V_\varrho \times \Lambda \to V_\varrho$ of the invariance equation provided by Corollary 2.5 belong to $\mathcal{C}^{\Sigma_{s^>,r^>}}$ with $s^>$ and $r^>$ satisfying*
  (1) If $A_p \ge d_p\eta$, $r^> = r$ and $s^> \le s$ satisfying (2.16).
  (2) If $d_p \le A_p < \eta d_p$ or $M < N$, $r^> \le r$, $s^> \le s$ satisfying (2.17).
  (3) If $A_p < d_p$ and $M \ge N$, $r^> \le r$, $s^> \le s$, $r^> + s^> \le r_*$ satisfying (2.18).
  (4) If $F \in \mathcal{C}^{\Sigma_{s,\infty}}$ and $A_p \ge d_p$, then $r^> = \infty$ and $s^> = s$.
  Moreover, if either F is real analytic or it belongs to $\mathcal{C}^{\Sigma_{s,\omega}}$ and $A_p > b_p$, then K is either real analytic or $K \in \mathcal{C}^{\Sigma_{s,\omega}}$ respectively.
- *Parametric version of Corollary 2.6: Let $K_x^j : V_\varrho \times \Lambda \to \mathbb{R}^n$ be $\mathcal{C}^{\Sigma_{s^>,r^>}}$ homogeneous functions of degree j with respect to x. We introduce $K_x^*(x, \lambda) = x + \sum_{j=2}^{\ell-N} K_x^j(x, \lambda)$ as in Corollary 2.6.*
  Then, the function $R^* : V_\varrho \times \Lambda \to \mathbb{R}^n$ provided by Corollary 2.6 belongs to $\mathcal{C}^{\Sigma_{s^>,r^>}}$. Moreover, if $\Delta R : V_\varrho \times \Lambda \to \mathbb{R}^n$ with $\Delta R(x, \lambda) = \mathcal{O}(\|x\|^\ell)$ uniformly in $\lambda \in \Lambda$, belongs to $\mathcal{C}^{\Sigma_{s^>,r^>}}$, then the function K satisfying the invariance equation (2.14) for $R = R^* + \Delta R$ given in Corollary 2.6 also belongs to $\mathcal{C}^{\Sigma_{s^>,r^>}}$.

Proof. For any fixed $\lambda_0 \in \Lambda$, the existence and uniqueness of $K(x, \lambda_0) = \mathcal{O}(\|x\|^{\ell-N+1})$ satisfying the invariance equation (2.14) is guaranteed by Corollary 2.5. To obtain the regularity with respect to the parameter we have to apply Theorem 2.8. To do so we need to discuss Hypothesis

HP. Since for any $\lambda \in \Lambda$, F has the form in (2.1), we have that $Df(x, y, \lambda) = \mathcal{O}(\|(x, y)\|^N)$ and $Dg(x, y, \lambda) = \mathcal{O}(\|(x, y)\|^M)$ but the bounds are not necessarily uniform in λ. Nevertheless, by continuity, for any $\lambda_0 \in \Lambda$ there exists an open ball centered at λ_0, $B_{\rho_0}(\lambda_0) \subset \mathbb{R}^{n'}$, in such a way that HP is satisfied when we restrict the domain of λ to $\Lambda_{\lambda_0} = \Lambda \cap B_{\rho_0}(\lambda_0)$. In addition, restricting ρ_0 if necessary, we can get the approximate solutions $K^{\leq}$, R satisfying items (a), (b) and (c) in Theorem 2.8, that is, with uniform bounds in $\lambda \in \Lambda_{\lambda_0}$.

In conclusion, $K \in \mathcal{C}^{\Sigma_{s>},r>}$ with $(x, \lambda) \in V_\varrho \times \Lambda_{\lambda_0}$. Since $K(\cdot, \lambda)$ is the unique solution of (2.14) of order $\mathcal{O}(\|x\|^{\ell-N+1})$, $K \in \mathcal{C}^{\Sigma_{s>},r>}$ in the full domain $(x, \lambda) \in V_\varrho \times \Lambda$. $\square$

2.3. *Existence results for invariant manifolds. The flow case*

We deduce the analogous result to Corollary 2.5 in the case of time periodic flows, that is, in the case of a flow with a parabolic periodic orbit. To study invariant objects associated to periodic orbits of vector fields (in our case invariant manifolds), one possibility is to consider a Poincaré map in a section transversal to the orbit and then apply the results for fixed points of maps. In this way, one gets the invariant manifolds $W^{\mathrm{s,u}}$ of the Poincaré map and, from them, the invariant manifolds of the periodic orbit by considering all the solutions starting in $W^{\mathrm{s,u}}$. Nevertheless this approach has a drawback: in applications, it is not easy to compute the Poincaré map. Hence, if one wants to compute effectively the invariant manifolds, it is better to have a statement already adapted to the vector field itself.

To shorten the exposition we deal directly with the parametric case. Let $U \subset \mathbb{R}^{n+m}$ be an open neighborhood of the origin, $\Lambda \subset \mathbb{R}^{n'}$ a set of parameters and $X : U \times \mathbb{R} \times \Lambda \to \mathbb{R}^{n+m}$ a T-periodic vector field:

$$\dot{z} = X(z, t, \lambda), \qquad X(z, t + T, \lambda) = X(z, t, \lambda) \tag{2.19}$$

with $z = (x, y) \in U$ having the form

$$X(z, t, \lambda) = X(x, y, t, \lambda) = \begin{pmatrix} p(x, y, \lambda) + f(x, y, t, \lambda) \\ q(x, y, \lambda) + g(x, y, t, \lambda) \end{pmatrix}, \tag{2.20}$$

where p, q, f and g are as in Section 2.2.1. We have this form after having translated the parabolic orbit to the origin.

Let $\varphi(t; t_0, x, y, \lambda)$ be the flow of (2.19). Given a subset $V \subset \mathbb{R}^n$, we define the stable set of the origin over V:

$$W_V^{\mathrm{s}} = \{(x, y) \in U : \varphi_x(t; t_0, x, y, \lambda) \in V,\ t \geq 0,\ \varphi(t; t_0, x, y, \lambda) \to 0 \text{ as } t \to \infty\}$$

and its local version, when we restrict W_V^{s} to the open ball B_ϱ:

$$W_{V,\varrho}^{\mathrm{s}} = \{(x, y) \in U : \varphi_x(t; t_0, x, y, \lambda) \in V_\varrho,\ t \geq 0,\ \varphi(t; t_0, x, \lambda) \to 0 \text{ as } t \to \infty\}.$$

In the case of flows, a parametrization $K(x, t, \lambda)$ is invariant by the flow if there exists a vector field $Y(x, t, \lambda)$ such that

$$X(K(x, t, \lambda), t, \lambda) = D_x K(x, t, \lambda) Y(x, t, \lambda) + \partial_t K(x, t, \lambda) \tag{2.21}$$

or, equivalently

$$\varphi(u;t,K(x,t,\lambda),\lambda) = K(\psi(u;t,x,\lambda),u,\lambda), \qquad \forall u \ge t,\ \forall (x,\lambda) \in V_\varrho \times \Lambda, \tag{2.22}$$

where φ and ψ are the flows of the vector fields X and Y, respectively.

In this section, we will write that a function f belongs to $\mathcal{C}^{\Sigma_{s,r}}$ if it satisfies definition (2.13) with $z = (x, y)$ and $\mu = (\lambda, t)$.

Theorem 2.10. *Let $X \in \mathcal{C}^{\Sigma_{s,r}}$ be a vector field of the form (2.20). Assume that Hypotheses Hλ and HP hold true for some $\varrho_0 > 0$ and $r > \max\{\ell_0, \ell_1\}$. Assume also that there exist $K^\le : V_{\varrho_0} \times \mathbb{R}/(T\mathbb{Z}) \times \Lambda \to U$ and $Y : V_{\varrho_0} \times \Lambda \to V_{\varrho_0}$ such that*

(a) $K^\le, Y \in \mathcal{C}^{\Sigma_{s^\le,r^\le}}$, for some $s^\le, r^\le \ge 1$.
(b) For $(i,j) \in \Sigma_{s^\le,r^\le}$, uniformly over Λ,

$$\Delta K^\le(x,t,\lambda) := K^\le(x,\lambda) - (x,0) = \mathcal{O}(\|x\|^2), \qquad D_\lambda^i D_x^j \Delta K^\le(x,t,\lambda) = \mathcal{O}(\|x\|^{2-j}),$$
$$\Delta Y(x,\lambda) := Y(x,\lambda) - x - p(x,0,\lambda) = \mathcal{O}(\|x\|^{N+1}), \quad D_\lambda^i D_x^j \Delta Y(x,\lambda) = \mathcal{O}(\|x\|^{N+1-j}).$$

(c) The invariance equation (2.21) is satisfied up to order ℓ, $\ell_0 < \ell \le r$:

$$X(K^\le(x,t,\lambda),t,\lambda) - D_x K^\le(x,t,\lambda) Y(x,\lambda) - \partial_t K^\le(x,t,\lambda) = \mathcal{O}(\|x\|^\ell), \tag{2.23}$$

uniformly in $\lambda \in \Lambda$.

Then, there exists $\varrho > 0$ small enough and a unique function $K^> : V_\varrho \times \mathbb{R}/(T\mathbb{Z}) \times \Lambda \to U$ such that $K^>(x,t,\lambda) = \mathcal{O}(\|x\|^{\ell-N+1})$ uniformly in (t,λ) and $K = K^\le + K^>$ satisfies the invariance equation (2.21) with the prescribed vector field Y (or, equivalently, (2.22) with $\psi(u;t,x,\lambda)$ the flow of $\dot{x} = Y(x,\lambda)$).

Moreover, $\psi(u;t,x,\lambda) \to 0$ as $u \to \infty$ and K_x is invertible for any fixed (t,λ). Then we have

$$\{K(x,t,\lambda)\}_{x \in V_\varrho \times \mathbb{R} \times \Lambda} \subset W^{\mathrm{s}}_{V,\varrho}. \tag{2.24}$$

Concerning the regularity of K, we have the same results as the ones stated in Theorem 2.8.

To finish this section we formulate the existence result for the flow case based on the approximate solutions provided in [4]. The proof follows the same lines as the proof of Corollary 2.5

Corollary 2.11. *Let $X \in \mathcal{C}^{\Sigma_{s,r}}$ be a vector field of the form (2.20). Assume that there exists $\varrho_0 > 0$ such that Hypotheses Hλ' and HP hold true and $r > \max\{\ell_0, \ell_1\}$.*

Then, there exist $\varrho > 0$ small enough, a map $K : V_\varrho \times \mathbb{R}/(T\mathbb{Z}) \times \Lambda \to U$ and a vector field $Y : V_\varrho \times \mathbb{R} \to V_\varrho$ solutions of the invariance equation (2.21) satisfying (2.24).

In addition, $K = K^\le + K^>$ with $K^\le$ and Y provided by Theorem 2.8 in [4].

The parametrization K and the vector field Y are $\mathcal{C}^1$ functions at the origin in the sense of Definition 2.2. The regularity on $V_\varrho \times \mathbb{R} \times \Lambda$ is the same as the one stated in Corollary 2.9.

3. An algorithm to compute approximations of the invariant manifolds

In this section we present the algorithm developed in [4] to compute approximate solutions of the invariance equations (1.1) and (2.21).

3.1. Cohomological equations in the case of maps

Let F be the map given by (2.1), which we assume to be of class $\mathcal{C}^r$, with r large enough. Taking advantage of the fact that F can be written as a Taylor polynomial plus some higher order remainder, we look for approximate solutions which are finite sum of homogeneous functions of increasing degree. Then, for any $j \le r - N + 1$, we look for $K^{\le j}$ and $R^{\le j+N-1}$ of the form

$$K^{\le j}(x) = \sum_{l=1}^{j} K^l(x), \qquad R^{\le j+N-1}(x) = x + \sum_{l=N}^{j+N-1} R^l(x), \tag{3.1}$$

with $K^1(x) = (x, 0)^\top$, $R^N(x) = p(x, 0)$ and $K^l, R^l \in \mathcal{H}^l$, satisfying

$$\begin{aligned} E^{>j}(x) &:= F \circ K^{\le j}(x) - K^{\le j} \circ R^{\le j+N-1}(x) \\ &= (E_x^{>j}, E_y^{>j})(x) = \Big(o\big(\|x\|^{j+N-1}\big), o\big(\|x\|^{j+L-1}\big)\Big), \end{aligned} \tag{3.2}$$

where the constant $L = \min\{N, M\}$ was introduced in (2.2). We stress that the superscripts in the above formula have two different meanings. While in $K^{\le j}$ and $R^{\le j+N-1}$ the superscript indicates that they are sums of homogeneous functions of degree less or equal than j and $j + N - 1$, respectively, in $E^{>j}$ denotes that it is the j-th error term. Of course, the order of $E^{>j}$ depends on j but also on N and M and, as it is indicated in formula (3.2), the x and y-components of $E^{>j}$ may have different orders.

If, by induction, we assume that $E^{>j-1} = (E_x^{j+N-1}, E_y^{j+L-1}) + \widehat{E}^{>j}$, where E_*^ℓ, $* = x, y$, is a homogeneous function of degree ℓ and

$$\widehat{E}^{>j}(x) = \Big(o\big(\|x\|^{j+N-1}\big), o\big(\|x\|^{j+L-1}\big)\Big), \tag{3.3}$$

then the functions $K^j =: (K_x^j, K_y^j)$ and R^{j+N-1} must satisfy

$$DK_x^j(x)p(x,0) - D_x p(x,0)K_x^j(x) - D_y p(x,0)K_y^j(x) + R^{j+N-1}(x) = E_x^{j+N-1}(x) \tag{3.4}$$

and, depending on the values of N and M,

$$\text{if } N < M, \qquad DK_y^j(x)p(x,0) = E_y^{j+L-1}(x), \tag{3.5}$$

$$\text{if } N = M, \qquad DK_y^j(x)p(x,0) - D_y q(x,0)K_y^j(x) = E_y^{j+L-1}(x), \tag{3.6}$$

$$\text{if } N > M, \qquad -D_y q(x,0)K_y^j(x) = E_y^{j+L-1}(x). \tag{3.7}$$

At this point it is worth to remark that one could try to find solutions of the above equations in the space of homogeneous polynomials of degree j and $j + N - 1$, respectively. In the case that

$K^{\le j-1}$ and $R^{\le j+N-2}$ are sums of homogeneous polynomials, the error term is also a homogeneous polynomial. But when $N > M$ it is clear that $K_y^j(x) = -D_y q(x,0)^{-1} E_y^{j+L-1}(x)$ cannot be, in general, a polynomial, but a rational function. When $N \le M$, equations (3.5) and (3.6) are $m\binom{j+L+n-2}{n-1}$ conditions while K_y^j, if assumed to be a polynomial, would have only $m\binom{j+n-1}{n-1}$ free coefficients. Hence, since $L \ge 2$, generically these equations only admit polynomial solutions in the case that $n = 1$ (which is the case studied in [3]). It is easy to construct examples where these obstructions do appear. See Section 6 in [4].

Now we summarize how we solve equations (3.4), (3.5), (3.6) and (3.7).

In the case $N > M$, since, as a consequence of hypothesis H2, $D_y q(x,0)$ is invertible, equation (3.7) is trivially solvable in the space of homogeneous functions of degree j.

In the case $N \le M$, let $\varphi(t,x)$ be the flow of

$$\dot{x} = p(x,0).$$

As a consequence of H3, $\varphi(t,x) \in V$, for all $x \in V$ and $t > 0$. We consider the homogeneous linear equations

$$\begin{aligned} \frac{d\psi}{dt}(t,x) &= D_x p(\varphi(t,x),0)\psi(t,x), \\ \frac{d\psi}{dt}(t,x) &= D_y q(\varphi(t,x),0)\psi(t,x) \end{aligned}$$

and we denote by $M_p(t,x)$ and $M_q(t,x)$ their fundamental matrices such that $M_p(0,x) = \mathrm{Id}$, $M_q(0,x) = \mathrm{Id}$, respectively. From Theorem 3.2 in [4], the unique homogeneous solution of equations (3.5) and (3.6) for K_y^j is given by

$$\begin{aligned} K_y^j(x) &= \int_\infty^0 E_y^{j+L-1}(\varphi(t,x))\,dt, && \text{if } N < M, \\ K_y^j(x) &= \int_\infty^0 M_q^{-1}(t,x) E_y^{j+L-1}(\varphi(t,x))\,dt, && \text{if } N = M. \end{aligned} \tag{3.8}$$

Theorem 3.2 in [4] ensures that the above formulas define homogeneous functions of degree j.

The homogeneous solution of (3.7), clearly unique, is

$$K_y^j(x) = \big(D_y q(x,0)\big)^{-1} E_y^{j+L-1}(x), \qquad \text{if } N > M.$$

As for (3.4), notice that it is always possible to solve it by choosing K_x^j an arbitrary homogeneous function of degree j and taking

$$R^{j+N-1}(x) = E_x^{j+L-1}(x) - D_y p(x,0) K_y^j(x) + D_x p(x,0) K_x^j(x) - DK_x^j(x) p(x,0). \tag{3.9}$$

However, we prove in [4] that, provided that r is large enough, there exists ℓ_* (which depends explicitly on the constants defined in (2.4)) such that if $\ell_* - N + 2 \leq j$, R^{j+N-1} can be chosen as an arbitrary homogeneous function of degree $j + N - 1$ and

$$K_x^j(x) = \int_\infty^0 M_p^{-1}(t,x)\big[E_x^{j+L-1}(\varphi(t,x)) - R^{j+N-1}(\varphi(t,x)) \\ - D_y p(\varphi(t,x),0)K_y^j(\varphi(t,x))\big]\,dt. \quad (3.10)$$

For instance, one can choose R^{j+N-1} to be 0, if $j \geq \ell_* - N + 2$, which implies that the function $R^{\leq j+N-1}$ in (3.1) can be taken as a *finite* sum of homogeneous functions.

3.2. Cohomological equations in the case of flows

Let $U \subset \mathbb{R}^{n+m}$ a neighborhood of the origin and $X : U \times \mathbb{R} \to \mathbb{R}^{n+m}$ be a T-periodic vector field of the form (2.20). We look for K and Y of the form

$$K^{\leq j}(x,t) = \sum_{l=1}^{j} K^{(l)}(x,t), \qquad Y^{\leq j+N-1}(x) = \sum_{l=N}^{j+N-1} Y^l(x)$$

with $K_1(x,t) = (x,0)^\top$, $Y^N(x) = p(x,0)$ and $K^{(l)}$ a sum of two homogeneous functions: one of degree l independent of t and the other one of order $\big(o(\|x\|^{j+N-1}), o(\|x\|^{j+L-1})\big)$. The homogeneous terms K^l are obtained by rearranging the sum above. They have to satisfy the invariance equation (2.21) up to some order j in the sense that the error term

$$E^{>j}(x,t) := X(K^{\leq j}(x,t),t) - DK^{\leq j}(x,t)Y^{\leq j+N-1}(x) - \partial_t K^{\leq j}(x,t)$$

satisfies

$$E^{>j}(x) = (E_x^{>j}, E_y^{>j})(x) = \Big(o\big(\|x\|^{j+N-1}\big), o\big(\|x\|^{j+L-1}\big)\Big). \quad (3.11)$$

If, by induction, we assume that (3.3) is satisfied (taking into account the time dependence) the functions $K^{(j)} = (K_x^{(j)}, K_y^{(j)})$ and Y^{j+N-1} must satisfy

$$DK_x^{(j)}(x,t)p(x,0) - D_x p(x,0)K_x^{(j)}(x,t) - D_y p(x,0)K_y^{(j)}(x,t) \\ + Y^{j+N-1}(x) + \partial_t K_x^{(j)}(x,t) - E_x^{j+N-1}(x,t) = o\big(\|x\|^{j+N-1}\big), \quad (3.12)$$

and

$$DK_y^{(j)}(x,t)p(x,0) - D_y q(x,0)K_y^{(j)}(x,t) + \partial_t K_y^{(j)}(x,t) - E_y^{j+L-1}(x,t) \\ = o\big(\|x\|^{j+L-1}\big). \quad (3.13)$$

Equation (3.13), depending on the values of N and M, reads

$$\begin{aligned}
&\text{if } N<M, \quad DK_y^{(j)}(x,t)p(x,0) && +\partial_t K_x^{(j)}(x,t)-E_y^{j+L-1}(x,t) \\
& && = o\big(\|x\|^{j+L-1}\big),\\
&\text{if } N=M, \quad DK_y^{(j)}(x,t)p(x,0) && -D_y q(x,0)K_y^{(j)}(x,t)+\partial_t K_x^{(j)}(x,t)-E_y^{j+L-1}(x,t) \\
& && = o\big(\|x\|^{j+L-1}\big),\\
&\text{if } N>M, && -D_y q(x,0)K_y^{(j)}(x,t)+\partial_t K_x^{(j)}(x,t)-E_y^{j+L-1}(x,t) \\
& && = o\big(\|x\|^{j+L-1}\big).
\end{aligned}$$

We remark that, unlike the case of equations (3.4) to (3.7), the functions $K^{(j)}$ and Y^{j+N-1} we obtain cancel out the error term in (3.12) and (3.13) but introduce new terms of higher order.

For a T-periodic function h, we denote by $\overline{h}$ its mean, that is,

$$\overline{h}(x)=\frac{1}{T}\int_0^T h(x,t)\,dt,$$

and $\widetilde{h}=h-\overline{h}$ its oscillatory part. If equations (3.12) and (3.13) are satisfied for some $K^{(j)}$ periodic, then it is clear that the mean $\overline{K^{(j)}}$ has to satisfy the equations

$$\begin{aligned}
D\overline{K_x^{(j)}}(x)p(x,0)-D_x p(x,0)\overline{K_x^{(j)}}(x)-D_y p(x,0)\overline{K_y^{(j)}}(x)& \\
+Y^{j+N-1}(x)-\overline{E_x^{j+N-1}}(x)&=o\big(\|x\|^{j+N-1}\big), \\
D\overline{K_y^{(j)}}(x)p(x,0)-D_y q(x,0)\overline{K_y^{(j)}}(x)-\overline{E_y^{j+L-1}}(x)&=o\big(\|x\|^{j+L-1}\big).
\end{aligned} \tag{3.14}$$

These equations can be solved in the same way as (3.4), (3.5), (3.6) and (3.7), in the previous section. We conclude that $\overline{K^{(j)}}$ and Y^{j+N-1} exist and they both have the appropriate orders, i.e., degree j and $j+N-1$ respectively.

Now we impose that

$$\partial_t \widetilde{K^{(j)}}(x,t)=(\widetilde{E_x^{j+N-1}}(x,t),\widetilde{E_y^{j+L-1}}(x,t)) \tag{3.15}$$

and that $\widetilde{K^{(j)}}$ has zero mean. Consequently, $\widetilde{K^{(j)}}(x)=\big(o\big(\|x\|^{j+N-1}\big),o\big(\|x\|^{j+L-1}\big)\big)$.

We conclude that $K^{(j)}=\overline{K^{(j)}}+\widetilde{K^{(j)}}$ and Y^{j+N-1} satisfy equations (3.12) and (3.13) and then (3.11) is satisfied.

Remark 3.1. The $K^{(j)}$ found are not homogeneous functions, but sums of homogeneous functions. Concretely, $K_x^{(j)}$ has a term of order j and another of order $j+N-1$. Analogously, $K_y^{(j)}$ has a term of order j and another of order $j+L-1$.

4. Example. The elliptic spatial restricted three body problem

We have pointed out in the previous section that, in general, the invariant manifolds of a parabolic fixed point do not have polynomial expansions if their dimension is greater than one, regardless of the regularity of the map. However, it may be possible that the system of equations defined by (3.4) and (3.5)–(3.7) admits polynomial homogeneous solutions. Here we take advantage of the expressions (3.8), (3.9) and (3.10) to show that this is the case of the *parabolic infinity* in the elliptic spatial restricted three body problem.

The spatial elliptic restricted three body problem is a simplified version of the spatial three body problem where one of the bodies is assumed to have zero mass while the other two, named the primaries, evolve describing Keplerian ellipses around their center of mass.

We introduce $\hat{q}(f) = (\rho(f)\cos f, \rho(f)\sin f, 0)$, where, for a given eccentricity $0 \le e < 1$,

$$\rho(f) = \frac{1-e^2}{1+e\cos f}.$$

Rescaling time and mass units, we can assume that the masses of the primaries are μ and $1-\mu$, respectively, and their positions are given by $q_1 = \mu\hat{q}$ and $q_2 = -(1-\mu)\hat{q}$, where f denotes the so-called true anomaly which satisfies

$$\frac{df}{dt} = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}.$$

Then, denoting by $q \in \mathbb{R}^3$ the position of the third body, the equations for q are

$$\ddot{q} = -(1-\mu)\frac{q-q_1}{r_1^3} - \mu\frac{q-q_2}{r_2^3},$$

where $r_i = \|q - q_i\|$, $i = 1, 2$. Introducing the momenta $p = \dot{q}$, this system is Hamiltonian with respect to

$$H(q, p, t) = \frac{\|p\|^2}{2} - U(q, t), \qquad U(q, t) = \frac{1-\mu}{r_1} + \frac{\mu}{r_2}.$$

Our aim is to study the parabolic invariant manifolds of infinity. To this end, we consider spherical coordinates (r, α, θ) in $\mathbb{R}^3$, namely $q = (r\cos\alpha\cos\theta, r\sin\alpha\cos\theta, r\sin\theta)$. Let (R, A, Θ) be their conjugated momenta, which can be obtained through a Mathieu transformation. They satisfy

$$p = m(r, \alpha, \theta)\begin{pmatrix} R \\ A \\ \Theta \end{pmatrix}, \qquad m(r, \alpha, \theta) = \begin{pmatrix} \cos\alpha\cos\theta & -\frac{\sin\alpha}{r\cos\theta} & -\frac{\cos\alpha\sin\theta}{r} \\ \sin\alpha\cos\theta & \frac{\cos\alpha}{r\cos\theta} & -\frac{\sin\alpha\sin\theta}{r} \\ \sin\theta & 0 & \frac{\cos\theta}{r} \end{pmatrix}.$$

The new Hamiltonian is

$$\hat{H}(r, \alpha, \theta, R, A, \Theta, t) = \frac{1}{2}\left(\frac{A^2}{r^2\cos^2\theta} + \frac{\Theta^2}{r^2} + R^2\right) - \hat{U}(r, \alpha, \theta, t),$$

with

$$
\begin{aligned}
\hat{U}(r, \alpha, \theta, t) =& \frac{1-\mu}{\sqrt{r^2 - 2\mu\rho(f) r\cos(\alpha - f)\cos\theta + \mu^2\rho^2(f)}} \\
&+ \frac{\mu}{\sqrt{r^2 + 2(1-\mu)\rho(f) r\cos(\alpha - f)\cos\theta + (1-\mu)^2\rho^2(f)}} \\
=& \frac{1}{r} - \frac{\mu(1-\mu)}{2}(1 - 3\cos(\alpha - f))\frac{\rho^2(f)\cos^2 f\cos^2\theta}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right).
\end{aligned}
\tag{4.1}
$$

To study the behavior of the system at $r = \infty$, we perform the non-canonical change of variables due to McGehee $r = 2/z^2$. Since $\dot{r} = R$ and the change does not involve the remaining variables, the equations of motion in the new variables are

$$
\begin{aligned}
\dot{z} &= -\frac{1}{4}z^3 R \\
\dot{\alpha} &= \partial_A \hat{H}_{|r=2/z^2} = \frac{Az^4}{4\cos^2\theta} \\
\dot{\theta} &= \partial_\Theta \hat{H}_{|r=2/z^2} = \frac{1}{4}\Theta z^4 \\
\dot{R} &= -\partial_r \hat{H}_{|r=2/z^2} = \frac{A^2 z^6}{8\cos^2\theta} + \frac{\Theta^2 z^6}{8} + \partial_r \hat{U}(2/z^2, \alpha, \theta, t) = -\frac{1}{4}z^4 + \mathcal{O}(z^6) \\
\dot{A} &= -\partial_\alpha \hat{H}_{|r=2/z^2} = \partial_\alpha \hat{U}(2/z^2, \alpha, \theta, t) = \mathcal{O}(z^6) \\
\dot{\Theta} &= -\partial_\theta \hat{H}_{|r=2/z^2} = -\frac{A^2 z^4 \sin\theta}{4\cos^3\theta} + \partial_\theta \hat{U}(2/z^2, \alpha, \theta, t) = -\frac{A^2 z^4\sin\theta}{4\cos^3\theta} + \mathcal{O}(z^6).
\end{aligned}
$$

Notice that the set $\{z = 0,\ R = 0\}$ is invariant and foliated by fixed points. We focus on those with $\theta = \Theta = 0$, $\alpha = \alpha_0$, $A = A_0$. To apply our theory, we perform the following local change of variables

$$
\hat{\theta} = \frac{\theta}{z}, \quad \hat{\Theta} = \frac{z\Theta}{\theta}, \quad \hat{\alpha} = \frac{\alpha - \alpha_0 + AR}{z}, \quad \hat{A} = \frac{A - A_0}{z},
$$

which transforms the system into

$$
\begin{aligned}
\dot{z} &= -\frac{1}{4}z^3 R & \dot{R} &= -\frac{1}{4}z^4 + z^6\mathcal{O}_0 \\
\dot{\hat{\alpha}} &= \frac{1}{4}z^2 R\hat{\alpha} + z^5\mathcal{O}_0 & \dot{\hat{A}} &= \frac{1}{4}\hat{A}z^2 R + z^5\mathcal{O}_0 \\
\dot{\hat{\theta}} &= \frac{1}{4}z^2 R\hat{\theta} + \frac{1}{4}z^3\hat{\theta}\hat{\Theta} & \dot{\hat{\Theta}} &= -\frac{1}{4}z^2 R\hat{\Theta} - \frac{1}{2}z^3\hat{\Theta}^2 + z^5\mathcal{O}_0,
\end{aligned}
$$

where all the terms up to degree 6 in the local variables are shown (we write $\mathcal{O}_k$ meaning $\mathcal{O}(\|(z, R, \hat{\alpha}, \hat{A}, \hat{\theta}, \hat{\Theta})\|^k)$). Notice that the leading terms are of degree 4. Finally, it will be convenient to introduce

$$u = \frac{1}{2}(z + R), \qquad v = \frac{1}{2}(z - R)$$

so that the system, reordering equations, becomes

$$\begin{aligned}
\dot{u} &= -\frac{1}{4}(u+v)^3 u + (u+v)^6 \mathcal{O}_0 \\
\dot{\hat{\Theta}} &= -\frac{1}{4}(u+v)^2(u-v)\hat{\Theta} - \frac{1}{2}(u+v)^3\hat{\Theta}^2 + (u+v)^5\mathcal{O}_0 \\
\dot{v} &= \frac{1}{4}(u+v)^3 v + (u+v)^6 \mathcal{O}_0 \\
\dot{\hat{\alpha}} &= \frac{1}{4}(u+v)^2(u-v)\hat{\alpha} + (u+v)^5\mathcal{O}_0 \\
\dot{\hat{A}} &= \frac{1}{4}(u+v)^2(u-v)\hat{A} + (u+v)^5\mathcal{O}_0 \\
\dot{\hat{\theta}} &= \frac{1}{4}(u+v)^2(u-v)\hat{\theta} + \frac{1}{2}(u+v)^3\hat{\theta}\hat{\Theta}.
\end{aligned} \tag{4.2}$$

We emphasize that the leading terms do not depend on t, but the remainders do depend 2π-periodically on t.

Let X denote the vector field defined by (4.2). We can write the vector field in the form (2.20) taking $x = (u, \hat{\Theta})$, $y = (v, \hat{\alpha}, \hat{A}, \hat{\theta})$ and

$$p(x,y) = \begin{pmatrix} -\frac{1}{4}(u+v)^3 u \\ -\frac{1}{4}(u+v)^2(u-v)\hat{\Theta} \end{pmatrix}, \qquad q(x,y) = \begin{pmatrix} \frac{1}{4}(u+v)^3 v \\ \frac{1}{4}(u+v)^2(u-v)\hat{\alpha} \\ \frac{1}{4}(u+v)^2(u-v)\hat{A} \\ \frac{1}{4}(u+v)^2(u-v)\hat{\theta} \end{pmatrix}. \tag{4.3}$$

Theorem 4.1. *Let W be a perturbation of $\hat{U}$ in (4.1) of the form $W = \hat{U} + V$, where*

$$V(r,\alpha,\theta,t) = \frac{1}{r^3}\hat{V}(r,\alpha,\theta,t)$$

(in spherical variables) is such that the equations of motion leave the plane $\theta = \Theta = 0$ invariant (that is, $\partial_\theta V_{|\theta=0} = 0$) and $\hat{V}$ is analytic in $1/r$ and the rest of its arguments.

Then, after the changes of variables described above, the equations of motion are given by (4.2). The origin is a parabolic fixed point. It has an analytic stable invariant two dimensional manifold which admits a parametrization of the form $K(x,t) = (x,0) + \tilde{K}(x,t)$, where

$$\tilde{K}(x,t) = \mathcal{O}(\|x\|^2), \qquad u > 0,\ \tilde{\Theta} > 0$$

such that

$$X(K(x,t),t) = DK(x,t)Y(x) + \partial_t K(x,t),$$

with $Y(x) = p(x,0) + \mathcal{O}(\|x\|^5)$ is a polynomial of degree 7.

The function $\tilde{K}(x,t)$ is 2π-periodic in t and, for all $\ell \geq 7$,

$$\tilde{K}(x,t) = \sum_{j=2}^{\ell} \breve{K}^j(x,t) + \mathcal{O}(\|x\|^{\ell+1}),$$

where $\breve{K}^j$ are homogeneous polynomials, with respect to x, of degree j. That is, the stable invariant manifold admits polynomial approximation up to any order.

Proof. System (4.2) satisfies hypotheses (a) and (b) of Theorem 2.10. Hence, in order to obtain the claim, we only need to check hypothesis (c). It is enough to find approximate solutions of the invariance equation

$$X(K(x,t),t) - DK(x,t)Y(x) - \partial_t K(x,t) = 0. \tag{4.4}$$

We show that there are indeed approximate solutions of this equation up to any order and that these solutions are sums of homogeneous polynomials.

We use the construction described in Section 3.2 to find approximate solutions of the above equation. The procedure applies in the region $\{u > 0, \hat{\Theta} > 0\}$.

The explicit expression of the flow of the vector field $p(x,0)$, with p on (4.3), is

$$\varphi(t,x) = \frac{1}{(1+\frac{3}{4}tu^3)^{1/3}} \begin{pmatrix} u \\ \hat{\Theta} \end{pmatrix}.$$

Let $M_p(t,x)$ and $M_q(t,x)$ be the fundamental matrices of the linear equations

$$\begin{aligned} \frac{d\psi}{dt}(t,x) &= D_x p(\varphi(t,x),0)\psi(t,x) \\ \frac{d\psi}{dt}(t,x) &= D_y q(\varphi(t,x),0)\psi(t,x) \end{aligned}$$

such that $M_p(0,x) = \mathrm{Id}$ and $M_q(0,x) = \mathrm{Id}$, respectively. We have that

$$M_p^{-1}(t,x) = \begin{pmatrix} \left(1+\frac{3}{4}tu^3\right)^{4/3} & 0 \\ \frac{3}{4}tu^2\hat{\Theta}\left(1+\frac{3}{4}tu^3\right)^{1/3} & \left(1+\frac{3}{4}tu^3\right)^{1/3} \end{pmatrix}$$

and

$$M_q(t,x) = \left(1+\frac{3}{4}tu^3\right)^{1/3} \mathrm{Id}_{4\times 4}.$$

Along this proof we will deal with several objects that will be homogeneous polynomials. Their superscripts will denote their degree. A slightly different notation is used for $E^{>j}$. See (3.2)-(3.3).

We write the vector field in (4.2) as $X = \sum_{l\geq 4} X^l$, where X^l depends 2π-periodically on t. Following the algorithm described in Section 3.2 with $N = M = 4$, we look for solutions of the equation (4.4) of the form

$$K(x,t)=\sum_{l\geq 1}K^{(l)}(x,t), \qquad Y(x)=\sum_{l\geq 4}Y^l(x),$$

where $K^{(l)}$ depends 2π-periodically in t and it is of the form

$$K^{(l)}(x,t)=K^l(x)+\widetilde{K}^{l+3}(x,t), \qquad \text{with } \widetilde{K}^{l+3}=\widetilde{K^{(l)}},$$

and $K^1(x)=(x,0)$, $\widetilde{K}^4(x,t)=0$, $Y^4(x)=p(x,0)$. The homogeneous functions $\breve{K}^l$ in the statement of the theorem are obtained by rearranging the sum above.

We recall that $x=(u,\hat{\Theta})$ and $y=(v,\hat{\alpha},\hat{A},\hat{\theta})$. It is clear from (4.2) that the homogeneous polynomials X^l satisfy that

$$\begin{aligned}
&X^5_\xi(x,0,t)=u^5\hat{X}^0_\xi(x,t), && \xi=\hat{\alpha},\hat{A},\hat{\theta},\\
&X^5_u(x,0,t)=X^5_v(x,0,t)=0, &&\\
&X^5_{\hat{\Theta}}(x,0,t)=-\frac{1}{2}u^3\hat{\Theta}^2+u^5\hat{X}^0_{\hat{\Theta}}(x,t), &&
\end{aligned} \tag{4.5}$$

and, for $l\geq 6$,

$$\begin{aligned}
&X^l(x,0,t)=u^5\hat{X}^{l-5}(x,t)\\
&X^l_\xi(x,0,t)=u^6\hat{X}^{l-6}_\xi(x,t), \qquad \xi=u,v.
\end{aligned}$$

The statement is a consequence of the following claim. We make the convention that $\mathcal{O}_j=0$ if $j<0$.

Claim 4.2.

(i) $K^j(x)=u^2\mathcal{O}_{j-2}$, $K^j_{u,v}(x)=u^3\mathcal{O}_{j-3}$, $j\geq 2$. $K^j_x=0$ *if* $2\leq j\leq 7$. $\widetilde{K}^{j+3}(x,t)=u^5\mathcal{O}_{j-2}$, $\widetilde{K}^{j+3}_{u,v}(x,t)=u^6\mathcal{O}_{j-2}$, $j\geq 3$.

(ii) $Y^5(x)=\begin{pmatrix} a_1u^5\\ u^3(a_2\hat{\Theta}^2+a_3u^2)\end{pmatrix}$, *with* $a_1,a_2,a_3\in\mathbb{R}$, $Y^j(x)=(u^6\mathcal{O}_{j-6},u^5\mathcal{O}_{j-5})^\top$, $6\leq j\leq 7$ *and* $Y^j=0$ *for* $j\geq 8$.

(iii) Denoting $K^{\leq j}=\sum_{l=1}^j(K^l+\widetilde{K}^{l+3})$, $Y^{\leq j}=\sum_{l=4}^j Y^l$,

$$E^{>j}(x,t)=X(K^{\leq j}(x,t),t)-DK^{\leq j}(x,t)Y^{\leq j+3}(x)-\partial_t K^{\leq j}(x,t),$$

and $E^{>j}=E^{j+4}+\hat{E}^{>j+1}$ *with* $E^{j+4}(x,t)=\mathcal{O}(\|x\|^{j+4})$, $\hat{E}^{>j+1}(x,t)=o(\|x\|^{j+4})$, *then,*

$$E^{j+4}(x,t)=u^5\mathcal{O}_{j-1}, \quad E^{j+4}_{u,v}(x,t)=u^6\mathcal{O}_{j-2}, \quad j\geq 2.$$

In (i) and (ii) the terms $\mathcal{O}_j$ are homogeneous polynomials *in x of degree j while in (iii) $\mathcal{O}_j$ are analytic functions in x of order j.*

The following fact will be used repeatedly without mention. Given any monomial $Z(x) = u^{j_1}\hat{\Theta}^{j_2}$ and denoting $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$, there exist $c_i \in \mathbb{R}$, depending only on j_1 and j_2, such that

$$\int_\infty^0 M_p^{-1}(t,x)Z(\varphi(t,x))e_1\,dt = c_1\frac{Z(x)}{u^3}e_1 + c_2\frac{Z(x)}{u^4}\hat{\Theta}e_2,$$
$$\int_\infty^0 M_p^{-1}(t,x)Z(\varphi(t,x))e_2\,dt = c_3\frac{Z(x)}{u^3}e_2$$

and, denoting $\{e'_j\}_{j=1,\dots,4}$ the canonical basis of $\mathbb{R}^4$, there exists $c \in \mathbb{R}$, depending only on j_1 and j_2, such that

$$\int_\infty^0 M_q^{-1}(t,x)Z(\varphi(t,x))e'_j\,dt = c\frac{Z(x)}{u^3}e'_j, \qquad j = 1,\dots,4.$$

Indeed, it suffices to make the change $s = tu^3$ in the integrals. Obviously, the previous integrals are only convergent when $j_1 + j_2 \geq 8$, for the first one, when $j_1 + j_2 \geq 5$, for the second one and $j_1 + j_2 \geq 3$ for the last one.

We prove the claim by induction. We start with the case $j = 2$.

According to the algorithm, using that $X^4 = (p,q)^\top$ and $Y^4(x) = p(x,0)$ in equations (3.14) for $j = 2$, the functions K^2 and Y^5 must satisfy

$$DK^2Y^4 - (DX^4 \circ K^1)K^2 + \begin{pmatrix}\mathrm{Id}\\ 0\end{pmatrix}Y^5 = \overline{E^5},$$

where $E^5 = X^5 \circ K^1$ denotes the terms of degree 5 of $E^{>1}$ and we recall that $\overline{Z}$ denotes the mean of a periodic function Z. Using (3.8), the equation for K_y^2 has the homogeneous solution

$$K_y^2(x) = \int_\infty^0 M_q^{-1}(t,x)\overline{E_y^5}(\varphi(t,x))\,dt. \tag{4.6}$$

Since, in view of (4.5), $\overline{X_y^5} \circ K^1(x) = \overline{X_y^5}(x,0) = a_0u^5$, we have that $K_y^2(x) = b_0u^2$, where $a_0, b_0 \in \mathbb{R}^4$. Furthermore, since $\overline{X_v^5} = 0$ and M_q is a diagonal matrix, we deduce that $K_v^2 = 0$.

Once K_y^2 is found, we take $K_x^2 = 0$ and choose appropriately Y^5, that is,

$$Y^5(x) = D_yX_x^4(x,0)K_y^2(x) + \overline{X_x^5} \circ K^1(x) = \begin{pmatrix} a_1u^5 \\ a_2u^3\hat{\Theta}^2 + a_3u^5\end{pmatrix},$$

with $a_1, a_2, a_3 \in \mathbb{R}$, where we have used that, since $K_v^2 = 0$, and

$$D_y X_x^4(x,0) = D_y p(x,0) = \begin{pmatrix} -\frac{3}{4}u^3 & 0 & 0 & 0 \\ -\frac{1}{4}u^2\hat{\Theta} & 0 & 0 & 0 \end{pmatrix}, \tag{4.7}$$

we have that $D_y p(x,0)K_y^2(x) = 0$. This accounts for the first part of (ii).

To cancel the oscillatory part of E^5 we use (3.15) and we choose $\widetilde{K}^5$ with zero mean such that

$$\partial_t \widetilde{K}^5(x,t) = \widetilde{E^5}(x,t).$$

From (4.5) we get that $\widetilde{K}^5(x,t) = u^5\mathcal{O}_0$ and $\widetilde{K}_u^5(x,t) = \widetilde{K}_v^5(x,t) = 0$.

With this choice of $K^{\le 2}$ and $Y^{\le 5}$ the algorithm ensures that the remainder $E^{>2}(x,t) = \mathcal{O}_6$. We have that

$$\begin{aligned} E^{>2}(x,t) =& X(K^{\le 2}(x,t),t) - DK^{\le 2}(x,t)Y^{\le 5}(x) - \partial_t K^{\le 2}(x,t) \\ =& X^6(x,0,t) + DX^5(x,0,t)K^2(x) + \frac{1}{2}D^2X^4(x,0)K^2(x)^{\otimes 2} \\ & - DK^2(x)Y^5(x) + u^5\mathcal{O}_1 \\ =& u^5\mathcal{O}_1. \end{aligned}$$

The last equality uses that $K_x^2 = 0$, $K_v^2 = 0$, $K_y^2(x) = u^2\mathcal{O}_0$, the particular form of Y^5, X^5, X^6 and the fact that $\frac{\partial^2 X^4}{\partial y^2}(x,0) = u\mathcal{O}_1$. Moreover, using that $K_x^2 = 0$, $K_v^2 = 0$, $X_{u,v}^5 = 0$ and $X_v^6(x,0,t) = u^6\mathcal{O}_0$ one obtains that

$$E_{u,v}^{>2}(x,t) = u^6\mathcal{O}_0. \tag{4.8}$$

This proves the claim for $j = 2$.

Now we assume that we have obtained $K^{\le j-1}$, $Y^{\le j+2}$ and $E^{>j-1}$, with $j \ge 3$, satisfying the induction hypotheses. The equation for K^j and Y^{j+3} is

$$DK^jY^4 - (DX^4 \circ K^1)K^j + \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix} Y^{j+3} = \overline{E^{j+3}}.$$

The function K_y^j is obtained as we did for K_y^2 in (4.6).

Since $E^{j+3}(x,t) = u^5\mathcal{O}_{j-2}$, we obtain that $K_y^j(x) = u^2\mathcal{O}_{j-2}$. By the same argument, using (4.8) and that M_q is a diagonal matrix, one has $K_v^j(x) = u^3\mathcal{O}_{j-3}$.

To find K_x^j and Y^{j+3} we proceed in two different ways according to whether $j \le 4$ or $j \ge 5$. The point is that for $j \ge 5$ we can take $Y^{j+3} = 0$ choosing appropriately K_x^j. However, for $j \le 4$, the integrals involved in the computation of K_x^j do not converge.

If $j \le 4$, we choose $K_x^j = 0$ and

$$Y^{j+3}(x) = D_yX_x^4 \circ K^1(x)K_y^j(x) + \overline{E_x^{j+3}}(x).$$

Formula (4.7) and the induction hypothesis gives that $Y^{j+3}(x) = (u^6\mathcal{O}_{j-3}, u^5\mathcal{O}_{j-2})^\top$.

Instead, if $j \geq 5$, we choose $Y^{j+3} = 0$ and

$$K_x^j(x) = \int_\infty^0 M_p^{-1}(t,x)\overline{E_x^{j+3}}(\varphi(t,x))\,dt.$$

The induction hypothesis on E^{j+3} gives $K_x^j(x) = (u^3\mathcal{O}_{j-3}, u^2\mathcal{O}_{j-2})^\top$.

We choose $\widetilde{K}^{j+3}$ with zero mean such that $\partial_t \widetilde{K}^{j+3}(x,t) = \widetilde{E}^{j+3}(x,t)$. Again from the induction hypothesis we get that $\widetilde{K}^{j+3}(x,t) = u^5\mathcal{O}_{j-2}$ and $\widetilde{K}^{j+3}_{u,v}(x,t) = u^6\mathcal{O}_{j-3}$.

Finally, we need to check the properties of E^{j+4}. From the definition of $E^{>j}$,

$$\begin{aligned} E^{>j}(x,t) =& E^{>j-1}(x,t) + X(K^{\leq j}(x,t),t) - X(K^{\leq j-1}(x,t),t) \\ & - (DK^{\leq j}(x,t)Y^{\leq j+3}(x) - DK^{\leq j-1}(x,t)Y^{\leq j+2}(x)) \\ & - (\partial_t K^{\leq j}(x,t) - \partial_t K^{\leq j-1}(x,t)) \\ =& E^{>j-1}(x,t) + T_1(x,t) - T_2(x,t) - T_3(x,t), \end{aligned}$$

where T_i are defined in the obvious way.

We have

$$T_1 = \int_0^1 DX(K^{\leq j-1} + s(K^j + \tilde{K}^{j+3}),t)(K^j + \tilde{K}^{j+3})\,ds.$$

Taking into account the structure of X in (4.2) a long but straightforward computation gives

$$T_1(x,t) = u^5\mathcal{O}_{j-2}, \quad (T_1)_{u,v}(x,t) = u^6\mathcal{O}_{j-3}.$$

For T_2 we have

$$T_2 = DK^{\leq j-1}Y^{j+3} + (DK^j + D\tilde{K}^{j+3})Y^{\leq j+3}.$$

A simple calculation gives that for $j \geq 3$,

$$\begin{aligned} DK^{\leq j-1}(x,t) &= \begin{pmatrix} 1\mathcal{O}_0 & u\mathcal{O}_0 & u^2\mathcal{O}_0 & u\mathcal{O}_0 & u\mathcal{O}_0 & u\mathcal{O}_0 \\ u^3\mathcal{O}_0 & 1\mathcal{O}_0 & u^3\mathcal{O}_0 & u^2\mathcal{O}_0 & u^2\mathcal{O}_0 & u^2\mathcal{O}_0 \end{pmatrix}^\top, \\ Y^{j+3}(x) &= \begin{pmatrix} u^6\mathcal{O}_0 \\ u^5\mathcal{O}_0 \end{pmatrix}, \\ (DK^{\leq j} + D\tilde{K}^{j+3})(x,t) &= \begin{pmatrix} u^2\mathcal{O}_0 & u\mathcal{O}_0 & u^2\mathcal{O}_0 & u\mathcal{O}_0 & u\mathcal{O}_0 & u\mathcal{O}_0 \\ u^3\mathcal{O}_0 & u^2\mathcal{O}_0 & u^3\mathcal{O}_0 & u^2\mathcal{O}_0 & u^2\mathcal{O}_0 & u^2\mathcal{O}_0 \end{pmatrix}^\top \end{aligned}$$

and

$$Y^{\leq j+3}(x) = \begin{pmatrix} u^5\mathcal{O}_0 \\ u^3\mathcal{O}_2 \end{pmatrix},$$

where here $\mathcal{O}_j$ denotes a polynomial in x of order j. This implies

$$T_2(x,t) = u^5\mathcal{O}_{j-2}, \qquad (T_2)_{u,v}(x,t) = u^6\mathcal{O}_{j-3}.$$

Finally, $T_3 = \partial_t \tilde{K}^{j+3}$ and the induction hypotheses gives (iii) for j.

Note however that some terms of T_1, T_2 and T_3 do not contribute because their order is less or equal than $j+3$ and are compensated by the choice of the K's and Y's. $\square$

5. Examples

In this section we provide examples showing that hypotheses H1, H2 and H3 are necessary for the existence of the invariant manifolds. We also show that the manifolds may be much less regular than the map.

5.1. A toy model

The first example corresponds to a map without stable invariant manifold but satisfying both H1 and H2.

Let φ be the flow of the equations in $\mathbb{R}^2 \times \mathbb{R}$

$$\dot{x}_1 = -x_1^2, \qquad \dot{x}_2 = -ax_1x_2, \qquad \dot{y} = bx_1y + x_2^3,$$

being $a, b > 0$, and $F(x,y) = \varphi(1; x, y)$, with $x = (x_1, x_2)$, its time 1 map.

Claim 5.1. *There exists $V \subset \mathbb{R}^2$, star-shaped with respect to the origin, such that F satisfies hypotheses H1 and H2 in V.*

If $b + 3a \leq 1$, F has no invariant stable manifold over V of the origin.

The map F has the form (2.1) with $p(x,y) = (-x_1^2, -ax_1x_2)$ and $q(x,y) = bx_1y$. We introduce

$$W = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : |x_2| < (1-a)x_1 < \frac{2}{a+1} \right\}.$$

First we note that the map F satisfies hypotheses H1 and H2 with the supremum norm in any open set V contained in W. Of course the constants A_p, B_q will depend on V. However we claim that there is no invariant set for F_x contained in W. As a consequence, hypothesis H3 can not be satisfied. Indeed, assume that $x^0 = (x_1, x_2) \in W$, and consider

$$x^n = F_x(x^{n-1}) = F_x^n(x^0) = \left(\frac{x_1}{1 + nx_1}, \frac{x_2}{\left(1 + nx_1\right)^a} \right).$$

The sequence $x^n \in W$ if and only if $x_1 \geq |x_2|\left(1 + nx_1\right)^{1-a}$, $\forall n \geq 0$, which is not true since $x_1 > 0$ and $a < 1$.

Now we check that the map F has no stable invariant manifold. Indeed, if such a manifold exists, then, for any (x,y) belonging to it, $F_y^n(x,y) \to 0$ as $n \to \infty$. Since

$$\varphi_y(t,x,y) = \left(1+tx_1\right)^b \left[y + x_2^3 \int_0^t \frac{1}{\left(1+sx_1\right)^{b+3a}}\, ds \right],$$

we deduce that

$$F_y^n(x,y) = \left(1+nx_1\right)^b \left[y + x_2^3 \int_0^n \frac{1}{\left(1+sx_1\right)^{b+3a}}\, ds \right].$$

Therefore, since $(1+nx_1)^b \to \infty$ as $n \to \infty$ a necessary condition for $F_y^n(x,y) \to 0$ as $n \to \infty$, is that

$$y = x_2^3 \int_\infty^0 \frac{1}{\left(1+sx_1\right)^{b+3a}}\, ds,$$

and the claim follows because the above integral is not convergent when $b+3a \le 1$.

5.2. *The loss of differentiability*

The following example shows that the invariant manifolds of a parabolic fixed point may be of finite order of differentiability. This maximum order of differentiability is attained when the manifold is written (locally) as a graph, since if the invariant manifold possesses a parametrization of the form given by Theorem 2.4 with some regularity, by performing a close to the identity change of variables, its representation as a graph will be also of the same regularity.

Let $a, b > 0$. Let F be the time 1 map of

$$\dot{x} = p(x), \qquad \dot{y} = q_1(x)y + g(x),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}$, p is such that the equation $\dot{x} = p(x)$ in polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ becomes

$$\dot{r} = -ar^5, \qquad \dot{\theta} = r^4 \sin 4\theta \tag{5.1}$$

(p is a homogeneous polynomial of degree 5) and

$$q_1(x) = b(x_1^2 + x_2^2)^2, \quad g(x) = 8(x_1^2 + x_2^2)x_1x_2(x_1^2 - x_2^2).$$

Claim 5.2. *Let $\nu \in (0,1)$. There exists $a_0 > 0$ such that, for any $a > a_0$, the map F satisfies hypotheses H1, H2 and H3 in V_ϱ, for some $\varrho > 0$, where*

$$V = \{x \in \mathbb{R}^2 : \nu|x_1| \le x_2\}.$$

Furthermore, for any $m, n \in \mathbb{N}$ satisfying $n > a_0$ and $2m > n+1$, the stable manifold over V of the origin with $a = 2n$ and $b = 4(2m-n-1)$ is only $2m-2 \ge 1$ times differentiable.

Proof. Let $\varphi(t,x)$ be the flow of $\dot{x} = p(x)$ and $M(t,x)$ the solution of $\dot{M} = q_1(\varphi(t,x))M$ such that $M(0,x) = 1$. The stable manifold (if it exists) has to be the graph of $y = h(x)$ with

$$h(x) = \int_{\infty}^{0} M^{-1}(t,x) g(\varphi(t,x))\, dt. \tag{5.2}$$

We note that, for any value of $a > 0$, the map $x \mapsto p(x)$ has exactly five invariant lines in the set $\{x_2 \geq 0\}$ corresponding to the values of $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi$.

It is straightforward to check that, taking $a > 0$ big enough, there exists $\varrho > 0$ small enough and a norm in $\mathbb{R}^2$ such that p satisfies $H1, H3$ in V_ϱ with the usual Euclidean norm $\|v\| = \sqrt{v_1^2 + v_2^2}$.

Moreover, a simple computation shows that $0 < A_p < b_p = a$. We recall that these constants were defined in (2.4).

Using polar coordinates (r,θ) in the (x_1,x_2)-plane, q_1 and g have the simpler expressions

$$q_1(r) = br^4, \qquad g(r,\theta) = 2r^6 \sin 4\theta.$$

In what follows we will write with the same letter a function $f(x)$ and its expression in polar coordinates $f(r,\theta) = f(r\cos\theta, r\sin\theta)$.

The stable manifold over V of the origin (which exists and it is $\mathcal{C}^1$) is the graph of $y = h(x)$ with h given in (5.2). Let $\varphi(t; r,\theta)$ be the flow associated to (5.1) in polar coordinates. We denote φ_r and φ_θ the first and second component respectively of φ when written in polar coordinates. Then

$$h(x) = 2\int_{\infty}^{0} [M_y(t,x)]^{-1} \left[\varphi_r(t; r,\theta)\right]^6 \sin(4\varphi_\theta(t; r,\theta))\, dt$$

with M_y the solution of the linear system $\dot{M}_y = b(\varphi_r(t; r,\theta))^4 M_y$ such that $M_y(0,r,\theta) = 1$.

We first note that, if $\theta = \pi/4, \pi/2, 3\pi/4$ (that is, x belongs to an invariant line), then $\varphi_\theta(t; r,\theta) = \theta$ and consequently $\sin(4\varphi_\theta(t; r,\theta)) \equiv 0$. This implies that the stable manifold evaluated at points with argument $\theta = \pi/4, \pi/2, 3\pi/4$ is $h(x) \equiv 0$. If the argument of x, $\theta \neq \pi/4, \pi/2, 3\pi/4$,

$$h(x) = 4c_\theta r^6 \int_{\infty}^{0} \frac{1}{\left(1 + 4atr^4\right)^{\frac{b}{4a} + \frac{6}{4} - \frac{1}{a}} \cdot \left[c_\theta^2 + \left(1 + 4atr^4\right)^{\frac{2}{a}}\right]}\, dt,$$

where

$$c_\theta = \frac{1 + \cos(4\theta)}{\sin(4\theta)} = \frac{x_1^2 - x_2^2}{2x_1x_2} = c_x$$

and (r,θ) are the polar coordinates of $x = (x_1,x_2)$. In particular, if $\theta = \pi/4, 3\pi/4$, then $c_x \equiv 0$ and hence the above expression for h is also valid in these invariant lines.

We perform the change of variables $(1+4atr^4)^{2/a} = c_x^2/w$ and we obtain that

$$h(x) = -c_x \frac{x_1^2+x_2^2}{4|c_x|^{a(\frac{b}{4a}+\frac{6}{4}+\frac{1}{a}-1)}} \int_0^{c_x^2} \frac{w^{\frac{a}{2}(\frac{b}{4a}+\frac{6}{4}+\frac{1}{a}-1)}}{w(w+1)}\, dw.$$

Now we take m, n as in the claim and choose a, b accordingly. Then

$$h(x) = -\frac{x_1^2+x_2^2}{4c_x^{2m-1}} \int_0^{c_x^2} \frac{w^{m-1}}{w+1}\, dw.$$

Using the elementary identity

$$\frac{w^{m-1}}{w+1} = \sum_{j=2}^{m} (-1)^j w^{m-j} + \frac{(-1)^{m+1}}{w+1},$$

we obtain

$$h(x) = -\frac{x_1^2+x_2^2}{4c_x^{2m-1}} \left(\sum_{j=2}^{m} (-1)^j \frac{c_x^{2(m-j+1)}}{m-j+1} + (-1)^{m+1} \log(c_x^2+1) \right).$$

Now we are going to look for the differentiability of h at points of the form $(0, x_2)$, $x_2 \neq 0$. To determine the regularity with respect to x_1, we only need to study the auxiliary function

$$\tilde{h}(x) = x_1^{2m-1} \left(\sum_{j=2}^{m} (-1)^j \frac{x_1^{-2(m-j+1)}}{m-j+1} + (-1)^{m+1} \log\left(\frac{1}{x_1^2} + 1 \right) \right).$$

This function is only $2m-2 \geq 1$ times differentiable. $\square$

6. Decomposition of V_ϱ

In this section we describe a decomposition of the set V_ϱ associated to a map of the form $\mathcal{R}(x) = x + p(x,0) + \mathcal{O}(\|x\|^{N+1})$. Moreover we will obtain a quantitative estimate of the rate of convergence of $\|\mathcal{R}^k(x)\|$ to 0 as $k \to \infty$.

We introduce the constant

$$\alpha = \frac{1}{N-1}.$$

For a given $\varrho > 0$, let $u > 0$ and $a_0 > 0$ be such that $a_0 u^{-\alpha} = \varrho$. Consider two sequences $a_k \in \mathbb{R}$, $k \geq 0$ and $b_k \in \mathbb{R}$, $k \geq 1$, such that

$$\frac{b_{k+1}}{(u+k+1)^\alpha} < \frac{a_k}{(u+k)^\alpha}, \qquad k \geq 0. \tag{6.1}$$

We introduce the sets

$$V_k = \left\{ x \in V_\varrho \ : \ \|x\| \in I_k := \left[\frac{b_{k+1}}{(u+k+1)^\alpha}, \frac{a_k}{(u+k)^\alpha} \right] \right\}. \tag{6.2}$$

Lemma 6.1. *Let p be the homogeneous polynomial defined in (2.1). Let $\mathcal{R} : V_\varrho \to \mathbb{R}^n$ be a continuous map such that $\mathcal{R}(x) - x - p(x,0) = \mathcal{O}(\|x\|^{N+1})$.*

Assume that p satisfies H1 and H3 and let $a_p \le b_p$ be the constants defined in (2.4).

Then for any $a < a_p$ and $b > b_p$, there exists ϱ small enough such that

(1) if $x \in V_\varrho$,

$$\|\mathcal{R}(x) - x\| \le b\|x\|^N, \qquad \|\mathcal{R}(x)\| \le \|x\|\big(1 - a\|x\|^{N-1}\big).$$

(2) Let $a_0, b_0, u > 0$ be such that $a_0^{N-1} = \alpha a^{-1}$, $b_0^{N-1} = \alpha b^{-1}$ and $a_0 u^{-\alpha} = \varrho$. There exist two sequences $a_k, b_k \in \mathbb{R}$, satisfying (6.1), such that $a_k = a_0\big(1+\mathcal{O}(k^{-\beta})\big)$, $b_k = b_0\big(1+\mathcal{O}(k^{-\beta})\big)$ for some $\beta > 0$. Moreover

$$\overline{V_\varrho}\backslash\{0\} = \bigcup_{k=0}^{\infty} V_k \quad \text{and} \quad \mathcal{R}(V_k) \subset V_{k+1}. \tag{6.3}$$

Consequently, if $x \in V_k$, then one has that

$$\frac{\alpha}{b(u+k+1+j)}\big(1+\mathcal{O}(k^{-\beta})\big) \le \|\mathcal{R}^j(x)\|^{N-1} \le \frac{\alpha}{a(u+k+j)}\big(1+\mathcal{O}(k^{-\beta})\big).$$

Proof. The proof of item (1) is straightforward from the definitions of a_p and b_p.

Now we check (2). We define the auxiliary functions of real variable, $\mathcal{R}_a(v) = v - av^N$ and $\mathcal{R}_b(v) = v - bv^N$. We first observe that, if ϱ is small enough,

$$\mathcal{R}_b(\|x\|) \le \|\mathcal{R}(x)\| \le \mathcal{R}_a(\|x\|).$$

Indeed, the right hand side inequality follows from the definition of a and the left hand side inequality is a straightforward consequence of the definition of b and the triangular inequality $\|\mathcal{R}(x)\| \ge \|x\| - \|\mathcal{R}(x) - x\|$.

For $k \ge 0$ we define the sequences a_k, b_k by the recurrences

$$\frac{a_{k+1}}{(u+k+1)^\alpha} = \mathcal{R}_a\left(\frac{a_k}{(u+k)^\alpha}\right), \qquad \frac{b_{k+1}}{(u+k+1)^\alpha} = \mathcal{R}_b\left(\frac{b_k}{(u+k)^\alpha}\right), \qquad k \ge 0,$$

and also $\bar{a}_k, \bar{b}_k$ by

$$\bar{a}_k = \frac{a_k}{(u+k)^\alpha}, \qquad \bar{b}_k = \frac{b_k}{(u+k)^\alpha}, \qquad k \ge 0.$$

We have that $a < b$. We choose ϱ small enough such that both $\mathcal{R}_a$ and $\mathcal{R}_b$ are monotonically increasing functions in $[0,\varrho]$ and $0 < \mathcal{R}_b(v) < \mathcal{R}_a(v) < v$, for $v \in (0,\varrho]$. From the choice of a_0, b_0 we have $\bar{b}_0 < \bar{a}_0$ and $\bar{a}_0 = \varrho$. We easily check by induction

$$0 < \bar{b}_k < \bar{a}_k, \qquad \bar{a}_{k+1} < \bar{a}_k, \qquad \bar{b}_{k+1} < \bar{b}_k \qquad \text{and} \qquad \lim_{k\to\infty} (\bar{a}_k^2 + \bar{b}_k^2) = 0.$$

As an immediate consequence, the sets V_k in (6.2) are well defined for this choice of sequences b_k and a_k and, in addition, equality (6.3) holds. Moreover, we note that if $u \in I_l = [\bar{b}_{l+1}, \bar{a}_l]$, then, by the definition of the sequences $\bar{a}_k, \bar{b}_k$, since $\mathcal{R}_a$ and $\mathcal{R}_b$ are increasing functions in $[0, \varrho]$ and $\mathcal{R}_b(v) \le \mathcal{R}_a(v)$,

$$\mathcal{R}_a(v) \in [\mathcal{R}_a(\bar{b}_{l+1}), \mathcal{R}_a(\bar{a}_l)] \subset [\mathcal{R}_b(\bar{b}_{l+1}), \mathcal{R}_a(\bar{a}_l)] = [\bar{b}_{l+2}, \bar{a}_{l+1}] = I_{l+1},$$
$$\mathcal{R}_b(v) \in [\mathcal{R}_b(\bar{b}_{l+1}), \mathcal{R}_b(\bar{a}_l)] \subset [\mathcal{R}_b(\bar{b}_{l+1}), \mathcal{R}_a(\bar{a}_l)] = [\bar{b}_{l+2}, \bar{a}_{l+1}] = I_{l+1}.$$

Therefore, if $x \in V_l$ (which is equivalent to $\|x\| \in I_l$), then $\mathcal{R}(x) \in I_{l+1}$ since $R_b(\|x\|) \le \|\mathcal{R}(x)\| \le \mathcal{R}_a(\|x\|)$.

In [3] it was proven that there exist two analytic function φ_a, φ_b of the form

$$\varphi_a(w) = \frac{a_0}{w^\alpha} + \mathcal{O}\left(\frac{1}{w^{\alpha+\beta}}\right), \qquad \varphi_b(w) = \frac{b_0}{w^\alpha} + \mathcal{O}\left(\frac{1}{w^{\alpha+\beta}}\right) \tag{6.4}$$

with $\beta > 0$ which conjugate both $\mathcal{R}_a$ and $\mathcal{R}_b$ to $w \mapsto w + 1$, namely

$$\mathcal{R}_a(\varphi_a(w)) = \varphi_a(w+1), \qquad \mathcal{R}_b(\varphi_b(w)) = \varphi_b(w+1). \tag{6.5}$$

Let w_k^a, w_k^b be such that $\varphi_a(w_k^a)(u+k)^\alpha = a_k$ and $\varphi_b(w_k^b)(u+k)^\alpha = b_k$. We observe that, by definition of a_k, b_k and (6.5)

$$\varphi_a(w_k^a) = \frac{a_k}{(u+k)^\alpha} = \mathcal{R}_a\left(\frac{a_{k-1}}{(u+k-1)^\alpha}\right) = \mathcal{R}_a(\varphi_a(w_{k-1}^a)) = \varphi_a(w_{k-1}^a + 1)$$

which implies (by the injective property of φ_a) that $w_k^a = w_{k-1}^a + 1 = w_0^a + k$. Analogously one can see that $w_k^b = w_0^b + k$. Now we notice that, by the form (6.4) of φ_a, φ_b, one has that

$$w_0^a = u + \mathcal{O}(u^{1-\beta}), \qquad w_0^b = u + \mathcal{O}(u^{1-\beta}).$$

Therefore,

$$\begin{aligned}
a_k &= (u+k)^\alpha \varphi_a(w_k^a) = (u+k)^\alpha \varphi_a(w_0^a + k) \\
&= \frac{a_0 (u+k)^\alpha}{\left[u + k + \mathcal{O}(u^{1-\beta})\right]^\alpha} + \mathcal{O}\left(\frac{(u+k)^\alpha}{\left[u+k+\mathcal{O}(u^{1-\beta})\right]^{\alpha+\beta}}\right) \\
&= \frac{a_0}{\left[1 + \mathcal{O}\big(u^{1-\beta}(u+k)^{-1}\big)\right]^\alpha} + \mathcal{O}\left(\frac{1}{(u+k)^\beta \left[1 + \mathcal{O}\big(u^{1-\beta}(u+k)^{-1}\big)\right]^{\alpha+\beta}}\right) \\
&= a_0 + \mathcal{O}\left(\frac{1}{(u+k)^\beta}\right).
\end{aligned}$$

Analogously, one checks that $b_k = b_0 + \mathcal{O}((u+k)^{-\beta})$ and the proof of the lemma is concluded. □

Remark 6.2. Note that as a simple consequence of this technical lemma, we have that for $x \in V_\varrho$, $\mathcal{R}^k(x) \to 0$ as $k \to \infty$. Hence, if we are able to prove the existence of a parametrization K satisfying the invariance equation $F \circ K - K \circ \mathcal{R} = 0$, since $F^k(K(x)) = K(\mathcal{R}^k(x))$, the image of K will represent a subset of the stable invariant manifold.

7. The invariant manifold. The differentiable case

In this section we prove Theorem 2.4 in the differentiable case. This is accomplished by stating and solving a fixed point equation in some appropriate Banach spaces. The proof follows along the same lines of the equivalent result in [3], but there are technical differences that prevent to apply directly that proof. However, these differences are not important enough to justify the inclusion of the whole proof. For this reason, in this section we include a series of technical lemmas, equivalent to those in [3], with the suitable hypothesis in our current case. We sketch their proofs when they are different enough from their counterpart in [3]. The existence of the manifold follows directly from this set of lemmas.

Along this section we will assume that all the hypotheses of Theorem 2.4 hold. We will denote by C a positive constant which may take different values at different places.

7.1. Preliminary facts

We take $\ell \in \mathbb{N}$ such that $\ell_0 < \ell \le r$ with ℓ_0 introduced in (2.5) and we decompose our map F into

$$F(x, y) = P(x, y) + G_\ell(x, y),$$

where P is the Taylor expansion of F up to degree $\ell - 1$ and $G_\ell(x) = o(\|x\|^{\ell-1})$. In fact, since $\ell \le r$, we actually have that $G_\ell(x) = \mathcal{O}(\|x\|^\ell)$. By hypothesis, there exist $K^\le$ and R, $\mathcal{C}^{r^\le}$ functions such that

$$P \circ K^\le - K^\le \circ R = T^\ell, \qquad T^\ell(x) = \mathcal{O}(\|x\|^\ell). \tag{7.1}$$

Since P is a polynomial and $K^\le$, R satisfy item (c) in Theorem 2.4, the remainder T^ℓ satisfies

$$D^j T^\ell(x) = \mathcal{O}(\|x\|^{\ell-j}), \qquad j = 0, \cdots, r^\le.$$

Finally, using that $D^j G_\ell$ is the Taylor's remainder of $D^j F$,

$$D^j G_\ell(x, y) = \mathcal{O}(\|(x, y)\|^{\ell-j}), \qquad j = 0, \cdots, r.$$

We will use these simple facts without special mention.

As a consequence of (7.1), the purpose of this section is to prove that there is only one solution $K^>$ of

$$F \circ (K^\le + K^>) - (K^\le + K^>) \circ R = 0. \tag{7.2}$$

We will see that equation (7.2) can be rewritten as a fixed point equation. Then, a solution of this fixed point equation will be found.

7.2. The Banach spaces and the main statement

Given E a Banach space, we will denote

$$\mathcal{X}_k^\nu(E) = \{h : V_\varrho \subset \mathbb{R}^n \to E : h \in \mathcal{C}^\nu,\ \max_{0\le j\le \nu} \sup_{x\in V_\varrho} \frac{\|D^j h(x)\|}{\|x\|^{k-j\eta}} < \infty\}$$

with $\eta = 1 - L + N$ defined in (2.2). This quantity was already introduced in [3], jointly with a motivating example showing that, if $K^>(x) = \mathcal{O}(\|x\|^k)$, then $DK^>(x)$ is not necessarily $\mathcal{O}(\|x\|^{k-1})$.

With this definition, if $h \in \mathcal{X}_k^\nu(E)$, then $Dh \in \mathcal{X}_{k-\eta}^{\nu-1}(L(\mathbb{R}^n; E))$. Thus we understand by $\|D^j h(x)\|$ the norm of the j-linear map induced by the norm in E.

We endow $\mathcal{X}_k^\nu$ with the norm

$$\|h\|_{\nu,k} = \max_{0\le j\le \nu} \sup_{x\in V_\varrho} \frac{\|D^j h(x)\|}{\|x\|^{k-j\eta}}$$

and it becomes a Banach space. We denote by $\mathcal{B}_k^\nu(\varsigma) \subset \mathcal{X}_k^\nu$ the open ball of radius ς.

Proposition 7.1. *Assume all the conditions in Theorem 2.4. Let $\ell \in \mathbb{N}$ be such that $\ell_0 < \ell \le r$ (the case $r = \infty$ is included). Then there exists $\varsigma_* > 0$ such that for any $\varsigma \ge \varsigma_*$ there exists ϱ small enough such that equation (7.2) has a unique solution $K^> : V_\varrho \to \mathbb{R}^{n+m}$ belonging to $\mathcal{B}_{\ell-N+1}^{r_\ell^>}(\varsigma)$ with $r_\ell^> \le \min\{r, r^\le\}$ and satisfying*

$$r_\ell^> \max\left\{\eta - \frac{A_p}{d_p}, 0\right\} < \ell - N + 1 - \frac{B_p}{a_p}.$$

Note that when $\eta d_p \le A_p$, the maximum differentiability degree is $r_\ell^> = \min\{r, r^\le\}$. In addition $r^> = r_\ell^>$ for $\ell = r$ is the value stated in Theorem 2.4.

In the next sections we prove this proposition by using the same scheme as in [3].

Next proposition proves the uniqueness statement of Theorem 2.4. This proposition ends the proof of Theorem 2.4 in the differentiable case.

Proposition 7.2. *Assume the hypotheses of Proposition 7.1. We denote by $\varrho_* > 0$ the corresponding quantity provided in Proposition 7.1 for the radius ς_*. Then equation (7.2) has a unique solution $K^> : V_{\varrho_*} \to \mathbb{R}^n$ in $\mathcal{X}_{\ell-N+1}^{r_\ell^>}$.*

Proof. Let $K_1 = K^\le + K_1^>$ and $K_2 = K^\le + K_2^>$ be two solutions of the invariance equation $F \circ K = K \circ R$ with $K_1^>, K_2^> \in \mathcal{X}_{\ell-N+1}^{r_\ell^>}$. We denote by V_{ϱ_0} their common domain (all the suprema will be taken in this domain) and we consider

$$\varsigma = \varsigma_* + \max\{\|K_1^>\|_{r_\ell^>,\ell-N+1}, \|K_2^>\|_{r_\ell^>,\ell-N+1}\} > \varsigma_*.$$

By Proposition 7.1, there exists $\varrho \le \varrho_0$ small enough and a unique function $K^> : V_\varrho \to \mathbb{R}^{n+m}$, belonging to $\mathcal{X}_{\ell-N+1}^{r_\ell^>}$ with norm $\|K^>\|_{r_\ell^>,\ell-N+1} \le \varsigma$. Since, for $i = 1, 2$, $\|K_i^>\|_{r_\ell^>,\ell-N+1} < \varsigma$

they have to coincide in V_ϱ. In addition, we can extend $K^>$ to V_{ϱ_*} by using the invariance equation. Indeed, let $K = K^\le + K^>$. First, we notice that by (2) of Lemma 6.1, there exists k such that $R^k(V_{\varrho_*}\setminus V_\varrho) \subset V_\varrho$. Second, the relation $K = F^{-k} \circ K \circ R^k$ extends K to V_{ϱ_*} and the result is proven. □

In $\mathbb{R}^{n+m}$ we will use the norm

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}, \qquad (x, y) \in \mathbb{R}^{n+m}, \tag{7.3}$$

where the chosen norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ are such that hypotheses H1, H2, and H3 hold.

7.3. A compilation of technical lemmas

The lemmas in this section are the translation to our current setting of the lemmas in [3]. We first present the following elementary properties of the Banach spaces $\mathcal{X}_k^\nu$.

Lemma 7.3. *The Banach spaces $\mathcal{X}_k^\nu$ satisfy:*

(1) Let $f(x) \in L(X_1, X_2)$ with $f \in \mathcal{X}_k^\nu$ and $g(x) \in X_1$ with $g \in \mathcal{X}_l^\nu$, then $f \cdot g \in \mathcal{X}_{k+l}^\nu$ and $\|f \cdot g\|_{\nu,k+l} \le 2^\nu \|f\|_{\nu,k}\|g\|_{\nu,l}$.
(2) Let $f : U \subset \mathbb{R}^{n+m} \to E$ be a $\mathcal{C}^\nu$ map, with E a Banach space, such that $\|D^l f(x)\| = \mathcal{O}(\|x\|^{j-l})$ for all $0 \le l \le \nu$. Then, for any map $g : V_\varrho \to U$ such that $g \in \mathcal{X}_1^i$ for some $0 \le i \le \nu$ we have that $f \circ g \in \mathcal{X}_j^i$.

For any $a < a_p, b > b_p$, we define the auxiliary constant

$$d = \begin{cases} a, & \text{if } A_p \le 0, \\ b, & \text{otherwise.} \end{cases}$$

From now on we fix values $a < a_p$, $b > b_p$ and $B > B_p$ such that if either a) $A_p > \eta d_p$, or b) $d_p < A_p < \eta d_p$ or c) $A_p < d_p$ then a) $A_p > \eta d$, b) $d < A_p < \eta d$ or c) $A_p < d$ respectively. We also choose the constants a, b such that the cases $A_p = \eta d$ or $A_p = d$ can be skipped even when either $A_p = \eta d_p = \eta b_p$ or $A_p = d_p$ respectively. Below we introduce k_0 and we further impose that

$$\begin{aligned} \ell_0 < k_0 := N - 1 + \frac{B}{a} + \max\left\{\eta - \frac{A_p}{d}, 0\right\} < \ell \le r, \\ \ell - N + 1 - \frac{B}{a} - r_\ell^> \max\left\{\eta - \frac{A_p}{d}, 0\right\} > 0. \end{aligned} \tag{7.4}$$

The first property holds because $\ell_0 < \ell \le r$. The second one holds by the definition of $r_\ell^>$ in Proposition 7.1. The constant k_0 depends on the values a, b, B but it can be chosen arbitrarily close to ℓ_0 (see (2.5) for the definition of ℓ_0).

7.3.1. Scaling

We perform a scaling in the y-variables by the change $S_\delta(x, y) = (x, \delta y)$. Then, equations (7.1) and (7.2) become

$$\tilde{P} \circ \tilde{K}^{\leq} - \tilde{K}^{\leq} \circ R = \tilde{T}_\ell \tag{7.5}$$

and

$$\tilde{F} \circ (\tilde{K}^{\leq} + \tilde{K}^{>}) - (\tilde{K}^{\leq} + \tilde{K}^{>}) \circ R = 0, \tag{7.6}$$

where $\tilde{P} = S_\delta^{-1} \circ P \circ S_\delta$, $\tilde{F} = S_\delta^{-1} \circ F \circ S_\delta$, $\tilde{K}^{\leq} = S_\delta^{-1} \circ K^{\leq}$ and $\tilde{K}^{>} = S_\delta^{-1} \circ K^{>}$.

We observe that

$$\tilde{P}_x(x, y) = x + p(x, 0) + p(x, \delta y) - p(x, 0) + \hat{f}(x, \delta y),$$

where, by hypothesis, $\tilde{p}(x, y) = p(x, \delta y) - p(x, 0)$ is a homogeneous polynomial of degree N and $\hat{f}(x, \delta y) = \mathcal{O}(\|(x, \delta y)\|^{N+1})$. We have that $\tilde{p}(x, y) = \hat{p}_{N-1}(x, y)y$, where

$$\hat{p}_{N-1}(x, y) = \delta \int_0^1 D_y p(x, \tau\delta y)\, d\tau$$

is a matrix whose coefficients are homogeneous polynomials of degree $N - 1$. It satisfies $\hat{p}_{N-1}(x, 0) = \delta D_y p(x, 0)$.

Lemma 7.4. *With B given in (7.4), there exist $\varrho, \delta > 0$ small enough such that*

$$\|(D\tilde{P})^{-1}(\tilde{K}^{\leq}(x))\| \leq 1 + B\|x\|^{N-1}, \qquad \textit{for all } x \in V_\varrho.$$

Proof. The proof of this lemma is analogous to Lemma 4.5 in [3]. However, we sketch it. Let $\varrho > 0$ be such that $\varrho^{1/2}\delta^{-1} < 1$. Taking into account the above considerations about the scaling, the norm of the matrix $(D\tilde{P})^{-1}(\tilde{K}^{\leq}(x))$ is

$$\|(D\tilde{P})^{-1}(\tilde{K}^{\leq}(x))\| \leq \max\{1 + (B_p + \mathcal{O}(\varrho) + \mathcal{O}(\delta))\|x\|^{N-1}, 1 - (B_q + \mathcal{O}(\delta^{-1}\varrho))\|x\|^{M-1}\}.$$

Recall that we are using in $\mathbb{R}^n$ the norm given in (7.3). Since $\varrho^{1/2}\delta^{-1} < 1$, taking ϱ, δ small enough, the constant B in (7.4) satisfies

$$\|(D\tilde{P})^{-1}(\tilde{K}^{\leq}(x))\| \leq \max\{1 + B\|x\|^{N-1}, 1 - (B_q + \mathcal{O}(\varrho^{1/2}))\|x\|^{M-1}\}.$$

To obtain the result, we need to check that $B\|x\|^{N-1} \geq -(B_q + \mathcal{O}(\varrho^{1/2}))\|x\|^{M-1}$. If $N \neq M$, the result follows from H2 and the smallness of ϱ. The case $N = M$, follows from $-B_q + \mathcal{O}(\varrho^{1/2}) < Na_p + \mathcal{O}(\varrho^{1/2}) \leq B_p + \mathcal{O}(\varrho^{1/2})$, by H2 and Lemma 2.1. Again taking ϱ small enough, we are done. □

From now on, we suppress the “tilde” from the scaled functions.

We fix $\delta, \varrho > 0$ small enough and a, b, B in (7.4) such that the conclusions of Lemma 6.1 applied to R and Lemma 7.4 hold true.

7.3.2. Weak contraction of the nonlinear terms

Since the fixed point is parabolic there is no contraction from the linear part of the map at the point. In the following lemma we measure the contraction provided by the nonlinear terms.

Lemma 7.5. *Let $V_k \subset V_\varrho$ be the sets defined in (6.2). There exists a constant $C > 0$, depending only on δ, ϱ and ℓ (which are fixed a priori), such that for any $k \geq 0$, $x \in V_k$ and $i \geq 0$*

$$\prod_{m=0}^{i} \|(DP)^{-1}(K^{\leq}(R^m(x)))\| \leq C \left(\frac{u+k+i}{u+k}\right)^{\alpha B a^{-1}}, \tag{7.7}$$

$$\|D[(DP)^{-1} \circ K^{\leq}](x)\| \leq C(u+k)^{-\alpha(L-2)}, \tag{7.8}$$

$$\|DR^i(x)\| \leq \prod_{m=0}^{i-1} \|DR \circ R^m(x)\| \leq C \left(\frac{u+k}{u+k+i}\right)^{\alpha A_p d^{-1}}. \tag{7.9}$$

Finally, if $A_p < d$

$$\|D^2 R^i(x)\| \leq C(u+k+i)^{\alpha} \left(\frac{u+k}{u+k+i}\right)^{2\alpha A_p d^{-1}} \tag{7.10}$$

and in the case $A_p > d = b$

$$\|D^2 R^i(x)\| \leq C(u+k)^{\alpha} \left(\frac{u+k}{u+k+i}\right)^{\alpha A_p d^{-1}}. \tag{7.11}$$

Remark 7.6. The proof of this lemma is analogous to the one of Lemma 4.6 in [3] using the estimates for $\|R^i(x)\|$ given in Lemma 6.1. However, the exponents in inequalities (7.7), (7.9)–(7.11) are different from their counterpart in [3] due to the fact that here the invariant manifold is not one dimensional. In particular, the constant analogous to $A_p d^{-1}$ was exactly N in [3]. We also are forced to separate the cases $A_p < d$ and $A_p > d$ in the bound of $\|D^2 R^j(x)\|$.

Proof. We begin with (7.7). By Lemma 6.1, if $x \in V_k$, then $R^m(x) \in V_{k+m}$. Therefore, using Lemma 7.4 and item (2) of Lemma 6.1 we have that

$$\|(DP)^{-1}(K^{\leq}(R^m(x)))\| \leq 1 + \frac{\alpha B}{a(u+k+m)}\big(1 + \mathcal{O}((k+m)^{-\beta})\big),$$

for $x \in V_k$. Then, since

$$\begin{aligned}\sum_{m=0}^{i} \log\Big(\|(DP)^{-1}(K^{\leq}(R^m(x)))\|\Big) &\leq \sum_{m=0}^{i} \log\left(1 + \frac{\alpha B}{a(u+k+m)}\big(1 + \mathcal{O}((k+m)^{-\beta})\big)\right) \\ &= \frac{\alpha B}{a} \sum_{m=0}^{i} \frac{1}{u+k+m}\big(1 + \mathcal{O}((k+m)^{-\beta})\big)\end{aligned}$$

$$= \frac{\alpha B}{a}\left[\log\left(\frac{u+k+i}{u+k}\right)+\mathcal{O}(k^{-\beta})\right],$$

and (7.7) is proven.

Bound (7.8) is a straightforward computation. To prove estimate (7.9) we first notice that since $R(x)=x+p(x,0)+\mathcal{O}(\|x\|^{N+1})$, by Lemma 6.1, if $x\in V_k$,

$$\|DR(x)\|\le 1-\frac{\alpha A_p}{d(u+k)}+\frac{C}{(u+k)^{1+\beta}}.$$

Then, using again Lemma 6.1,

$$\|DR^i(x)\|\le\prod_{m=0}^{i-1}\|DR\circ R^m(x)\|\le\prod_{m=0}^{i-1}\left(1-\frac{\alpha A_p}{d(u+k+m)}+\frac{C}{(u+k+m)^{1+\beta}}\right).$$

Finally, estimate (7.9) follows from the fact that

$$\sum_{m=0}^{i-1}\log\left(1-\frac{\alpha A_p}{d(u+k+m)}+\frac{C}{(u+k+m)^{1+\beta}}\right)\le\frac{\alpha A_p}{d}\log\left(\frac{u+k}{u+k+i}\right)+\frac{C}{(u+k)^{1+\beta}}.$$

To bound $\|D^2R^i(x)\|$ we first note that

$$\|D^2R^i(x)\|=\|D(\prod_{m=0}^{i-1}DR\circ R^m)\|\le\sum_{m=0}^{i-1}\|D^2R\circ R^m\|\|DR^m\|\prod_{l=0}^{i-1}\|DR\circ R^l\|\|DR\circ R^m\|^{-1}.$$

Then, taking into account that $\|DR\circ R^m(x)\|\ge 1/2$ and that,

$$\|D^2R(R^m(x))\|\le C\|R^m(x)\|^{N-2},$$

using again 6.1 of Lemma 6.1, we have that

$$\begin{aligned}\|D^2R^i(x)\|\le& C\prod_{l=0}^{i-1}\|DR\circ R^l\|\sum_{m=0}^{i-1}\|R^m(x)\|^{N-2}\|DR^m\|\\ \le& C\prod_{l=0}^{i-1}\|DR\circ R^l\|\sum_{m=0}^{i-1}\frac{(u+k)^{\alpha A_p d^{-1}}}{(u+k+m)^{\alpha A_p d^{-1}+\alpha(N-2)}}.\end{aligned}$$

Now we distinguish two cases. First, when $A_p>d=b$, we have $\alpha A_p d^{-1}+\alpha(N-2)>1$ and then

$$\sum_{m=0}^{i-1}\frac{(u+k)^{\alpha A_p d^{-1}}}{(u+k+m)^{\alpha A_p d^{-1}+\alpha(N-2)}}\le C(u+k)^{\alpha}.$$

This bound together with (7.9), implies (7.11) in this case. On the other hand, when $A_p<d$,

$$\sum_{m=0}^{i-1} \frac{(u+k)^{\alpha A_p d^{-1}}}{(u+k+m)^{\alpha A_p d^{-1}+\alpha(N-2)}} \leq C(u+k)^{\alpha}(u+k+i)^{\alpha(1-A_p d^{-1})}$$

and, using again (7.9), we get (7.10). □

7.3.3. Operators for higher order derivatives and their inverses

Now we proceed to rewrite equation (7.6), which we recall here

$$F \circ (K^{\leq} + K^{>}) - (K^{\leq} + K^{>}) \circ R = 0, \tag{7.12}$$

as a fixed point equation. We emphasize that we have skipped the symbol ˜ of our notation, although we work with the rescaled map. That is, since $K^{\leq}$ satisfies (7.5): $P \circ K^{\leq} - K^{\leq} \circ R = T^{\ell}$, $K^{>}$ has to satisfy

$$\begin{aligned}\big(DP \circ K^{\leq}\big)K^{>} - K^{>} \circ R = \\ - T^{\ell} - G_{\ell} \circ \big(K^{\leq} + K^{>}\big) - P \circ \big(K^{\leq} + K^{>}\big) + P \circ K^{\leq} + \big(DP \circ K^{\leq}\big)K^{>}.\end{aligned}$$

To shorten the notation, we introduce the operators

$$\mathcal{L}^0(S) = \big(DP \circ K^{\leq}\big)S - S \circ R \tag{7.13}$$

and

$$\mathcal{F}(K) = -T^{\ell} - G_{\ell} \circ \big(K^{\leq} + K\big) - P \circ \big(K^{\leq} + K\big) + P \circ K^{\leq} + \big(DP \circ K^{\leq}\big)K. \tag{7.14}$$

Then equation (7.12) for $K^{>}$ can be rewritten as

$$\mathcal{L}^0(K^{>}) = \mathcal{F}(K^{>}). \tag{7.15}$$

The formal inverse of $\mathcal{L}^0$ is

$$\mathcal{S}^0(T) = \sum_{i=0}^{\infty} \left[\prod_{m=0}^{i} \big(DP\big)^{-1} \circ K^{\leq} \circ R^m\right] T \circ R^i \tag{7.16}$$

and consequently, we can formally write equation (7.15) as the fixed point equation

$$K^{>} = \mathcal{S}^0 \circ \mathcal{F}(K^{>}). \tag{7.17}$$

Following the same arguments as the ones in the proof of Lemma 4.9 in [3], one can check that the operator $\mathcal{S}^0 : \mathcal{X}^0_{\ell} \to \mathcal{X}^0_{\ell-N+1}$ is continuous. Therefore, the operator $\mathcal{L}^0$, introduced in (7.13), is suitable to prove the existence of a continuous invariant manifold. In order to obtain the higher order derivatives, we introduce the operators

$$\mathcal{L}^j(S) = \big(DP \circ K^{\leq}\big)S - S \circ R(DR)^j, \qquad j \geq 1.$$

The key property is that if S is a $\mathcal{C}^\nu$ solution of $\mathcal{L}^0(S) = T$, with T a $\mathcal{C}^\nu$ function, then $D^j S$ is a solution of

$$\mathcal{L}^j(H) = T^j, \qquad 0 \le j \le \nu,$$

where T^j is defined by the recurrence relation

$$\begin{aligned} T^0 &= T, \\ T^{j+1} &= DT^j - D(DP \circ K^{\le})D^j S + jD^j S \circ R(DR)^{j-1}D^2 R. \end{aligned}$$

Recall the parameters $L = \min\{N, M\}$ and $\eta = 1 + N - L$ defined in (2.2). The following lemma is analogous to Lemma 4.7 in [3], with an appropriate change in the hypothesis.

Lemma 7.7. *Let $\ell > N - 1 + Ba^{-1}$ and $j \ge 0$ be such that*

$$\ell - N + 1 - \frac{B}{a} - j\left(\eta - \frac{A_p}{d}\right) > 0.$$

Then, the operators $\mathcal{L}^j : \mathcal{X}^0_{\ell-N+1-j\eta} \to C^0$, $j \ge 0$, are well defined, continuous and one to one.

Proof. Since $R(x) = x + p(x, 0) + \mathcal{O}(\|x\|^{N+1})$, $N \ge 2$, and $a_p > 0$, $\|R(x)\| \le \|x\|$ and then $\mathcal{L}^j$ is well defined and continuous.

Let $j \ge 0$ and $S \in \mathcal{X}^0_{\ell-N+1-j\eta}$ be such that $\mathcal{L}^j(S) = 0$, that is, $S = \left(DP \circ K^{\le}\right)^{-1} S \circ R(DR)^j$, or, using this condition iteratively,

$$S = \left(\prod_{m=0}^{i} \left(DP\right)^{-1} \circ K^{\le} \circ R^m\right) S \circ R^{i+1}(DR^{i+1})^j, \qquad i \ge 0.$$

Now, using that $\|S \circ R^{i+1}(x)\| \le C\|S\|_{0,\ell-N+1-j\eta}\|R^{i+1}(x)\|^{\ell-N+1-j\eta}$ and Lemmas 6.1 and 7.5, we obtain that, for $x \in V_k$,

$$\|S(x)\| \le C\|S\|_{0,\ell-N+1-j\eta} \frac{(u+k)^{\alpha(jA_p d^{-1} - Ba^{-1})}}{(u+k+i)^{\alpha(\ell-N+1-Ba^{-1}-j(\eta-A_p d^{-1}))}}.$$

By hypothesis, the right hand side of the above expression tends to 0 when i tends to ∞, which implies that $S = 0$ and, consequently, that $\mathcal{L}^j$ is one to one. □

A formal inverse of the operator $\mathcal{L}^j$ obtained recursively from $\mathcal{L}^j(S) = T$ is given by the formula

$$\mathcal{S}^j(T) = \sum_{i \ge 0} \left(\prod_{m=0}^{i} (DP)^{-1} \circ K^{\le} \circ R^m\right) T \circ R^i \cdot (DR^i)^j. \tag{7.18}$$

Notice that $\mathcal{S}^j$ acts on j-linear maps. If this formula is absolutely convergent, it is a simple computation to check that $\mathcal{L}^j(\mathcal{S}^j(T)) = T$.

In the next lemma, equivalent to Lemma 4.9 in [3] with adjusted hypotheses, we check that $\mathcal{S}^j$ is indeed well defined and bounded between appropriate spaces.

Lemma 7.8. *Assume that* $\ell > N - 1 + Ba^{-1}$ *and that* $j \geq 0$ *satisfies*

$$\ell - N + 1 - \frac{B}{a} - j\left(\eta - \frac{A_p}{d}\right) > 0.$$

Then the operator $\mathcal{S}^j : \mathcal{X}^0_{\ell-j\eta} \to \mathcal{X}^0_{\ell-j\eta-N+1}$ *is well defined and bounded. Also we have* $\mathcal{L}^j \circ \mathcal{S}^j = \mathrm{Id}$ *on* $\mathcal{X}^0_{\ell-j\eta}$.

Moreover, if $\ell > k_0$, *with* k_0 *defined in* (7.4) *and* $j \geq 0$ *is such that*

$$\ell - k_0 - j\left(\eta - \frac{A_p}{d}\right) > 0,$$

the operator $\mathcal{S}^j : \mathcal{X}^1_{\ell-j\eta} \to \mathcal{X}^1_{\ell-j\eta-N+1}$ *is well defined and*

$$D\left(\mathcal{S}^j(T)\right) = \mathcal{S}^{j+1}(\tilde{T}), \qquad \text{if } T \in \mathcal{X}^1_{\ell-j\eta},$$

where

$$\tilde{T} = DT - D(DP \circ K^{\leq})\mathcal{S}^j(T) + j\mathcal{S}^j(T) \circ R(DR)^{j-1}D^2R.$$

Proof. Let $T \in \mathcal{X}^0_{\ell-j\eta}$ and $S = \mathcal{S}^j(T)$. Following the same lines as the ones in the proof of Lemma 4.9 in [3], a direct computation shows that, for $x \in V_k$,

$$\|S(x)\| \leq C\|T\|_{0,\ell-j\eta} \sum_{i\geq 0} \frac{(u+k)^{\alpha(jA_pd^{-1}-Ba^{-1})}}{(u+k+i)^{\alpha(\ell-j(\eta-A_pd^{-1})-Ba^{-1})}}.$$

Therefore, since by hypothesis $\alpha(\ell - j(\eta - A_pd^{-1}) - Ba^{-1}) > 1$, if $x \in V_k$,

$$\|S(x)\| \leq C\|T\|_{0,\ell-j\eta}(u+k)^{-\alpha(\ell-j\eta-N+1)} \leq C\|x\|^{\ell-j\eta-N+1}\|T\|_{0,\ell-j\eta}.$$

Hence $\|S\|_{0,\ell-j\eta-N+1} \leq \|T\|_{0,\ell-j\eta}$, that is, $\mathcal{S}^j : \mathcal{X}^0_{\ell-j\eta} \to \mathcal{X}^0_{\ell-j\eta-N+1}$ is well defined and bounded. It also proves that $\mathcal{L}^j \circ \mathcal{S}^j = \mathrm{Id}$ on $\mathcal{X}^0_{\ell-j\eta}$.

Following the proof of Lemma 4.9 in [3], we argue that, if $\mathcal{S}^j(T)$ is differentiable and its derivative belongs to $\mathcal{X}^0_{\ell-(j+1)\eta-N+1}$, then $D\mathcal{S}^j(T) = \mathcal{S}^{j+1}(\tilde{T})$. The trick is to check that both are solutions of the same equation $\mathcal{L}^{j+1}(H) = \tilde{T}$ belonging to $\mathcal{X}^0_{\ell-(j+1)\eta-N+1}$. Indeed, first we note that if $T \in \mathcal{X}^1_{\ell-j\eta}$, then $\tilde{T} \in \mathcal{X}^0_{\ell-(j+1)\eta}$ provided $DT \in \mathcal{X}^0_{\ell-(j+1)\eta}$, $D(DP \circ K^{\leq}) \in \mathcal{X}^0_{L-2}$, $D^2R \in \mathcal{X}^0_{N-2}$ and the definition of η. This implies that $\mathcal{S}^{j+1}(\tilde{T}) \in \mathcal{X}^0_{\ell-(j+1)\eta-N+1}$. It only remains to check that $D(\mathcal{S}^j(T))$ is a solution of $\mathcal{L}^{j+1}(H) = \tilde{T}$ which can be proven by taking derivatives in $\mathcal{L}^j(\mathcal{S}^j(T)) = T$. Hence the uniqueness result, Lemma 7.7, proves that $D(\mathcal{S}^j(T)) = \mathcal{S}^{j+1}(\tilde{T})$.

Now we prove that $\mathcal{S}^j(T)$ is differentiable and belongs to $\mathcal{X}^0_{\ell-(j+1)\eta-N+1}$. In order to do so, we take derivatives formally in (7.18). We have $D(\mathcal{S}^j(T)) = S_1 + S_2 + S_3$, where

$$S_1 = \sum_{i\ge 0}\left(\prod_{m=0}^{i}(DP)^{-1}\circ K^{\le}\circ R^m\right) DT\circ R^i (DR^i)^{j+1},$$

$$S_2 = \sum_{i\ge 0}\left(\prod_{m=0}^{i}(DP)^{-1}\circ K^{\le}\circ R^m\right) T\circ R^i j (DR^i)^{j-1} D^2R^i,$$

$$S_3 = \sum_{i\ge 0}\sum_{m=0}^{i}\left(\prod_{l=0}^{m-1}(DP)^{-1}\circ K^{\le}\circ R^l\right) D\Big((DP)^{-1}\circ K^{\le}\circ R^m\Big)$$
$$\times\left(\prod_{l=m+1}^{i}(DP)^{-1}\circ K^{\le}\circ R^l\right) T\circ R^i (DR^i)^j,$$

and check that the above expressions are absolutely convergent, belong to $\mathcal{X}^0_{\ell-(j+1)\eta-N+1}$ and are bounded.

Since $DT\in\mathcal{X}^0_{\ell-(j+1)\eta}$, then, by the first part of the lemma, $S_1 = \mathcal{S}^{j+1}(DT)$ belongs to $\mathcal{X}^0_{\ell-(j+1)\eta-N+1}$ and we are done with S_1.

Next we deal with S_2. Let $x\in V_k$. Assume that $A_p > d = b$. Then, by Lemma 7.5, we have that:

$$\|S_2(x)\| \le C\|T\|_{1,\ell-j\eta}\sum_{i\ge 0}\frac{(u+k)^{\alpha(jA_pd^{-1}+1-Ba^{-1})}}{(u+k+i)^{\alpha(\ell-j(\eta-A_pd^{-1})-Ba^{-1})}}.$$

Since $\ell - j(\eta - A_pd^{-1}) - Ba^{-1} > N-1$, the sum is convergent and we obtain

$$\|S_2(x)\| \le C\frac{\|T\|_{1,\ell-j\eta}}{(u+k)^{\alpha(\ell-j\eta-N)}} = C\|x\|^{\ell-j\eta-N}\|T\|_{1,\ell-j\eta} \le C\|x\|^{\ell-(j+1)\eta-N+1}\varrho^{\eta-1}\|T\|_{1,\ell-j\eta}$$

which implies that $S_2\in\mathcal{X}^0_{\ell-N+1-(j+1)\eta}$. Here we have used that $\eta\ge 1$. If $A_p < b$, then again by Lemma 7.5,

$$\|S_2(x)\| \le C\|T\|_{1,\ell-j\eta}\sum_{i\ge 0}\frac{(u+k)^{\alpha((j+1)A_pd^{-1}-Ba^{-1})}}{(u+k+i)^{\alpha(\ell-(j+1)(\eta-A_pd^{-1})-Ba^{-1}+\eta-1)}}.$$

Proceeding as in the previous case, one gets that $\|S_2(x)\| \le K\|x\|^{\ell-(j+1)\eta-N+1}\varrho^{\eta-1}\|T\|_{1,\ell-j\eta}$ and the study for S_2 is finished.

Finally we consider S_3. Using again Lemma 7.5, if $x\in V_k$ we have that

$$\|S_3(x)\| \le C\|T\|_{1,\ell-j\eta}\sum_{i\ge 0}\frac{(u+k)^{\alpha((j+1)A_pd^{-1}-Ba^{-1})}}{(u+k+i)^{\alpha(\ell-j(\eta-A_pd^{-1})-Ba^{-1})}}\sum_{m=0}^{i}\frac{1}{(u+k+m)^{1-\alpha(\eta-A_pd^{-1})}}.$$

We have different estimates for the sum with respect to m if either $A_p d^{-1} \le \eta$ or $A_p d^{-1} > \eta$. Nevertheless, the sum with respect to i is convergent provided ℓ satisfies the current hypothesis. Performing straightforward computations, we obtain that

$$\|S_3(x)\| \le C \frac{\|T\|_{1,\ell-j\eta}}{(u+k)^{\alpha(\ell-(j+1)\eta-N+1)}} \le C\|T\|_{1,\ell-j\eta}\|x\|^{\ell-(j+1)\eta-N+1}$$

and the lemma is proven. □

The last result of this section is the following.

Proposition 7.9. *Let $r^{\le}$ be the differentiability degree of $K^{\le}$ and R assumed in Theorem 2.4. Take $\ell > N-1+Ba^{-1}$ and ν such that $0 \le \nu \le r^{\le}$ and*

$$\ell - N + 1 - \frac{B}{a} - \nu \max\left\{\eta - \frac{A_p}{d}, 0\right\} > 0. \tag{7.19}$$

Then,

$$\mathcal{S}^0 : \mathcal{X}^\nu_\ell \to \mathcal{X}^\nu_{\ell-N+1} \quad \text{and} \quad \mathcal{S}^1 : \mathcal{X}^\nu_{\ell-\eta} \to \mathcal{X}^\nu_{\ell-\eta-N+1}$$

are bounded linear operators.

Proof. The proof of this proposition is analogous to the corresponding one Proposition 4.10 in [3]. Let $T \in \mathcal{X}^\nu_\ell \subset \mathcal{X}^0_\ell$. The key point of the proof is to deduce that

$$D[\mathcal{S}^{j-1}(T^{j-1})] = \mathcal{S}^j(T^j), \qquad 1 \le j \le \nu, \tag{7.20}$$

being $\{T^j\}_{0\le j\le\nu}$ the sequence defined inductively by

$$\begin{aligned} T^0 &= T, \\ T^{j+1} &= DT^j - D(DP \circ K^{\le})\mathcal{S}^j(T^j) + j\mathcal{S}^j(T^j) \circ R(DR)^{j-1}D^2R, \end{aligned}$$

for $0 \le j \le \nu - 1$. Indeed, one checks by induction that T^j belongs to $\mathcal{X}^1_{\ell-j\eta}$ if $j \le \nu-1$. For $j = \nu$ we have that $T^\nu \in \mathcal{X}^0_{\ell-\nu\eta}$ and therefore, by Lemma 7.8, $\mathcal{S}^j(T^j) \in \mathcal{X}^1_{\ell-j\eta-N+1}$ and $\mathcal{S}^\nu(T^\nu) \in \mathcal{X}^0_{\ell-\nu\eta}$. Note that, if $j \le \nu-1$ with ν satisfying (7.19), then

$$\ell - k_0 - j\left(\eta - \frac{A_p}{d}\right) \ge \ell - N + 1 - \frac{B}{a} - (j+1)\max\left\{\eta - \frac{A_p}{d}, 0\right\} > 0.$$

Then, for $j \le \nu-1$, the results of Lemma 7.8 on the operators $\mathcal{S}^j : \mathcal{X}^1_{\ell-j\eta} \to \mathcal{X}^1_{\ell-j\eta-N+1}$ apply.

Applying iteratively (7.20) we have that $D^j[\mathcal{S}^0(T)] = S^j(T^j) \in \mathcal{X}^0_{\ell-j\eta-N+1}$, for $j \le \nu$ and, hence, $\mathcal{S}^0(T) \in \mathcal{X}^\nu_{\ell-N+1}$.

Finally, to prove that the operator $\mathcal{S}^0 : \mathcal{X}^\nu_\ell \to \mathcal{X}^\nu_{\ell-N+1}$ is bounded, we refer the reader to [3], Proposition 4.10. The proof that $\mathcal{S}^1 : \mathcal{X}^\nu_{\ell-\eta} \to \mathcal{X}^\nu_{\ell-\eta-N+1}$ is also bounded is very similar to the one for $\mathcal{S}^0$. □

7.4. End of the proof of Proposition 7.1. Fixed point equation

Using Proposition 7.9 we are able to prove that the fixed point equation (7.17),

$$K^{>} = \mathcal{S}^0 \circ \mathcal{F}(K^{>}), \tag{7.21}$$

is well defined in the appropriate Banach spaces and it is a contraction. Concretely, we prove Proposition 7.1. That is, that there exists a unique solution $K^{>}$ of equation (7.21) belonging to $\mathcal{X}^{r_\ell^{>}}_{\ell-N+1}$ for any $\ell_0 < \ell \le r$. To do so, we follow the same steps as the ones in Section 4.10 of [3]. We sketch them without proofs, only given the essential information. The main tool is Lemma 7.3.

Let $\varrho > 0$ be such that all the results in Sections 7.3.3 and 7.3.2 are valid. Recall that we settled this quantity at the end of Section 7.3.1 satisfying the results in Section 6 and (7.4) for a, b and B.

Since $\mathcal{F}(0) = -T^\ell - G_\ell \circ K^{\le}$, using that T^ℓ and G_ℓ are $\mathcal{C}^r$ functions, that $K^{\le}$ is a $\mathcal{C}^{r^{\le}}$ function and the definition of $r_\ell^{>}$, we have that $\mathcal{F}(0) \in \mathcal{C}^{\min\{r,r^{\le}\}} \subset \mathcal{C}^{r_\ell^{>}}$. Then $\mathcal{F}(0) \in \mathcal{X}^{r_\ell^{>}}_{\ell}$ and, since $\nu = r_\ell^{>}$ satisfies the condition stated in Proposition 7.9, see (7.4), $\mathcal{S}^0 \circ \mathcal{F}(0) \in \mathcal{X}^{r_\ell^{>}}_{\ell-N+1}$. We also have that

$$\|\mathcal{S}^0 \circ \mathcal{F}(0)\|_{r_\ell^{>},\ell-N+1}\| \le \|\mathcal{S}^0\| \big(\|T^\ell\|_{r_\ell^{>},\ell} + \|G_\ell \circ K^{\le}\|_{r_\ell^{>},\ell}\big) =: \frac{\varsigma_*}{2}.$$

Since the domain of $K^{\le}$ is $V_\varrho \subset V_{\varrho_0}$ we will work with this domain in the spaces $\mathcal{X}^\nu_k$.

We will find the solution $K^{>}$ of equation (7.21) in $\mathcal{B}^{r_\ell^{>}-1,\varsigma}_{\ell-N+1} \subset \mathcal{X}^{r_\ell^{>}-1}_{\ell-N+1}$, the ball of radius $\varsigma \ge \varsigma_*$. First we note that for any $\varsigma \ge \varsigma_*$ there exists ϱ' small enough such that if $K^{>} \in \mathcal{B}^{r_\ell^{>}-1,\varsigma}_{\ell-N+1}$ and $x \in V_{\varrho'}$, then $(K^{\le} + K^{>})(x) \in U$, the domain of F. Indeed, we deduce this property because U is an open set, $\mathrm{dist}(V_{\varrho'}, \partial U) > 0$ and $\|(K^{\le} + K^{>})(x) - x\| \le C(\varrho')^2$ with $C > 0$ a constant. Note that ϱ' depends on ς, ϱ and $K^{\le}$.

As usual in the differentiable case, we first prove the existence of a solution belonging to $\mathcal{C}^{r_\ell^{>}-1}$ defined on $V_{\varrho'}$. To do so, it only remains to check that the operator $\mathcal{F} : \mathcal{B}^{r_\ell^{>}-1,\varsigma}_{\ell-N+1} \to \mathcal{X}^{r_\ell^{>}-1}_{\ell-N+1}$ is a contraction. The proof of this result follows from the analogous result in [3] and in fact we obtain the same bound for the Lipschitz constant

$$\mathrm{lip}(\mathcal{F}) \le C(\varrho')^{\ell-2N-L},$$

with C independent of ϱ', but depending on ς and ϱ.

As a consequence, equation (7.21) has a solution $K^{>} : V_{\varrho'} \to \mathbb{R}^{n+m}$. Applying the linear operator $\mathcal{L}^0$ we obtain that equation (7.15) has a unique differentiable solution $K^{>}$. This implies that $K = K^{\le} + K^{>}$ and R are $\mathcal{C}^{r_\ell^{>}-1}$ solutions of the invariance equation (7.2).

Following the same arguments as the ones given in [3] we deduce that, if $r = \infty$ the parametrization $K = K^{\le} + K^{>}$ is also a $\mathcal{C}^\infty$ as well as R is. Moreover, the arguments to prove the sharp regularity can be also applied in this new context. Hence we obtain $\mathcal{C}^{r_\ell^{>}}$ parametrizations.

Until now the function $K = K^{\le} + K^{>}$ is defined on $V_{\varrho'}$ with $\varrho' \le \varrho$. However, since ϱ is small enough to assure that R satisfies the conclusions of Lemma 6.1, we can use the invariance equation to extend the domain of K to V_ϱ as we did in the proof of Corollary 7.2. Indeed, let $k \in \mathbb{N}$ be such that $R^k(V_\varrho \backslash V_{\varrho'}) \subset V_{\varrho'}$. Then $K = F^{-k} \circ K \circ R^k$ extends K to V_ϱ.

Remark 7.10. We have proven that the domain V_ϱ of K and R depends on ℓ, $K^{\leq}$ and on the constants a, b, B as well as a_p, b_p, A_p, B_p.

8. Dependence on parameters

In this section we prove Theorem 2.8 about the dependence of the invariant manifold on parameters. Along this section we will assume all the conditions stated in this theorem. We will proceed in a similar way as in the proof of Theorem 2.4.

8.1. Preliminary facts. Consequences of the previous results

As a consequence of Lemma 2.7, if Hypothesis Hλ holds true, then H1, H2 and H3 are satisfied for any $\lambda \in \Lambda$. Then, using Proposition 7.1 with $\ell = r$, we have that for any $\lambda \in \Lambda$, there exists ϱ_λ such that the invariance equation

$$F(K^{\leq}(x,\lambda) + K^{>}(x,\lambda),\lambda) - K^{\leq}(R(x,\lambda),\lambda) + K^{>}(R(x,\lambda),\lambda) = 0$$

has a solution $K^{>}(\cdot,\lambda) \in \mathcal{X}_{\ell-N+1}^{r_\ell^>}$ defined on V_{ϱ_λ}. However we emphasize that

- the degree of differentiability $r_\ell^>$ does not depend on λ and
- ϱ_λ can be taken independent on λ provided the constants a_p, b_p, A_p, B_p, B_q are independent on the parameter (see Remark 7.10). Then $K^>$ is defined over $V_\varrho \times \Lambda$.

In addition, we already know that for any λ, $K^>(\cdot,\lambda)$ is the unique solution belonging to $\mathcal{X}_{\ell-N+1}^{r_\ell^>}$ of the fixed point equation (7.21):

$$K^> = \mathcal{S}^0 \circ \mathcal{F}(K^>) \tag{8.1}$$

being $\mathcal{S}^0$ and $\mathcal{F}$ defined by (7.16) and (7.14), respectively.

It is important to remark that all the functions involved, $P, T^\ell, G_\ell, K^{\leq}, K, R$, and T, depend on both, x, λ, but, abusing notation, we only indicate the composition with respect to the x variable. For instance R^2 means $R(R(x,\lambda),\lambda)$ and $G_\ell \circ \big(K^{\leq} + K\big)$ means $G_\ell(K^{\leq}(x,\lambda) + K(x,\lambda),\lambda)$.

We restate Theorem 2.8 in a functional setting using the space $\mathcal{C}^{\Sigma_{\sigma,\nu}}$ introduced in (2.13). We also introduce the Banach space

$$\mathcal{Y}_k^{\sigma,\nu} = \big\{ f : \mathcal{U} \times \Lambda \to \mathbb{R}^l : f \in \mathcal{C}^{\Sigma_{\sigma,\nu-\sigma}} \max_{i,j \in \Sigma_{\sigma,\nu-\sigma}} \sup_{(x,\lambda)\in \mathcal{U}\times\Lambda} \frac{\|D_\lambda^i D_x^j f(x,\lambda)\|}{\|x\|^{k+i-(i+j)\eta}} < \infty \big\}$$

for $\nu \geq \sigma$, endowed with the norm

$$\|f\|_{\nu,k}^\sigma = \max_{i,j \in \Sigma_{\sigma,\nu-\sigma}} \sup_{(x,\lambda)\in \mathcal{U}\times\Lambda} \frac{\|D_\lambda^i D_x^j f(x,\lambda)\|}{\|x\|^{k+i-(i+j)\eta}}.$$

Note that $\mathcal{Y}_k^{\sigma,\nu+\sigma} \subset \mathcal{C}^{\Sigma_{\sigma,\nu}}$. The differentiability conclusions of Theorem 2.8 are a direct consequence of the following proposition.

Proposition 8.1. *Assume all the conditions in Theorem 2.8. Let $\ell \in \mathbb{N}$ be such that $\max\{\ell_0, \ell_1\} < \ell \le r$ with ℓ_0 and ℓ_1 defined in (2.5) and (2.15) respectively. Then the solution $K^> : V_\varrho \times \Lambda \to \mathbb{R}^{n+m}$ of the fixed point equation (8.1) belongs to $\mathcal{Y}_{\ell-N+1}^{s_\ell^>, \nu_\ell^>}$ with $\nu_\ell^> = r_\ell^> + s_\ell^>$, $r_\ell^> \le \min\{r, r^\le\}$, $s_\ell^> \le \min\{s, s^\le\}$ and*

$$\ell - N + 1 - \frac{B_p}{a_p} - (\nu_\ell^> - i)\max\left\{\eta - \frac{A_p}{d_p}, 0\right\} - i(\eta - 1) > 0, \qquad 0 \le i \le s_\ell^>.$$

The remaining part of this section is devoted to prove this result. The procedure is similar to the one we have followed in Section 7. First we study the product and composition of functions belonging to the functional spaces $\mathcal{Y}_k^{\sigma,\nu}$. Then, we study the linear operator $\mathcal{S}^0$ defined on $\mathcal{Y}_\ell^{\sigma,\nu}$ and, finally, we apply the fixed point theorem to obtain a solution $K^>$ of the fixed point equation (8.1) belonging to $\mathcal{Y}_{\ell-N+1}^{\sigma,\nu}$ with appropriate values of σ and ν. With standard arguments we check the sharp regularity of the solutions.

8.2. Technical lemmas

Next lemma, whose proof we skip, is analogous to Lemma 7.3 for $\mathcal{Y}_k^{\sigma,\nu}$.

Lemma 8.2. *The Banach spaces $\mathcal{Y}_k^{\sigma,\nu}$ satisfy:*

(1) Let $f(x,\lambda) \in L(X_1, X_2)$ with $f \in \mathcal{Y}_k^{\sigma,\nu}$ and $g(x,\lambda) \in X_1$ with $g \in \mathcal{Y}_l^{\sigma,\nu}$, then $f \cdot g \in \mathcal{Y}_{k+l}^{\sigma,\nu}$ and $\|f \cdot g\|_{\nu,k+l}^\sigma \le 2^\nu \|f\|_{\nu,k}^\sigma \|g\|_{\nu,l}^\sigma$.
(2) Let $f : U \times \Lambda \subset \mathbb{R}^{n+m+n'} \to E$ be a $\mathcal{C}^{\Sigma_{\sigma,\nu-\sigma}}$ map and E a Banach space such that $\|D_\lambda^{l'} D_x^l f(x,\lambda)\| = \mathcal{O}(\|x\|^{j-l})$ for all $(l', l) \in \Sigma_{\sigma,\nu-\sigma}$. Then, for any map $g : V_\varrho \times \Lambda \to U$ such that $g \in \mathcal{Y}_1^{i',i}$ for some $(i', i) \in \Sigma_{\sigma,\nu}$ we have that $f \circ (g, \mathrm{Id}) \in \mathcal{Y}_j^{i',i}$.

We need to establish the dependence on λ of $\mathcal{S}^0(K)$.

8.2.1. Differentiability with respect to λ of the linear operator $\mathcal{S}^0$

We first note that all the results stated in the previous sections are valid uniformly in $\lambda \in \Lambda$ for functions belonging to $\mathcal{Y}_\ell^{0,\nu}$. This is due to Hypothesis HP and to the fact that the constants a_p, b_p, etcetera, defined in (2.12) are independent of $\lambda \in \Lambda$ and therefore, all the bounds in the previous sections are uniform with respect to $\lambda \in \Lambda$. The uniformity with respect to $\lambda \in \Lambda$ of Lemmas 7.5 and 7.7 and Proposition 7.9 is summarized in the following lemma.

Lemma 8.3. *We have that:*

(i) All the bounds in Lemma 7.5 hold true with constants C independent of $\lambda \in \Lambda$.
(ii) Under the hypotheses of Lemma 7.7, the formula

$$\mathcal{L}^0(S) = \left(DP \circ K^\le\right)S - S \circ R$$

defines an operator $\mathcal{L}^0 : \mathcal{Y}_{\ell-N+1}^{0,0} \to \mathcal{C}^0$, continuous and one to one.

(iii) If the conditions for ν, ℓ *of Proposition 7.9 are satisfied, then*

$$\mathcal{S}^0 : \mathcal{Y}^{0,\nu}_\ell \to \mathcal{Y}^{0,\nu}_{\ell-N+1} \quad \text{and} \quad \mathcal{S}^1 : \mathcal{Y}^{0,\nu}_{\ell-\eta} \to \mathcal{Y}^{0,\nu}_{\ell-\eta-N+1}$$

are bounded linear operators.

Now we state and prove the differentiability results with respect to the parameter λ.

Lemma 8.4. *Let* ℓ, ν, σ *be such that* $\ell_0 < \ell \le r$, $\sigma \le s^{\le}$, $1 \le \sigma \le \nu \le r^{\le} + s^{\le}$ *and*

$$\ell - N + 1 - \frac{B}{a} - (\nu - i)\max\left\{\eta - \frac{A_p}{d}, 0\right\} - i(\eta - 1) > 0, \qquad 0 \le i \le \sigma. \tag{8.2}$$

We have that:

(1) (Low order regularity) The linear operator $\mathcal{S}^0 : \mathcal{Y}^{1,\nu}_\ell \to \mathcal{Y}^{1,\nu}_{\ell-N+1}$ *is bounded if* ℓ, ν *satisfy condition (8.2) with* $\sigma = 1$. *In addition,*

$$D_\lambda \mathcal{S}^0(T) = \mathcal{S}^0(\tilde{T}) \tag{8.3}$$

with

$$\tilde{T} = -D_\lambda(DP \circ K^{\le})\mathcal{S}^0(T) + D_x[\mathcal{S}^0(T)] \circ R \cdot D_\lambda R + D_\lambda T.$$

(2) (Higher order regularity) The linear operator $\mathcal{S}^0 : \mathcal{Y}^{\sigma,\nu}_\ell \to \mathcal{Y}^{\sigma,\nu}_{\ell-N+1}$ *is bounded.*

Proof. We have to check first that for any $T \in \mathcal{Y}^{1,\nu}_{\ell-N+1}$,

$$\mathcal{S}^0(T) \in \mathcal{Y}^{0,\nu}_{\ell-N+1}, \qquad D_\lambda \mathcal{S}^0(T) \in \mathcal{Y}^{0,\nu-1}_{\ell-N+1-(\eta-1)}.$$

The first relation, which corresponds to $\sigma = 0$, follows from Lemma 8.3. To deal with the second one, we proceed as in the proof of Lemma 7.8. We take derivatives with respect to λ formally and we check that the different factors we obtain, which will be infinite sums, are absolutely convergent, belong to $\mathcal{Y}^{0,\nu-1}_{\ell-N+1-(\eta-1)}$ and are bounded. Indeed, we formally decompose $D_\lambda \mathcal{S}^0(T) = S_1 + S_2$ with

$$S_1 = \sum_{i=0}^{\infty} \left[\prod_{j=0}^{i} (DP)^{-1} \circ K^{\le} \circ R^j\right] \left[D_\lambda T \circ R^i + (D_x T \circ R^i) D_\lambda R^i\right],$$

$$S_2 = \sum_{i=0}^{\infty} \sum_{m=0}^{i} \left(\prod_{l=0}^{m-1} (DP)^{-1} \circ K^{\le} \circ R^l\right) D_\lambda \left((DP)^{-1} \circ K^{\le} \circ R^m\right)$$
$$\times \left(\prod_{l=m+1}^{i} (DP)^{-1} \circ K^{\le} \circ R^l\right) T \circ R^i.$$

It can be checked by induction that, if $i \geq 2$,

$$D_\lambda R^i = D_\lambda R \circ R^{i-1} + \sum_{j=1}^{i-1} (D_x R^j \circ R^{i-j}) D_\lambda R \circ R^{i-j-1}.$$

Therefore, from item (i) of Lemma 8.3, if $(x, \lambda) \in V_k \times \Lambda$,

$$\begin{aligned}
\|D_\lambda R^i(x,\lambda)\| &\leq \frac{C}{(u+k+i)^{\alpha N}} + \frac{C}{(u+k+i)^{\alpha A_p d^{-1}}} \sum_{j=1}^{i-1} \frac{1}{(u+k+j)^{\alpha(N-A_p d^{-1})}} \\
&\leq \frac{C}{(u+k+i)^{\alpha A_p d^{-1}} (u+k)^{\alpha(N-A_p d^{-1})-1}} + \frac{C}{(u+k+i)^{\alpha N - 1}}
\end{aligned}$$

with C independent of λ. Then, if $x \in V_k$ and $\lambda \in \Lambda$, using the definition of $\mathcal{S}^0$,

$$\begin{aligned}
\|S_1(x,\lambda)\| &\leq C \|\mathcal{S}^0(D_\lambda T)(x)\| + C \|T\|^1_{\nu,\ell} \frac{1}{(u+k)^{\alpha B a^{-1}}} \\
&\quad \times \sum_{i=0}^{\infty} \left(\frac{(u+k)^{-\alpha(1-A_p d^{-1})}}{(u+k+i)^{\alpha(\ell-\eta-Ba^{-1}+A_p d^{-1})}} + \frac{1}{(u+k+i)^{\alpha(\ell-\eta-1)}} \right) \\
&\leq C \|T\|^1_{\nu,\ell} (u+k)^{-\alpha(\ell-N+1-(\eta-1))},
\end{aligned}$$

where we have used (iii) of Lemma 8.3 to bound $\|\mathcal{S}^0(D_\lambda T)(x)\|$. Then,

$$\|S_1(x,\lambda)\| \leq C \|T\|^1_{\nu,\ell} \|x\|^{\ell-N+1-(\eta-1)}, \qquad x \in V_\varrho \tag{8.4}$$

uniformly in $\lambda \in \Lambda$.

To deal with S_2, we first note that if $x \in V_k$ and $m \in \mathbb{N}$, then

$$\|D_\lambda\big((D_x P)^{-1} \circ K^{\leq} \circ R^m\big)(x,\lambda)\| \leq \frac{C}{(u+k+m)^{\alpha(L-1)}}.$$

Then, using Lemma 7.5,

$$\|S_2(x,\lambda)\| \leq C \|T\|^1_{\nu,\ell} \sum_{i=0}^{\infty} \frac{(u+k)^{-\alpha B a^{-1}}}{(u+k+i)^{\alpha(\ell-Ba^{-1})}} \sum_{m=0}^{i} \frac{1}{(u+k+m)^{\alpha(L-1)}}.$$

If $\alpha(L-1) < 1$, then, since $\eta = 1 + N - L$,

$$\|S_2(x,\lambda)\| \leq C \|T\|^1_{\nu,\ell} \sum_{i=0}^{\infty} \frac{(u+k)^{-\alpha B a^{-1}}}{(u+k+i)^{\alpha(\ell-Ba^{-1}+L-1-N+1)}} \leq C \|T\|^1_{\nu,\ell} (u+k)^{-\alpha(\ell-\eta+1-N+1)}$$

and we are done in this case. When $\alpha(L-1) = 1$, in other words $\eta = 1$, we take a positive quantity $\varepsilon > 0$, such that $\alpha(L-1+\varepsilon) > 1$ and $\ell - Ba^{-1} - \varepsilon > N - 1$ (this last condition can be fulfilled by hypothesis). Then

$$\|S_2(x,\lambda)\| \le C\|T\|^1_{\nu,\ell} \sum_{i=0}^{\infty} \frac{(u+k)^{-\alpha Ba^{-1}}}{(u+k+i)^{\alpha(\ell-Ba^{-1}-\varepsilon)}} \sum_{m=0}^{i} \frac{1}{(u+k+m)^{1+\alpha\varepsilon}}$$
$$\le C\|T\|^1_{\nu,\ell} \sum_{i=0}^{\infty} \frac{(u+k)^{-\alpha(Ba^{-1}+\varepsilon)}}{(u+k+i)^{\alpha(\ell-Ba^{-1}-\varepsilon)}} \le C\|T\|^1_{\nu,\ell}(u+k)^{-\alpha(\ell-N+1)}.$$

In any case, $\|S_2(x,\lambda)\| \le C\|T\|^1_{\nu,\ell}\|x\|^{\ell-N+1-(\eta-1)}$. This bound together with the corresponding one for S_1 in (8.4), leads us to conclude that $D_\lambda \mathcal{S}^0(T) \in \mathcal{Y}^{0,0}_{\ell-N+1-(\eta-1)}$.

On the one hand, $D_\lambda \mathcal{S}^0(T)$ and $\mathcal{S}^0(\tilde{T})$ belong to $\mathcal{Y}^{0,0}_{\ell-N+1-(\eta-1)}$ and both are solutions of the same linear equation $\mathcal{L}^0 H = \tilde{T}$. Since, by (ii) of Lemma 8.3, $\mathcal{L}^0$ is injective,

$$D_\lambda \mathcal{S}^0(T) = \mathcal{S}^0(\tilde{T}).$$

On the other hand, it is clear that $\tilde{T} \in \mathcal{Y}^{0,\nu-1}_{\ell-\eta+1}$ because $T \in \mathcal{Y}^{1,\nu}_{\ell}$. Therefore, using (iii) of Lemma 8.3, $\mathcal{S}^0(\tilde{T}) \in \mathcal{Y}^{0,\nu-1}_{\ell-\eta+1-N+1}$ and consequently, $D_\lambda \mathcal{S}^0(T)$ belongs to $\mathcal{Y}^{0,\nu-1}_{\ell-\eta+1-N+1}$. This ends the proof of the first item of the lemma.

To deal with the second item, we perform an induction procedure. Let $T \in \mathcal{Y}^{\sigma,\nu}_{\ell}$ and $S = \mathcal{S}^0(T)$. We have to prove that $S \in \mathcal{Y}^{\sigma,\nu}_{\ell-N+1}$. The cases $\sigma = 0, 1$ are already proven. Assume that $S \in \mathcal{Y}^{\sigma-1,\nu}_{\ell-N+1}$ for $\sigma \le s^{\le}$. We define recursively for $0 \le i \le \sigma - 1$:

$$\begin{aligned} S^i &= D^i_\lambda S, \\ T^i &= -D_\lambda(DP \circ K^{\le})S^{i-1} + D_x S^{i-1} \circ R \cdot D_\lambda R + D_\lambda T^{i-1}. \end{aligned}$$

Note that, using (8.3), $S^i = D^i_\lambda S = \mathcal{S}^0(T^i)$. Moreover, since $S \in \mathcal{Y}^{\sigma-1,\nu}_{\ell-N+1}$,

$$S^i \in \mathcal{Y}^{\sigma-1-i,\nu-i}_{\ell-N+1-i(\eta-1)}, \qquad D_x S^{i-1} \in \mathcal{Y}^{\sigma-i,\nu-i}_{\ell-N+1-\eta-i(\eta-1)}.$$

Using that $\eta = N - L + 1$ and the above properties,

$$D_\lambda(DP \circ K^{\le})S^{i-1} \in \mathcal{Y}^{\sigma-i,\nu-i+1}_{\ell-i(\eta-1)}, \qquad D_x S^{i-1} \circ R \cdot D_\lambda R \in \mathcal{Y}^{\sigma-i,\nu-i}_{\ell-i(\eta-1)}$$

so that, by recurrence one gets $T^i \in \mathcal{Y}^{\sigma-i,\nu-i}_{\ell-i(\eta-1)}$, if $0 \le i \le \sigma - 1$. We take now $i = \sigma - 1$ and we obtain that

$$T^{\sigma-1} \in \mathcal{Y}^{1,\nu-(\sigma-1)}_{\ell-(\sigma-1)(\eta-1)}.$$

Using item 1) we deduce that $D^{\sigma-1}_\lambda S = S^{\sigma-1} = \mathcal{S}^0(T^{\sigma-1}) \in \mathcal{Y}^{1,\nu-(\sigma-1)}_{\ell-(\sigma-1)(\eta-1)-N+1}$ and therefore,

$$D^\sigma_\lambda S = D_\lambda \mathcal{S}^0(T^{\sigma-1}) \in \mathcal{Y}^{0,\nu-\sigma}_{\ell-\sigma(\eta-1)-N+1}$$

which implies that $S \in \mathcal{Y}^{\sigma,\nu}_{\ell-N+1}$. □

8.3. End of the proof of Proposition 8.1

We point out that, since $K^{\leq}$ and R satisfy b) of Theorem 2.8, if $(x,\lambda)\in V_\varrho\times\Lambda$,

$$D_\lambda^i D_x^j T^\ell(x,\lambda)=\mathcal{O}(\|x\|^{\ell-j}),\qquad (i,j)\in\Sigma_{s^{\leq},r^{\leq}},$$

and, since G_ℓ is the Taylor's remainder (with respect to the (x,y) variable) of $F\in\mathcal{C}^{\Sigma_{s,r}}$,

$$D_\lambda^i D_x^j G_\ell(x,y,\lambda)=\mathcal{O}(\|(x,y)\|^{\ell-j}),\qquad (i,j)\in\Sigma_{s,r}.$$

Moreover, these bounds are uniform on $\lambda\in\Lambda$.

Standard arguments allow us to apply the fixed point theorem to obtain the existence of a solution $K^{>}$ of the fixed point equation (8.1) belonging to $\mathcal{Y}_{\ell-N+1}^{s_\ell^{>},\nu_\ell-1}$. Finally we recover the last derivative as in the analogous result in [3].

9. The analytic case

In this section we deal with the conclusions of Theorems 2.4 and 2.8 in the analytic case. We assume that F, of the form (2.11), is a real analytic map, that $A_p>d_p=b_p$ and that $K^{\leq}$, R are real analytic functions in the complex extension $\Omega(\varrho,\gamma)\times\Lambda(\gamma)$ of $V_\varrho\times\Lambda$ given by

$$\begin{aligned}\Omega(\varrho,\gamma)&:=\{x\in\mathbb{C}^n:\operatorname{Re}x\in V_\varrho,\ \ \|\operatorname{Im}x\|\leq\gamma\|\operatorname{Re}x\|\},\\ \Lambda(\gamma)&:=\{\lambda\in\mathbb{C}^{n'}:\operatorname{Re}\lambda\in\Lambda,\ \ \|\operatorname{Im}\lambda\|\leq\gamma^2\}\end{aligned}$$

with the norm $\|\cdot\|$ in $\mathbb{C}^n$ as

$$\|x\|=\max\{\|\operatorname{Re}x\|,\|\operatorname{Im}x\|\}.$$

We note that, if $x\in\Omega(\varrho,\gamma)$ with $\gamma\leq1$, then $\|x\|=\max\{\|\operatorname{Re}x\|,\|\operatorname{Im}x\|\}=\|\operatorname{Re}x\|$. We will use this fact along this section without special mention.

It is clear that the facts in Section 7.1 also hold in this new setting, as well as the reformulation of the problem as a fixed point equation, $K^{>}=\mathcal{S}^0\circ\mathcal{F}(K^{>})$ (see (8.1)), with $\mathcal{S}^0$ and $\mathcal{F}$ defined in (7.16) and (7.14). Therefore, it is enough to prove that the fixed point equation has an analytic solution.

The first thing we need to control is the weak contraction of the nonlinear terms in the analytic case. We first need to prove an analogous result to Lemma 6.1 to decompose $\Omega(\varrho,\gamma)$ properly. For that, for a given $\varrho>0$, we consider $u>0$ and $a_0>0$ such that $a_0u^{-\alpha}=\varrho$ and sequences $a_k\in\mathbb{R}$, $k\geq0$, and $b_k\in\mathbb{R}$, $k\geq1$, satisfying condition (6.1). Moreover, for any $\gamma\leq1$, we introduce

$$\Omega_k=\left\{x\in\Omega(\varrho,\gamma):\|x\|\in I_k:=\left[\frac{b_{k+1}}{(u+k+1)^\alpha},\frac{a_k}{(u+k)^\alpha}\right]\right\}=\{x\in\Omega(\varrho,\gamma):\operatorname{Re}x\in V_k\},\tag{9.1}$$

where the sets V_k where introduced in (6.2).

Lemma 9.1. *Let p be the homogeneous polynomial with respect to (x, y) defined by (2.11). Let $\mathcal{R} : \Omega(\varrho, \gamma) \to \mathbb{C}^n$ be an analytic map such that $\mathcal{R}(x, \lambda) - x - p(x, 0, \lambda) = \mathcal{O}(\|x\|^{N+1})$ uniformly in Λ.*

Assume that there exists $\varrho_0 > 0$ such that p satisfies the corresponding conditions in $H\lambda$ and, moreover, $A_p > b_p$.

Then for any $a < a_p$ and $b > b_p$, there exist $\varrho_1, \gamma_1 > 0$ such that for any $\gamma \le \gamma_1$ and $\varrho \le \varrho_1$ the following claims hold.

(1) If $(x, \lambda) \in \Omega(\varrho, \gamma) \times \Lambda(\gamma)$,

$$\|\mathcal{R}(x, \lambda) - x\| \le b\|x\|^N, \qquad \|\mathcal{R}(x, \lambda)\| \le \|x\|\big(1 - a\|x\|^{N-1}\big).$$

(2) The set $\Omega(\varrho, \gamma)$ is invariant by $\mathcal{R}$, that is, $\mathcal{R}(\Omega(\varrho, \gamma)) \subset \Omega(\varrho, \gamma)$.

(3) Let $\{a_k\}, \{b_k\}$ be the two sequences defined in Lemma 6.1 and Ω_k defined in (9.1). We have that

$$\overline{\Omega(\varrho, \gamma)}\backslash\{0\} = \cup_{k=0}^{\infty}\Omega_k \quad \textit{and} \quad \mathcal{R}(\Omega_k) \subset \Omega_{k+1}.$$

Consequently, if $x \in \Omega_k$, then one has that

$$\frac{\alpha}{b(u + k + 1 + j)}\big(1 + \mathcal{O}(k^{-\beta})\big) \le \|\mathcal{R}^j(x)\|^{N-1} \le \frac{\alpha}{a(u + k + j)}\big(1 + \mathcal{O}(k^{-\beta})\big).$$

Proof. We first note that, if $\chi(x, \lambda)$ is a real analytic function,

$$\begin{aligned}\chi(x, \lambda) =& \chi(\mathrm{Re}\, x, \mathrm{Re}\, \lambda) + \mathrm{i}D\chi(\mathrm{Re}\, x, \mathrm{Re}\, \lambda)[\mathrm{Im}\, x, \mathrm{Im}\, \lambda] \\ & - \int_0^1 (1 - \mu)D^2\chi(x(\mu), \lambda(\mu))[\mathrm{Im}\, x, \mathrm{Im}\, \lambda]^2\, d\mu,\end{aligned}$$

with $x(\mu) = \mathrm{Re}\, x + \mathrm{i}\mu \mathrm{Im}\, x$ and $\lambda(\mu) = \mathrm{Re}\, \lambda + \mathrm{i}\mu\mathrm{Im}\, \lambda$.

In addition, if $\chi, D_\lambda\chi, D^2_\lambda\chi = \mathcal{O}(\|x\|^k)$, we have that, if $\lambda \in \Lambda(\gamma)$:

$$\chi(x, \lambda) = \chi(\mathrm{Re}\, x, \mathrm{Re}\, \lambda) + \mathrm{i}D_x\chi(\mathrm{Re}\, x, \mathrm{Re}\, \lambda)\mathrm{Im}\, x + \gamma^2\mathcal{O}(\|x\|^k). \tag{9.2}$$

The first item is a direct consequence of the above expression, for $\chi(x, \lambda) = \mathcal{R}(x, \lambda) - x$, the definition (2.12) of a_p, b_p and that $\chi(x, \lambda) = p(x, 0, \lambda) + \mathcal{O}(\|x\|^{N+1})$. The second one is also a consequence of (9.2). Indeed, on the one hand, if $\gamma \le \gamma_1$ and $\varrho \le \varrho_1$, writing $\mathcal{R}(x, \lambda) = x + \chi(x, \lambda)$,

$$\begin{aligned}\mathrm{dist}(\mathrm{Re}\, \mathcal{R}(x, \lambda), (V_\varrho)^c) &= \mathrm{dist}(\mathrm{Re}\, x + p(\mathrm{Re}\, x, 0, \mathrm{Re}\, \lambda), (V_\varrho)^c) - C\gamma^2\|x\|^N - C\|x\|^{N+1} \\ &\ge \|x\|^N(a_V - \mathcal{O}(\gamma_1^2, \varrho_1)) \ge \frac{a_V}{2}\|x\|^N,\end{aligned}$$

taking γ_1, ϱ_1 small enough. On the other hand,

$$\begin{aligned}
\|\mathrm{Re}\,\mathcal{R}(x,\lambda)\| \geq& \|\mathrm{Re}\,x\|(1-(b_p+\mathcal{O}(\gamma_1^2+\varrho_1^2))\|x\|^{N-1}),\\
\|\mathrm{Im}\,\mathcal{R}(x,\lambda)\| \leq& \|(\mathrm{Id}+D_x p(\mathrm{Re}\,x,0,\mathrm{Re}\,\lambda))\mathrm{Im}\,x\| + C\gamma^2\|x\|^N\\
\leq& \gamma(1-(A_p+\mathcal{O}(\gamma_1,\varrho_1))\|x\|^{N-1})\|\mathrm{Re}\,x\|
\end{aligned}$$

and then if $A_p > b_p$, taking ϱ_1, γ_1 small enough, $\|\mathrm{Im}\,\mathcal{R}(x,\lambda)\| \leq \gamma\|\mathrm{Re}\,\mathcal{R}(x,\lambda)\|$ for any $\gamma \leq \gamma_1$.

Finally, the third item is a consequence of Lemma 6.1, item (2) and the fact that $x \in \Omega_k$ if and only if $\mathrm{Re}\,x \in V_k$ and $\|\mathrm{Im}\,x\| \leq \gamma\|\mathrm{Re}\,x\| = \|x\|$. □

Let $U(\varrho,\gamma) = \Omega(\varrho,\gamma) \times \Lambda(\gamma)$. We define the Banach space of analytic functions

$$\mathcal{Z}_k = \{h : U(\varrho,\gamma) \to \mathbb{C}^{n+m},\ \text{real analytic, such that } \|h\|_k < \infty\},$$

where

$$\|h\|_k = \sup_{(x,\lambda)\in U(\varrho,\gamma)} \frac{\|h(x,\lambda)\|}{\|x\|^k}.$$

From formula (9.2) applied to $(DP)^{-1}(K^{\leq})(x)$ one can easily prove that Lemma 7.4 holds true for $x \in \Omega(\varrho,\gamma)$. As a consequence, if the scaling parameter is small, bound (7.7) in Lemma 7.5 is also true for $x \in \Omega_k$.

A proof analogous to the ones of Lemmas 7.7 and 7.8 for the continuous case proves that a) the operator $\mathcal{L}^0 : \mathcal{Z}_\ell \to \mathcal{C}^\omega$, where $\mathcal{C}^\omega$ is the space of analytic functions on $U(\varrho,\gamma)$, is continuous and one to one, and b) the linear operator $\mathcal{S}^0 : \mathcal{Z}_\ell \to \mathcal{Z}_{\ell-N+1}$ is well defined and bounded provided $\ell - N + 1 - Ba^{-1} > 0$. In addition, in the same way as in Lemma 8.3, we obtain that there are bounds of the norms of $\mathcal{S}^0$ uniform in $\lambda \in \Lambda$.

Finally, one easily checks that the operator $\mathcal{S}^0 \circ \mathcal{F}$ is contractive on a suitable open ball of $\mathcal{Z}_{\ell-N+1}$. We skip the details which are very similar to the ones in [3]. This ends the proof in the analytic case.

It only remains to deal with the $\mathcal{C}^{\Sigma_{s,\omega}}$ case. We first note that, for any $\lambda \in \Lambda$ fixed, $K(\cdot,\lambda)$ is analytic in $\Omega(\varrho,\gamma)$ for ϱ,γ small enough independent of λ. Moreover, since $\mathcal{C}^{\Sigma_{s,\omega}} \subset \mathcal{C}^{\Sigma_{s,\infty}}$, given $F \in \mathcal{C}^{\Sigma_{s,\omega}}$ we also have that $K \in \mathcal{C}^{\Sigma_{s,\infty}}$. Therefore, $K \in \mathcal{C}^{\Sigma_{s,\omega}}$.

10. The flow case

In this section we prove Theorem 2.10, the analogous result of Theorem 2.8 for flows.

The proof is performed in two steps in Sections 10.1 and 10.2 below. The first step is to see that the Poincaré map F associated to the periodic vector field X in (2.20) has an invariant parametrization K and a reparametrization R satisfying the invariance equation $F \circ K = K \circ R$. To do so we apply Theorem 2.8. The second step is to check that the invariance condition (2.22) for flows:

$$\varphi(u; t, K(x,t,\lambda),\lambda) - K(\psi(u;t,x,\lambda),u,\lambda) = 0 \qquad (10.1)$$

is satisfied for K, where φ is the flow of X and ψ is the flow of a vector field Y on $\mathbb{R}^n$ to be determined.

We assume that the vector field $X \in \mathcal{C}^{\Sigma_{s,r}}$ where in the definition (2.13) of $\mathcal{C}^{\Sigma_{s,r}}$ we take $z = (x, y)$ and $\mu = (t, \lambda)$. We will denote by D_z and D_μ the derivatives with respect to these variables.

10.1. From flows to maps

Assume that $X \in \mathcal{C}^{\Sigma_{s,r}}$ is a T-periodic vector field of the form (2.20)

$$X(x, y, t, \lambda) = \begin{pmatrix} p(x, y, \lambda) + f(x, y, t, \lambda) \\ q(x, y, \lambda) + g(x, y, t, \lambda) \end{pmatrix}, \tag{10.2}$$

that p satisfies Hλ and let $K^{\leq}, Y \in \mathcal{C}^{\Sigma_{s^{\leq}, r^{\leq}}}$ satisfying items (a), (b) and (c) in Theorem 2.10. In particular we have that condition (2.23) is satisfied, namely,

$$X(K^{\leq}(x, t, \lambda), t, \lambda) - DK^{\leq}(x, t, \lambda)Y(x, \lambda) - \partial_t K^{\leq}(x, t, \lambda) = \mathcal{O}(\|x\|^{\ell})$$

for a given ℓ such that $\ell_0 < \ell \leq r$.

We denote by $\varphi(u; t, x, y, \lambda)$ and $\psi(u; t, x, \lambda)$ the associated flows of $\dot{z} = X(z, t, \lambda)$, $z = (x, y)$, and $\dot{x} = Y(x, \lambda)$ respectively. For $t \in \mathbb{R}$ and $u \in [t, t + T]$,

$$\varphi(u; t, K^{\leq}(x, t, \lambda), \lambda) - K^{\leq}(\psi(u; t, x, \lambda), u, \lambda) = \mathcal{O}(\|x\|^{\ell}), \tag{10.3}$$

uniformly in u, λ. The proof is a consequence of Gronwall's lemma, (2.22) and the $\mathcal{C}^0$ dependence of $K^{\leq}$ with respect to t.

We introduce the Poincaré maps $F(x, y, t, \lambda) = \varphi(t + T; t, x, y, \lambda)$ and $R(x, \lambda) = \psi(T; 0, x, \lambda) = \psi(t + T; t, x, \lambda)$. Applying (10.3) to $u = t + T$, we obtain that

$$F(K^{\leq}(x, t, \lambda), t, \lambda) - K^{\leq}(R(x, \lambda), t, \lambda) = \mathcal{O}(\|x\|^{\ell}). \tag{10.4}$$

We want to apply Theorem 2.8, so we have to check the setting and hypotheses of that theorem for F.

By Hypothesis HP and, since X is of the form (10.2), for any $(x, y) \in B_\rho$, we have $\|X(x, y, t, \lambda)\| \leq C\rho^N$. Then, on the one hand, the flow $\varphi(u; t, x, y, \lambda)$ is well defined for $u \in [t, t + T]$ if $(x, y) \in B_\varrho$ and ϱ is small enough. On the other hand, by Gronwall's lemma,

$$\|\varphi(u; t, x, y, \lambda)\| \leq C\|(x, y)\|, \qquad (u, x, y, \lambda) \in [t, t + T] \times B_\varrho \times \Lambda. \tag{10.5}$$

Now we check that F has the form (2.11). Applying Taylor's theorem to $\varphi(u; t, x, y, \lambda)$, with respect to u:

$$F(x,y,\lambda,t) = \varphi(t+T;t,x,y,\lambda) = \begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} p(x,y,\lambda)+f(x,y,t,\lambda) \\ q(x,y,\lambda)+g(x,y,t,\lambda) \end{pmatrix}$$
$$+ \int_t^{t+T} (t+T-u) D_z X(\varphi(u;t,x,y,\lambda),u,\lambda) X(\varphi(u;t,x,y,\lambda),u,\lambda)\, du$$
$$+ \int_t^{t+T} (t+T-u) D_t X(\varphi(u;t,x,y,\lambda),u,\lambda)\, du.$$

Using bound (10.5) in the above formula for the Poincaré map F, we see that F has the form (2.11) and satisfies Hλ for any fixed $t \in \mathbb{R}$ since p does not depend on t. Moreover, using that f and g are periodic with respect to t, $D^2_{(x,y)} f$, $D^2_{(x,y)} g$ are bounded and they satisfy Hypothesis HP. We also have that the remainder $(\tilde f, \tilde g) = F - \mathrm{Id} - (Tp, Tq)$ satisfies Hypothesis HP.

Concerning the items of Theorem 2.8, (a) follows from the hypotheses and general regularity results for flows, (b) for $K^{\le}$ also follows from hypothesis and (c) have already been obtained in (10.4).

It remains to check that $R(x,\lambda) = \psi(T;0,x,\lambda)$ satisfies (b) in Theorem 2.8. Namely, defining $\Delta R(x,\lambda) := R(x,\lambda) - x - Tp(x,0,\lambda)$ we have to check that, uniformly in $\lambda \in \Lambda$,

$$D^j_\lambda D^i_x \Delta R(x,\lambda) = \mathcal{O}(\|x\|^{N+1-i}), \qquad (i,j) \in \mathcal{C}^{\Sigma_{s^\le, r^\le}}.$$

These bounds are consequence of the following elementary result, whose proof we omit.

Lemma 10.1. *Let $Z : V_{\varrho_0} \times \Lambda \to \mathbb{R}^n$ be a vector field of the form $Z(x,\lambda) = Z_0(x,\lambda) + Z_1(x,\lambda)$. Let $\chi(t;x,\lambda)$ be its flow.*

Let $\sigma \ge 0$ and $\nu \ge 2$. Assume that $Z_0, Z_1 \in \mathcal{C}^{\Sigma_{\sigma,\nu}}$ and that there exist $l > k \ge 2$ such that, for all $(i,j) \in \mathcal{C}^{\Sigma_{\sigma,\nu}}$:

$$D^i_\lambda D^j_x Z_0(x,\lambda) = \mathcal{O}(\|x\|^{k-j}), \qquad D^i_\lambda D^j_x Z_1(x,\lambda) = \mathcal{O}(\|x\|^{l-j})$$

uniformly in $\lambda \in \Lambda$.

Then for any $u_0 > 0$ there exists ϱ small enough such that, if $x \in V_{\varrho/2}$ and $u \in [0,u_0]$, the flow χ satisfies $\chi(u;x,\lambda) = x + uZ_0(x,\lambda) + \widetilde{Z}_1(u,x,\lambda) \in V_\varrho$ with

$$D^i_\lambda D^j_x \widetilde{Z}_1(u,x,\lambda) = \mathcal{O}(\|x\|^{k+1-j}), \qquad (i,j) \in \Sigma_{\sigma,\nu}$$

uniformly in $(u,\lambda) \in [0,u_0] \times \Lambda$.

Summarizing, let $\max\{\ell_0,\ell_1\} < \ell \le r$, $K^{\le}$ and Y be such that (10.3) holds true. Applying Theorem 2.8 to the Poincaré map $F(x,y,t,\lambda) = \varphi(t+T;t,x,y,\lambda)$ with $R(x,\lambda) = \psi(t+T;t,x,\lambda)$, we obtain a solution $K = K^{\le} + K^{>} \in \mathcal{C}^{\Sigma_{s^>, r^>}}$ of the invariance condition

$$F(K(x,t,\lambda),t,\lambda) = K(\psi(t+T;t,x,\lambda),t,\lambda) \tag{10.6}$$

with $K^{>}(x,t,\lambda) = \mathcal{O}(\|x\|^{\ell-N+1})$ uniformly in λ. Moreover, by the uniqueness of the solution, $K^{>}$ (and consequently K) is periodic with respect to t.

10.2. From maps to periodic flows

In this section we prove that the parametrization K found in the previous Section 10.1 satisfies the invariance condition (10.1) for flows. To avoid cumbersome notations, in this section we will skip the dependence on λ.

Using the properties of general solutions of vector fields, the definitions of F and R and (10.6) we obtain

$$K(x,s) = \varphi(s; s+T, K(R(x),s)), \qquad R(\psi(s;t,x)) = \psi(s;t,R(x)).$$

We define

$$\mathcal{K}_s(x,t) = \varphi(t; s, K(\psi(s;t,x),s)).$$

We have $\mathcal{K}_t(x,t) = K(x,t)$ and

$$\begin{aligned} F(\mathcal{K}_s(x,t),t) &= \varphi(t+T; s, K(\psi(s;t,x),s)) = \varphi(t+T; s+T, K(\psi(s;t,R(x)),s)) \\ &= \varphi(t; s, K(\psi(s;t-T,x,s))), \\ \mathcal{K}_s(R(x),t)) &= \varphi(t; s, K(\psi(s;t,R(x),s))) = \varphi(t; s, K(\psi(s;t-T,x),s)). \end{aligned}$$

Consequently, $\mathcal{K}_s(x,t)$ satisfies the invariant condition (10.1) for any s.

Applying again Taylor's theorem,

$$\begin{aligned} \mathcal{K}_s(x,t) =& \varphi(t; s, K(\psi(s,t,x),s)) = \varphi(t; s, K^{\leq}(\psi(s;t,x),s)) \\ &+ \int_0^1 D\varphi(t; s, K^{\leq}(\psi(s;t,x),s) + wK^{>}(\psi(s;t,x),s))K^{>}(\psi(s;t,x),s)\, dw \end{aligned}$$

and, applying equality (10.3) to $\psi(s;t,x)$,

$$\begin{aligned} \mathcal{K}_s(x,t) - K^{\leq}(x,t) = \mathcal{O}(\|x\|^{\ell}) &+ \int_0^1 DK^{\leq}(x + w(\psi(s;t,x)-x),t)[\psi(s;t,x)-x]\, dw \\ &+ \int_0^1 D\varphi(t; s, K^{\leq}(\psi(s;t,x),s) + wK^{>}(\psi(s;t,x),s))K^{>}(\psi(s;t,x),s)\, dw. \end{aligned}$$

Therefore, since $\psi(s;t,0) = 0$ and $\psi(s;t,x) = x + \mathcal{O}(\|x\|^N)$, we have that $\mathcal{K}_s(x,t) - K^{\leq}(x,t) = \mathcal{O}(\|x\|^{\ell-N+1})$ and this implies, by the uniqueness statement in Theorem 2.8 that $\mathcal{K}_s(x,t) = K(x,t)$. Then

$$K(\psi(s;t,x),s) = \varphi(s; t, \mathcal{K}_s(x,t)) = \varphi(s; t, K(x,t))$$

and the proof is complete.

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